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THE GENERATING FUNCTION OF ASSOCIATED NUMBERS AND THE REPRESENTING FORMULA OF $\Phi(n)$ WITH COMBINATORIAL FORMULAS OF $\Phi(n, k)$ AND $C_k(n)$

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ABSTRACT: In this paper,our idea from graphical theory,the authors have proposed the concept that is called as one associated number with $N(K_n, k)$ (see [1]),denoted by $\phi(n, k)$. By means of combinatorial methods and mechanical proof of computer, we present the generating function of $\phi(n, k)$, give the recurrence relation of $C_k(n)$, derive series of combinatorial formulas of $\phi(n, k)$ and series of combinatorial formulas of $C_k(n)$, finally, solve the representing formula of $\phi(n)$.

KEYWORDS: Associated numbers, $\phi(n)$, The number of chains $C_k(n)$, Pell numbers.

1. INTRODUCTION

In order to the number $N(K_n, k)$ (the number of $S^{(n)}$-factors with exactly k components in K_n, see [1]), the authors give the definition that is called as one associated number is as follows

Definition 1: For any $k, n \in N$,

$$\phi(n, k) = \sum_{\substack{\sum_{i=1}^{n} ib_i = n \\ \sum_{i=1}^{n} b_i = k}} \frac{k!}{b_1!} \prod_{i \geq 2} \frac{1}{b_i!(i!)^{b_i}},$$

where k is the number of components of $S^{(n)}$-factors with exactly k components in the complete graph K_n, then $\phi(n, k)$ is called as one associated number with $N(K_n, k)$. (Also see LiMin Yang [1] and [2]).

Let $\phi(n)$ be the number of all associated numbers, namely, $\phi(n) = \sum_{k=1}^{n} \phi(n, k)$.

In [1], we gave the recurrence relation of $\phi(n, k)$ and some combinatorial formulas. In [2], LiMin Yang discussed the number of Fubini formulas by means of $\phi(n, k)$. In this paper, the authors will continue to research other new problems. We will present main results as follows:

(1) generating function of $\phi(n, k)$ is $\dfrac{y+1}{2y^2} e^{-2x}$;

(2) the recurrence relation of $C_k(n)$;

(3) series of combinatorial identities of $\phi(n, k)$;

(4) series of combinatorial identities of $C_k(n)$;

(5) the representing formula of $\phi(n)$.

Here combinatorial identities are referring to all kinds of numbers, for examples, involving Lucas number, Pell number, Fibonacci number and Chebishev numbers (or Chebishev polynomials).

2. LEMMAS

For any $k, n \in N$, when $k > n$, $\phi(n, k) = 0$, when $n \geq 1$, $\phi(n, 0) = 0$, when $k \geq 1$, $\phi(0, k) = 0$, and $\phi(0, 0) = 1$.

Lemma 1 [1]: For any $k, n \in N$, there exists the recurrence equality

$$\phi(n, k) = \frac{k}{n}\left[\phi(n-1, k-1) + \phi(n-1, k)\right].$$

Some special values of $\phi(n, k)$ are given as follows:

$\phi(n, n) = 1$, $\phi(n, 1) = \frac{1}{n!}$, $\phi(n, 2) = \frac{2}{n!}(2^{n-1} - 1)$, $\phi(n, n-1) = \frac{n-1}{2}$, $\phi(n, 3) = \frac{3}{n!}(2^{n-1} - 1) + \frac{3 \cdot 3!}{n!}\left[4 \cdot 3^{n-1} - 2^{n+1} - 2^{n-1} + 2^{n-3} - 8(n-1)^2 + \binom{n}{2} - 3\right].$

Lemma 2 [1]: For any associated number $\phi(n, k)$, $k, n \in N$, if $G(t) = \sum_{k=0}^{\infty} g(k)t^k$, $g(k)$ is one complex coefficient, $k \geq 1$, then we have the combinatorial formula

$$\sum_{k=0}^{\infty} \phi(n,k)\frac{1}{k!}G^{(k)}t^k = \sum_{k=0}^{\infty}\frac{g(k)}{n!}k^n t^k,$$

where $G^{(k)}(t)$ is differential of $G(t)$ of order k.

Lemma 3: There exists the combinatorial formula $n!\phi(n, k) = k!S(n, k)$, where $S(n, k)$ is the Stirling number of the second kind.

Proof: Omitted. (see[2])

3. THE GENERATING FUNCTION OF ϕ (*n, k*)

Theorem 1: If $Z = \varphi(x, y) = \sum_{n,k\geq 0} \phi(n,k)\, x^n y^k$, and $\phi(n, k)$ is any associated number, then there exists the generating function as follows

$$Z = \frac{y+1}{2y^2} e^{-2x}.$$

Proof:

Let $\varphi(x, y) = \sum_{n,k\geq 0} \phi(n, k)\, x^n y^k$, $\frac{\partial \varphi(x,y)}{\partial x} = \sum_{n\geq 1,k\geq 0} n\phi(n,k)\, x^{n-1} y^k + \sum_{n\geq 1} n\phi(n, 0)\, x\, n^{-1}$, and by Lemma 1 $\phi(n, k) = \frac{k}{n}[\phi(n-1, k-1) + \phi(n-1, k)]$, when $n \geq 1$, $\phi(n, 0) = 0$, and $\phi(0, 0) = 1$, then

$$\begin{aligned}
\frac{\partial \varphi(x,y)}{\partial x} &= \sum_{n\geq 1,k\geq 1} k[\phi(n-1,k-1) + \phi(n-1,k)]x^{n-1}y^k + 0 \\
&= \sum_{n\geq 1,k\geq 1} k\phi(n-1,k-1)x^{n-1}y^k + \sum_{n\geq 1,k\geq 1} k\phi(n-1,k)x^{n-1}y^k \\
&= y\frac{\partial}{\partial y}\sum_{n\geq 1,k\geq 1} \phi(n-1,k-1)x^{n-1}y^k + y\frac{\partial}{\partial y}\sum_{n\geq 1,k\geq 1} \phi(n-1,k)x^{n-1}y^k \\
&= y\frac{\partial}{\partial y} y \sum_{n\geq 1,k\geq 1} \phi(n-1,k-1)x^{n-1}y^{k-1} + y\frac{\partial}{\partial y}\sum_{p\geq 0,k\geq 1} \phi(p,k)x^p y^k \\
&= y\frac{\partial}{\partial y} y\varphi(x,y) + y\frac{\partial}{\partial y}[\sum_{p\geq 0,k\geq 0} \phi(p,k)x^p y^k - \sum_{p\geq 0} \phi(p,0)x^p] \\
&= y\varphi(x,y) + y^2\frac{\partial}{\partial y}\varphi(x,y) + y\frac{\partial}{\partial y}[\varphi(x,y) - 1] \\
&= y\varphi(x,y) + y^2\frac{\partial \varphi(x,y)}{y} + y\frac{\partial \varphi(x,y)}{\partial y}
\end{aligned}$$

So that we derive the equation

$$\frac{\partial \varphi(x,y)}{\partial x} - (y^2 + y)\frac{\partial \varphi(x,y)}{\partial y} = y\varphi(x,y).$$

Because of $Z = \varphi(x, y)$, then the equation is as follows

$$\frac{\partial Z}{\partial x} - (y^2 + y)\frac{\partial Z}{\partial y} = yZ.$$

The solution of the equation is proved by mechanical proof of computer as the following

$$Z = \frac{F_1\left[\frac{(y+1)e^{-x}}{y}\right]e^{-x}}{y} = F_1\frac{y+1}{y^2}e^{-2x}.(*)$$

$$Z = \varphi(x, y) = \sum_{n\ge 0, k\ge 0}\phi(n,k)x^n y^k$$

$$= 1 + \sum_{k\ge 1}\phi(0,k)y^k + \sum_{n\ge 1, k\ge 0}\phi(n,k)x^n y^k = 1 + \sum_{n\ge 1, k\ge 0}\phi(n,k)x^n y^k$$

$Z(0, 1) = 1 + \sum_{n\ge 1, k\ge 0}\phi(n, k)0^n 1^k = 1$, by the equating (*), $1 = F_1\frac{1+1}{1^2}e^{-2*0}, F_1 = \frac{1}{2}$.

So that we gain the main result

$$Z = \frac{1}{2}\frac{y+1}{y^2}e^{-2x} = \frac{y+1}{2y^2}e^{-2x}.$$

The proof is completed.

Definition 2: $C_k(n)$ is the number of chains $\phi = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_k = [n]$, or alternatively the number of ordered partitions $(S_1, S_2 - S_1, S_3 - S_2, \cdots, [n] - S_{k-1})$ of $[n]$ into k(non-empty) blocks. (see[6]).

Let $C(n)$ denote the total number of ordered partitions of set $[n]$, namely, $C(n) = \sum_{k=1}^{n} C_k(n)$.

Theorem 2: If $C_k(n)$ is the number of chains $\phi = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_k = [n]$, then there exists the recurrence relation

$$C_k(n) = kC_{k-1}(n-1) + kC_k(n-1),\ k, n \in N.$$

Proof: Because $C_k(n)$ is the number of chains, then $C_k(n) = k!S(n, k)$ (see[6]). For $S(n.k) = \frac{n!}{k!}\phi(n, k)$, then $C_k(n) = n!\phi(n, k)$. By Lemma 1, $\phi(n, k) = \frac{k}{n}[\phi(n-1, k-1)$

+ $\phi(n-1, k)]$, then $C_k(n) = n!\ \frac{k}{n}[\phi(n-1, k-1) + \phi(n-1, k)] = k[(n-1)!\phi(n-1, k-1)] + k[(n-1)!\ \phi(n-1, k)]$, and $C_{k-1}(n-1) = (n-1)!\phi(n-1, k-1)$, $C_k(n-1) = (n-1)!\phi(n-1, k)$. Finally, we prove the recurrence relation

$$C_k(n) = kC_{k-1}(n-1) + kC_k(n-1),\ k, n \in N.$$

Theorem 3: Suppose $C(n)$ is the total number of ordered partitions of set $[n]$, then $\phi(n) = \frac{1}{n!}C(n)$.

Proof: By Lemma 3, then $\phi(n) = \sum_{k=1}^{n}\phi(n,k) = \sum_{k=1}^{n}\frac{k!}{n!}S(n,k) = \frac{1}{n!}\sum_{k=1}^{n} k!S(n, k) = \frac{1}{n!}C(n)$.

The proof is completed.

4. SERIES OF COMBINATORIAL IDENTITIES ON $\phi(n, k)$

A Pell number P_n is satisfied as follows generating function

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2},$$

P_n may be interpreted combinatorially as the number of tilings of a $1\times(n-1)$ rectangle with tiles of size 1×1 and 1×2, where each 1×1 tile can be red or blue.

Theorem 4: Suppose $\phi(n, k)(n, k \in N)$ are associated numbers, and $P_n(n \geq 1)$ are Pell numbers, then

$$\sum_{n=0}^{\infty}\phi(n,k)\left[\frac{\alpha}{\alpha-\beta}\cdot\frac{1}{(\alpha-t)^{k+1}}-\frac{\beta}{\alpha-\beta}\frac{1}{(\beta-t)^{k+1}}\right]t^k = \frac{1}{n!}\sum_{n=0}^{\infty}P_k k^n t^k,$$

where $\alpha=-1+\sqrt{2}, \beta=-1-\sqrt{2}$

Proof: When $1-2t-t^2=0$, $t^2+2t-1=0$, let $\alpha=-1+\sqrt{2}$, $\beta=-1-\sqrt{2}$, and $t^2+2t-1=(t-\alpha)(t-\beta)$, then

$$\frac{t}{1-2t-t^2}=\frac{-t}{(t-\alpha)(t-\beta)}=-\frac{(t-\alpha)+\alpha}{(t-\alpha)(t-\beta)}=\frac{1}{\beta-t}+\frac{\alpha}{\alpha-\beta}\left[\frac{1}{\alpha-t}-\frac{1}{\beta-t}\right].$$

$G(t)=\frac{t}{1-2t-t^2}, G^{(k)}(t)=\frac{k!}{(\beta-t)^{k+1}}+\frac{\alpha}{\alpha-\beta}\left[\frac{k!}{(\alpha-t)^{k+1}}-\frac{k!}{(\beta-t)^{k+1}}\right]$, on the other hand,

$\sum_{n=0}^{\infty} P_k t^k = \frac{t}{1-2t-t^2}$, $P_k(k \geq 1)$ are Pell numbers, $P_0 = 1$, $g(k) = P_k$, for $k \geq 1$. By Lemma 2, then we derive the relation between associated numbers and Pell numbers as follows

$$\sum_{n=0}^{\infty} \phi(n,k) \frac{1}{k!}\left[\frac{k!}{(\beta-t)^{k+1}} + \frac{\alpha}{\alpha-\beta}\left(\frac{k!}{(\alpha-t)^{k+1}} - \frac{k!}{(\beta-t)^{k+1}}\right)\right] t^k = \sum_{k=1}^{\infty} P_k \frac{k^n}{n!} t^k$$

$$\sum_{n=0}^{\infty} \phi(n,k)\left[\frac{\alpha}{\alpha-\beta}\cdot\frac{1}{(\alpha-t)^{k+1}} - \frac{\beta}{\alpha-\beta}\frac{1}{(\beta-t)^{k+1}}\right] t^k = \frac{1}{n!}\sum_{n=0}^{\infty} P_k k^n t^k,$$

where $\alpha = -1+\sqrt{2}, \beta = -1-\sqrt{2}$

Corollary 1: If P_n is any Pell number, then there exists the recurrence relation

$$\sum_{2l+m=n} (-1)^l (m+1) P_{m+1} = \frac{1}{4}(n+1)P_n + \frac{1}{4}(3n+4)\, P_{n+1},$$

where n is non-negative integers.

Proof: Because of

$$\begin{aligned}\frac{1}{(1-2x-x^2)^2} &= \frac{1}{1+x^2}\left(\frac{x}{1-2x-x^2}\right)' \\ &= \frac{1}{1-(-x^2)}\left(\sum_{n=0}^{\infty} P_n x^n\right)' \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \sum_{n=1}^{\infty} nP_n x^{n-1} \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \sum_{n=0}^{\infty} (n+1)P_{n+1} x^n \\ &= \sum_{k=0}^{\infty}\left[\sum_{2l+m=k} (-1)^l (m+1) P_{m+1}\right] x^k\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{2l+m=n}(-1)^l(m+1)P_{m+1}\right]x^n,$$

and $\frac{1}{(1-2x-x^2)^2} = \sum_{n=0}^{\infty} \frac{1}{4}$ $((n + 1)P_n + (3n + 4)P_n+1)$ x^n(see [5]), then there exists the recurrence relation

$$\sum_{2l+m=n}(-1)^l(m+1)P_{m+1} = \frac{1}{4}\left((n+1)P_n + (3n+4)P_{n+1}\right) = \frac{1}{4}(n+1)P_n + \frac{1}{4}(3n+4)P_{n+1}.$$

The proof is completed.

Theorem 5: There exists the equality between Chebishev polynomial $U_k(x)$ of the second kind and associated numbers $\phi(n, k)$ as follows

$$\frac{1}{\gamma_1-\gamma_2}\sum_{k=1}^{\infty}\phi(n,k)\left[\frac{1}{(\gamma_2-t)^{k+1}} - \frac{1}{(\gamma_1-t)^{k+1}}\right]t^k = \frac{1}{n!}\sum_{k=1}^{\infty}U_k(x)k^n t^k,$$

where $\gamma_1 = x+\sqrt{x^2-1}, \gamma_2 = x-\sqrt{x^2-1}, x\in(-\infty,-1)\cup(1,+\infty)$.

Proof: Suppose $t^2 - 2t + 1 = 0$, then $\gamma_1 = x + \sqrt{x^2-1}$, $\gamma_2 = x - \sqrt{x^2-1}$, $x \in (-\infty, -1) \cup (1, +\infty)$,

$$\frac{1}{1-2tx+t^2} = \frac{1}{(t-\gamma_1)(t-\gamma_2)} = \frac{1}{\gamma_1-\gamma_2}\left(\frac{1}{\gamma_2-t} - \frac{1}{\gamma_1-t}\right), \quad G(t) = \frac{1}{1-2tx+t^2},$$

$$G^{(k)}(t) = \frac{1}{\gamma_1-\gamma_2}\left[\frac{k!}{(\gamma_2-t)^{k+1}} - \frac{k!}{(\gamma_1-t)^{k+1}}\right] = \frac{k!}{\gamma_1-\gamma_2}\left[\frac{1}{(\gamma_2-t)^{k+1}} - \frac{1}{(\gamma_1-t)^{k+1}}\right].$$

On the other hand, $G(t)$ is the generating function of Chebishev polynomial $U_k(x)$ of the second kind,

$$G(t) = \sum_{k\geq 0}U_k(x)t^k = \frac{1}{1-2tx+t^2}(see[3]), \quad g(k) = U_k(x), \quad k\geq 1,$$

by Lemma 2, then we have the equality as the following

$$\sum_{k=1}^{\infty}\phi(n,k)\frac{1}{k!}\frac{k!}{\gamma_1-\gamma_2}\left[\frac{1}{(\gamma_2-t)^{k+1}}-\frac{1}{(\gamma_1-t)^{k+1}}\right]t^k=\sum_{k=1}^{\infty}U_k(x)\frac{k^n}{n!}t^k,$$

$$\frac{1}{\gamma_1-\gamma_2}\sum_{k=1}^{\infty}\phi(n,k)\left[\frac{1}{(\gamma_2-t)^{k+1}}-\frac{1}{(\gamma_1-t)^{k+1}}\right]t^k=\frac{1}{n!}\sum_{k=1}^{\infty}U_k(x)k^nt^k,$$

where $\gamma_1 = x + \sqrt{x^2-1}$, $\gamma_2 = x - \sqrt{x^2-1}$, $x \in (-\infty, -1) \cup (1, +\infty)$.

Corollary 2: $U_k(x) = \dfrac{\gamma_1^{k+1}-\gamma_2^{k+1}}{2\sqrt{x^2-1}}$, where $\gamma_1 = x + \sqrt{x^2-1}$, $\gamma_2 = x - \sqrt{x^2-1}$, $x \in (-\infty, -1) \cup (1, +\infty)$, $k \in N$

Proof: The authors give a simple proof for Corollary 2. By the course of Theorem 5, and

$$\gamma_1\gamma_2 = 1,$$

$$\frac{1}{1-2tx+t^2} = \frac{1}{\gamma_1-\gamma_2}\left(\frac{1}{\gamma_2-t}-\frac{1}{\gamma_1-t}\right)=\frac{1}{\gamma_1-\gamma_2}\left(\frac{\gamma_1}{1-\gamma_1 t}-\frac{\gamma_2}{1-\gamma_2 t}\right)$$

$$= \frac{1}{\gamma_1-\gamma_2}\left(\gamma_1\sum_{k=0}^{\infty}\gamma_1^k t^k-\gamma_2\sum_{k=0}^{\infty}\gamma_2^k t^k\right)$$

$$= \frac{1}{\gamma_1-\gamma_2}\sum_{k=0}^{\infty}\left(\gamma_1^{k+1}-\gamma_2^{k+1}\right)t^k$$

$$= \sum_{k=0}^{\infty}\frac{\gamma_1^{k+1}-\gamma_2^{k+1}}{2\sqrt{x^2-1}}t^k,$$

$$\frac{1}{1-2tx+t^2} = \sum_{k=0}^{\infty}U_k(x)t^k,$$

then we have the formula $U_k(x)=\dfrac{\gamma_1^{k+1}-\gamma_2^{k+1}}{2\sqrt{x^2}-1}$, where $\gamma_1 = x + \sqrt{x^2-1}$, $\gamma_2 = x -$

$\sqrt{x^2-1}$, $x \in (-\infty, -1) \bigcup (1, +\infty)$, $k \in N$.

Corollary 3: There exists the identity

$$\frac{1}{n!}\sum_{k=1}^{\infty} k^n \sum_{l=0}^{k+1}\left[1+(-1)^{k+2-l}\right]\binom{k+1}{l} 2^l \sqrt{3}^{k+1-l} t^k = \sum_{k=1}^{\infty}\phi(n,k)\left[\frac{1}{(2-\sqrt{3}-t)^{k+1}} - \frac{1}{(2+\sqrt{3}-t)^{k+1}}\right] t^k.$$

Proof: Let $x = 2$, then $\gamma_1 = 2 + \sqrt{3}$, $\gamma_2 = 2 - \sqrt{3}$.

$$U_k(2) = \frac{(2+\sqrt{3})^{k+1} - (2-\sqrt{3})^{k+1}}{2\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}\left[\sum_{l=0}^{k+1}\binom{k+1}{l} 2^l \sqrt{3}^{k+1-l} - \sum_{l=0}^{k+1}(-1)^{k+1-l}\binom{k+1}{l} 2^l \sqrt{3}^{k+1-l}\right]$$

$$= \frac{1}{2\sqrt{3}}\sum_{l=0}^{k+1}\left[1+(-1)^{k+2-l}\right]\binom{k+1}{l} 2^l \sqrt{3}^{k+1-l}.$$

By Theorem 7, we have the identity

$$\frac{1}{n!}\sum_{k=1}^{\infty} k^n \sum_{l=0}^{k+1}\left[1+(-1)^{k+2-l}\right]\binom{k+1}{l} 2^l \sqrt{3}^{k+1-l} t^k = \sum_{k=1}^{\infty}\phi(n,k)\left[\frac{1}{(2-\sqrt{3}-t)^{k+1}} - \frac{1}{(2+\sqrt{3}-t)^{k+1}}\right] t^k.$$

Corollary 4: There exists the identity

$$\frac{1}{n!}\sum_{k=1}^{\infty} k^n \sum_{l=0}^{k+1}\left[1+(-1)^{k+2-l}\right]\binom{k+1}{l} 3^l (2\sqrt{2})^{k+1-l} t^k = \sum_{k=1}^{\infty}\phi(n,k)\left[\frac{1}{(3-2\sqrt{2}-t)^{k+1}} - \frac{1}{(3+2\sqrt{2}-t)^{k+1}}\right] t^k.$$

Proof: Let $x = 3$, then $\gamma_1 = 3 + 2\sqrt{2}$, $\gamma_1 = 3 - 2\sqrt{2}$.

$$U_k(3) = \frac{(3+2\sqrt{2})^{k+1} - (3-2\sqrt{2})^{k+1}}{4\sqrt{2}}$$

$$= \frac{1}{4\sqrt{2}}\left[\sum_{l=0}^{k+1}\binom{k+1}{l} 3^l (2\sqrt{2})^{k+1-l} - \sum_{l=0}^{k+1}(-1)^{k+1-l}\binom{k+1}{l} 3^l (2\sqrt{2})^{k+1-l}\right]$$

$$= \frac{1}{4\sqrt{2}}\sum_{l=0}^{k+1}\left[1+(-1)^{k+2-l}\right]\binom{k+l}{l} 3^l (2\sqrt{2})^{k+1-l}.$$

By Theorem 5, then we have the identity

$$\frac{1}{n!}\sum_{k=1}^{\infty} k^n \sum_{l=0}^{k+1}[1+(-1)^{k+2-l}]\binom{k+1}{l}3^l(2\sqrt{2})^{k+1-l}t^k = \sum_{k=1}^{\infty}\phi(n,k)\left[\frac{1}{(3-2\sqrt{2}-t)^{k+1}}-\frac{1}{(3+2\sqrt{2}-t)^{k+1}}\right]t^k.$$

Theorem 6: There exists the equality

$$\sum_{k=1}^{\infty}\phi(n,k)\left[\binom{d+k-2}{k-1}+t\binom{d+k-2}{k}\right]\frac{t^k}{(1-t)^{d+k}} = \frac{1}{n!}\sum_{k=1}^{\infty} D(k-1;(a))k^n t^k,$$

where $(a) = (1, 1, \cdots, 1)$; the number of 1 in (a) is d.

Proof: Let $G(t) = \frac{t}{(1-t)^d}$, $d \in N$, and $G(t) = \frac{1}{(1-t)^d} - \frac{1}{(1-t)^{d-1}}$, $G^{(k)}(t) = \frac{\langle d\rangle_k}{(1-t)^{d+1}} - \frac{\langle d-1\rangle_k}{(1-t)^{d+1-1}}$, on the other hand ,

$$G(t) = \frac{t}{(1-t)^d} = t\left(\sum_{n=0}^{\infty} t^n\right)^d = t\sum_{k=0}^{\infty}\Big(\sum_{\substack{x_1+x_2+\cdots+x_d=k\\ x_i\geq 0\\ 1\leq i\leq d}} 1\Big)t^k,$$

the equation $a_1x + a_2x + \cdots + a_dx_d = n$, the number of solutions of non-negative integers is denoted by $D(n; (a))$, $(a) = (a_1, a_2, \cdots, a_d)$ (see Louis Comtet [3]), then $G(t) = t\sum_{k=0}^{\infty} D(k; (a))t^k$, where $(a) = (1, 1, \cdots, 1)$, the number of 1 in (a) is d, $G(t) = \sum_{k=0}^{\infty} D(k-1; (a))t^k$, $g(k) = D(k-1; (a))$, $k \geq 1$.

By Lemma 2, then we have the equality

$$\sum_{k=1}^{\infty}\phi(n,k)\frac{1}{k!}\left[\frac{\langle d\rangle_k}{(1-t)^{d+k}} - \frac{\langle d-1\rangle_k}{(1-t)^{d+k-1}}\right]t^k = \sum_{k=1}^{\infty} D(k-1;(a))\frac{k^n}{n!}t^k,$$

$$\sum_{k=1}^{\infty}\phi(n,k)\frac{1}{(1-t)^{d+k-1}}\left[\frac{\binom{d+k-1}{k}}{(1-t)} - \binom{d+k-2}{k}\right]t^k = \sum_{k=1}^{\infty} D(k-1;(a))\frac{k^n}{n!}t^k,$$

$$\sum_{k=1}^{\infty}\phi(n,k)\left[\binom{d+k-2}{k-1}+t\binom{d+k-2}{k}\right]\frac{t^k}{(1-t)^{d+k}}=\frac{1}{n!}\sum_{k=1}^{\infty}D(k-1;(a))k^n t^k,$$

The proof is completed.

Corollary 5[2]: If $\phi(n)$ is the number of all associated numbers, then $\phi(n) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{k^n}{2^{k+1}}$.

Proof: Let $d = 1$, by Theorem 8, then we have the relation equality

$$\sum_{k=1}^{\infty}\phi(n,k)\left[\binom{k-1}{k-1}+t\binom{k-1}{k}\right]\frac{t^k}{(1-t)^{1+k}}=\frac{1}{n!}\sum_{k=1}^{\infty}D(k-1;(a))k^n t^k,$$

where $(a) = (1)$.

Because of $(a) = (1)$, and $D(k - 1; (a)) = 1$, then we have the equality

$$\sum_{k=1}^{\infty}\phi(n,k)\frac{t^k}{(1-t)^{1+k}}=\frac{1}{n!}\sum_{k=1}^{\infty}k^n t^k,$$

and $k > n$, $\phi(n, k) = 0$, let $t = \frac{1}{2}$, so that we gain

$$\phi(n) = \sum_{k=1}^{n}\phi(n,k)=\frac{1}{n!}\sum_{k=1}^{\infty}\frac{k^n}{2^{k+1}}.$$

Corollary 6: If $C(n)$ is the total number of chains, then $C(n) = \sum_{k=1}^{\infty}\frac{k^n}{2^{k+1}}$.

Proof: Because of $C_k(n) = k!S(n, k)$, $C(n) = \sum_{k=1}^{n}k!S(n,k)=\sum_{k=1}^{n}n!\phi(n,k) = n!\sum_{k=1}^{n}\phi(n, k)$, by Corollary 5, then

$$C(n)=n!\frac{1}{n!}\sum_{k=1}^{\infty}\frac{k^n}{2^{k+1}}=\sum_{k=1}^{\infty}\frac{k^n}{2^{k+1}}.$$

Corollary 7: There exists the combinatorial identity

$$\sum_{k=1}^{n} \Delta^k O^n = \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}.$$

Proof: Because of $\Delta^k O^n = k!S(n, k)$(*see*[6]), then $\sum_{k=1}^{n} \Delta^k O^n = \sum_{k=1}^{n} k!S(n, k)$.

By Corollary 6, $\sum_{k=1}^{\infty} k!S(n,k) = \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}$, then there exists the combinatorial identity $\sum_{k=1}^{n} \Delta^k O^n = \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}$.

Theorem 7: If $\phi(n, k)$ is any associated number, F_k is the k–th Fibonacci number, $k \geq 1$, $n \in N$, then

$$\sum_{k=1}^{\infty} \phi(n,k)\left[\frac{1}{(\gamma_1 - t)^{k+1}} - \frac{1}{(\gamma_2 - t)^{k+1}}\right] t^k = \sqrt{5}\sum_{k=1}^{\infty} F_k \frac{k^n}{n!} t^k,$$

where $\gamma_1 = \dfrac{-1+\sqrt{5}}{2}, \gamma_2 = \dfrac{-1-\sqrt{5}}{2}$.

Proof: Let $G(t) = \dfrac{1}{1-t-t^2} = \dfrac{1}{\gamma_1 - \gamma_2}\left(\dfrac{1}{\gamma_1 - t} - \dfrac{1}{\gamma_2 - t}\right)$, and $\gamma_1 = \dfrac{-1+\sqrt{5}}{2}$, $\gamma_2 = \dfrac{-1-\sqrt{5}}{2}$, $G^{(k)}(t) = \dfrac{1}{\gamma_1 - \gamma_2}\left[\dfrac{k!}{(\gamma_1 - t)^{k+1}} - \dfrac{k!}{(\gamma_2 - t)^{k+1}}\right]$, $k \geq 1$.

When $G(t) = \dfrac{1}{1-t-t^2} = \sum_{k=0}^{\infty} F_k t^k$, F_k is any k–th Fibonacci number, $k \geq 1$, then $g(k) = F_k$, $k \geq 1$, $F_k = \dfrac{\beta^{k+1} - \alpha^{k+1}}{\sqrt{5}}, \alpha = \dfrac{1-\sqrt{5}}{2}, \beta = \dfrac{1+\sqrt{5}}{2}$.

By Lemma 2, then we have the relation equality

$$\sum_{k=1}^{\infty} \phi(n,k) \frac{1}{k!} \frac{1}{\gamma_1 - \gamma_2}\left[\frac{k!}{(\gamma_1 - t)^{k+1}} - \frac{k!}{(\gamma_2 - t)^{k+1}}\right] t^k = \sum_{k=1}^{\infty} F_k \frac{k^n}{n!} t^k,$$

$$\sum_{k=1}^{\infty}\phi(n,k)\left[\frac{1}{(\gamma_1-t)^{k+1}}-\frac{1}{(\gamma_2-t)^{k+1}}\right]t^k=\sqrt{5}\sum_{k=1}^{\infty}F_k\frac{k^n}{n!}t^k,$$

where $\gamma_1=\frac{-1+\sqrt{5}}{2},\gamma_2=\frac{-1-\sqrt{5}}{2}$.

Corollary 8: If F_n is the n–th Fibonacci number, then there exists the recurrence relation

$$\sum_{l+m=n}(-1)^l 2^l(m+1)F_{n+1}=\frac{1}{5}(n+1)F_n+\left(\frac{3}{5}n+1\right)F_{n+1}.$$

Proof:

$$\begin{aligned}\frac{1}{(1-x-x^2)^2} &= \frac{1}{1+2x}\left(\frac{1}{1-x-x^2}\right)'=\sum_{n=0}^{\infty}(-2)^n x^n\left(\sum_{n=0}^{\infty}F_n x^n\right)' \\ &= \sum_{n=0}^{\infty}(-1)^n 2^n x^n\sum_{n=1}^{\infty}nF_n x^{n-1} \\ &= \sum_{n=0}^{\infty}(-1)^n 2^n x^n\sum_{n=0}^{\infty}(n+1)F_{n+1}x^n \\ &= \sum_{k=0}^{\infty}\left[\sum_{l+m=k}(-1)^l 2^l(m+1)F_{m+1}\right]x^k \\ &= \sum_{n=0}^{\infty}\left[\sum_{l+m=k}(-1)^l 2^l(m+1)F_{m_+1}\right]x^n,\end{aligned}$$

and

$$\frac{1}{(1-x-x^2)^2}=\sum_{n=0}^{\infty}\left[\left(\frac{1}{5}n+\frac{1}{5}\right)F_n+\left(\frac{3}{5}n+1\right)F_{n+1}\right]x^n(see[5]).$$

Then there exists the recurrence relation

$$\sum_{l+m=n}(-1)^l 2^l(m+1)F_{m+1}=\frac{1}{5}(n+1)F_n+\left(\frac{3}{5}n+1\right)F_{n+1}.$$

The proof is completed.

Theorem 8: Suppose $T_k(x)$ is Chebishev polynomial of the first kind, and

$$\sum_{k\geq 0} T_k(x)t^k = \frac{1-tx}{1-2tx+t^2},$$

then

$$\sum_{k=1}^{\infty} \phi(n,k)\left\{\frac{1}{\gamma_1-\gamma_2}\left[\frac{(1-x\gamma_2)}{(\gamma_2-t)^{k+1}}+\frac{(x\gamma_2-1)}{(\gamma_1-t)^{k+1}}\right]+\frac{x}{(\gamma_2-t)^{k+1}}\right\}t^k = \sum_{k=1}^{\infty} T_k(x)\frac{k^n}{n!}t^k,$$

where $\gamma_1 = x+\sqrt{x^2+1}, \gamma_2 = x-\sqrt{x^2-1}, x\in(-\infty,-1)\bigcup(1,+\infty)$.

Proof: Let $1-2tx+t^2=0$, $\gamma_1 = x+\sqrt{x^2-1}$, $\gamma_2 = x-\sqrt{x^2-1}$, $x\in(-\infty,-1)\bigcup(1,+\infty)$,

$$\frac{tx}{1-2tx+t^2} = \frac{xt}{(t-\gamma_1)(t-\gamma_2)} = \frac{x(t-\gamma_1+\gamma_1)}{(t-\gamma_1)(t-\gamma_2)}$$

$$= \frac{x\gamma_1}{\gamma_1-\gamma_2}\left(\frac{1}{\gamma_2-t}-\frac{1}{\gamma_1-t}\right)-\frac{x}{\gamma_2-t},$$

$$\frac{1}{1-2tx+t^2} = \frac{1}{(t-\gamma_1)(t-\gamma_2)}$$

$$= \frac{1}{\gamma_1-\gamma_2}\left(\frac{1}{\gamma_2-t}-\frac{1}{\gamma_1-t}\right), \text{ then}$$

$$\frac{1-tx}{1-2tx+t^2} = \frac{1}{1-2tx+t^2}-\frac{tx}{1-2tx+t^2}$$

$$= \frac{1}{\gamma_1-\gamma_2}\left(\frac{1}{\gamma_2-t}-\frac{1}{\gamma_1-t}\right)-\frac{x\gamma_1}{\gamma_1-\gamma_2}\left(\frac{1}{\gamma_2-t}-\frac{1}{\gamma_1-t}\right)+\frac{x}{\gamma_2-t}.$$

Let $G(t)=\dfrac{1-tx}{1-2tx+t^2}$, then

$$G_k(t) = \frac{1}{\gamma_1 - \gamma_2}\left[\frac{k!}{(\gamma_2 - t)^{k+1}} - \frac{k!}{(\gamma_1 - t)^{k+1}}\right] - \frac{x\gamma_1}{\gamma_1 - \gamma_2}\left[\frac{k!}{(\gamma_2 - t)^{k+1}} - \frac{k!}{(\gamma_1 - t)^{k+1}}\right] + \frac{xk!}{(\gamma_2 - t)^{k+1}}.$$

On the other hand, $T_k(x)$ is the Chebishev polynomial of the first kind, its generation function $\sum_{k=0}^{\infty} T_k(x)t^k = \frac{1-tx}{1-2tx-x^2} = G(t)$, then $g(k) = T_k(x)$, $k \geq 1$.

By Lemma 2, then we derive the identity between $\phi(n, k)$ and the Chebishev polynomial $T_k(x)$ of the first kind as follows

$$\sum_{k=1}^{\infty} \phi(n,k)\frac{1}{k!}\left\{\frac{1}{\gamma_1 - \gamma_2}\left[\frac{k!}{(\gamma_2 - t)^{k+1}} - \frac{k!}{(\gamma_1 - t)^{k+1}}\right] - \frac{x\gamma_1}{\gamma_1 - \gamma_2}\left[\frac{k!}{(\gamma_2 - t)^{k+1}} - \frac{k!}{(\gamma_1 - t)^{k+1}}\right]\right.$$

$$\left. + \frac{xk!}{(\gamma_2 - t)^{k+1}}\right\} t^k = \sum_{k=1}^{\infty} T_k(x)\frac{k^n}{n!}t^k,$$

$$\sum_{k=1}^{\infty} \phi(n,k)\left\{\frac{1}{\gamma_1 - \gamma_2}\left[\frac{1 - x\gamma_2}{(\gamma_2 - t)^{k+1}} + \frac{x\gamma_2 - 1}{(\gamma_1 - t)^{k+1}}\right] + \frac{x}{(\gamma_2 - t)^{k+1}}\right\} t^k = \sum_{k=1}^{\infty} T_k(x)\frac{k^n}{n!}t^k$$

$$\sum_{k=1}^{\infty} \phi(n,k)\left\{\frac{1 - x\gamma_2}{\gamma_1 - \gamma_2}\left[\frac{1}{(\gamma_2 - t)^{k+1}} - \frac{1}{(\gamma_1 - t)^{k+1}}\right] + \frac{x}{(\gamma_2 - t)^{k+1}}\right\} t^k = \sum_{k=1}^{\infty} T_k(x)\frac{k^n}{n!}t^k$$

where $\gamma_1 = x + \sqrt{x^2 - 1}$, $\gamma_2 = x - \sqrt{x^2 - 1}$.

Corollary 9: If $T_n(x)$ is the Chebishev polynomial of the first kind, then $T_n(x) = \frac{\gamma_1^n + \gamma_2^n}{2}$, $\gamma_1 = x + \sqrt{x^2 - 1}$, $\gamma_2 = x - \sqrt{x^2 - 1}$, $x \in (-\infty, -1) \cup (1, +\infty)$, $n \in N$.

Proof: Here we give a method to prove the Corollary 9 by the course of Theorem 8. By Theorem 8,

$$G(t) = \frac{1-tx}{1-2tx+t^2} = \frac{1}{\gamma_1 - \gamma_2}\left(\frac{1}{\gamma_2 - t} - \frac{1}{\gamma_1 - t}\right) - \frac{x\gamma_1}{\gamma_1 - \gamma_2}\left(\frac{1}{\gamma_2 - t} - \frac{1}{\gamma_1 - t}\right) + \frac{x}{\gamma_2 - t},$$

$\gamma_1\gamma_2 = 1$ and $\gamma_1 - \gamma_2 = 2\sqrt{x^2 - 1}$, then

$$G(t) = \frac{1}{\gamma_1-\gamma_2}\left[\frac{1}{\gamma_2}\sum_{k=0}^{\infty}\frac{t^k}{\gamma_2^k}-\frac{1}{\gamma_1}\sum_{k=0}^{\infty}\frac{t^k}{\gamma_1^k}\right]$$

$$-\frac{x\gamma_1}{\gamma_1-\gamma_2}\left[\frac{1}{\gamma_2}\sum_{k=0}^{\infty}\frac{t^k}{\gamma_2^k}-\frac{1}{\gamma_1}\sum_{k=0}^{\infty}\frac{t^k}{\gamma_1^k}\right]+\frac{x}{\gamma_2}\sum_{k=0}^{\infty}\frac{t^k}{\gamma_2^k}$$

$$= \frac{1}{\gamma_1-\gamma_2}\sum_{k=0}^{\infty}\left(\frac{1}{\gamma_2^{k+1}}-\frac{1}{\gamma_1^{k+1}}-\frac{x\gamma_1}{\gamma_2^{k+1}}+\frac{x\gamma_1}{\gamma_1^{k+1}}\right)t^k+x\sum_{k=0}^{\infty}\frac{t^k}{\gamma_2^{k+1}}$$

$$= \frac{1-x\gamma_1}{\gamma_1-\gamma_2}\sum_{k=0}^{\infty}\left(\frac{1}{\gamma_2^{k+1}}-\frac{1}{\gamma_1^{k+1}}\right)t^k+x\sum_{k=0}^{\infty}\frac{t^k}{\gamma_2^{k+1}}$$

$$= \sum_{k=0}^{\infty}\left(\frac{1-x\gamma_2}{\gamma_1-\gamma_2}\frac{1}{\gamma_2^{k+1}}-\frac{1-x\gamma_1}{\gamma_1-\gamma_2}\frac{1}{\gamma_1^{k+1}}\right)t^k$$

$$= \sum_{k=0}^{\infty}\frac{(1-x\gamma_2)\gamma_1^{k+1}-(1-x\gamma_1)\gamma_2^{k+1}}{(\gamma_1-\gamma_2)(\gamma_1\gamma_2)^{k+1}}$$

$$= \sum_{k=0}^{\infty}\frac{(1-x\gamma_2)\gamma_1^{k+1}-(1-x\gamma_1)\gamma_2^{k+1}}{2\sqrt{x^2-1}}t^k,$$

and $G(t) = \sum_{k=0}^{\infty} T_k(x)t^k$, so that

$$T_k(x) = \frac{(1-x\gamma_2)\gamma_1^{k+1}-(1-x\gamma_1)\gamma_2^{k+1}}{2\sqrt{x^2-1}}.$$

Finally, we derive the Chebishev polynomial of the first kind

$$T_n(x) = \frac{(1-x\gamma_2)\gamma_1^{n+1}-(1-x\gamma_1)\gamma_2^{n+1}}{2\sqrt{x^2-1}} = \frac{(\gamma_1-x)\gamma_1^n+(x-\gamma_2)\gamma_2^n}{2\sqrt{x^2-1}}$$

$$= \frac{\sqrt{x^2-1}\gamma_1^n+\sqrt{x^2-1}\gamma_2^n}{2\sqrt{x^2-1}\gamma_1^n} = \frac{\gamma_1^n+\gamma_2^n}{2},$$

where $\gamma_1 = x + \sqrt{x^2-1}$, $\gamma_2 = x - \sqrt{x^2-1}$, $n \in N$.

This completes the proof.

5. SERIES OF COMBINATORIAL IDENTITIES ON $C_k(n)$

Theorem 9: There exists the combinatorial identity

$$\sum_{k=1}^{\infty} C_k(n)\left[\frac{\alpha}{(\alpha-t)^{k+1}} - \frac{\beta}{(\beta-t)^{k+1}}\right]t^k = 2\sqrt{2}\sum_{k=1}^{\infty} P_k k^n t^k,$$

where $P_k (k \geq 1)$ are Pell numbers, $\alpha = -1 + \sqrt{2}$ and $\beta = -1 - \sqrt{2}$.

Proof: Because of $\phi(n, k) = \frac{k!}{n!} S(n, k) = \frac{1}{n!} C_k(n)$ and by *Theorem* 4, then

$$\sum_{k=1}^{\infty} C_k(n)\left[\frac{\alpha}{(\alpha-t)^{k+1}} - \frac{\beta}{(\beta-t)^{k+1}}\right]t^k = 2\sqrt{2}\sum_{k=1}^{\infty} P_k k^n t^k,$$

where $P_k (k \geq 1)$ are Pell numbers, $\alpha = -1 + \sqrt{2}$ and $\beta = -1 - \sqrt{2}$.

Theorem 10: There exists the combinatorial identity

$$\sum_{k=1}^{\infty} C_k(n)\left[\frac{1}{(\gamma_2-t)^{k+1}} - \frac{1}{(\gamma_1-t)^{k+1}}\right]t^k = 2\sqrt{x^2-1}\sum_{k=1}^{\infty} U_k(x) k^n t^k,$$

where $U_k(x) = \frac{\gamma_1^{k+1}-\gamma_2^{k+1}}{2\sqrt{x^2-1}}$, $\gamma_1 = x + \sqrt{x^2-1}$, $\gamma_2 = x - \sqrt{x^2-1}$, and $x \in (-\infty, -1) \cup (1, \infty)$.

Proof: Because of $\phi(n, k) = \frac{1}{n!} C_k(n)$ and by *Theorem* 7, then

$$\sum_{k=1}^{\infty} C_k(n)\left[\frac{1}{(\gamma_2-t)^{k+1}} - \frac{1}{(\gamma_1-t)^{k+1}}\right]t^k = 2\sqrt{x^2-1}\sum_{k=1}^{\infty} U_k(x) k^n t^k,$$

where $U_k(x) = \frac{\gamma_1^{k+1}-\gamma_2^{k+1}}{2\sqrt{x^2-1}}$, $k \in N$, $\gamma_1 = x + \sqrt{x^2-1}$, $\gamma_2 = x - \sqrt{x^2-1}$, and $x \in (-\infty, -1) \cup (1, \infty)$.

Corollary 10: There exists the combinatorial identity

$$\sum_{k=1}^{\infty} k^n \sum_{l=0}^{k+1} \left[1+(-1)^{k+2-l}\right]\binom{k+1}{l} 2^l \sqrt{3}^{k+1-l} t^k$$

$$= \sum_{k=1}^{\infty} C_k(n) \left[\frac{1}{(2-\sqrt{3}-t)^{k+1}} - \frac{1}{(2+\sqrt{3}-t)^{k+1}}\right] t^k.$$

Proof: Because of $\phi(n, k) = \frac{1}{n!} C_k(n)$ and by *Corollary* 3,then

$$\sum_{k=1}^{\infty} k^n \sum_{l=0}^{k+1} \left[1+(-1)^{k+2-l}\right]\binom{k+1}{l} 2^l \sqrt{3}^{k+1-l} t^k$$

$$= \sum_{k=1}^{\infty} C_k(n) \left[\frac{1}{(2-\sqrt{3}-t)^{k+1}} - \frac{1}{(2+\sqrt{3}-t)^{k+1}}\right] t^k.$$

Corollary 11: There exists the combinatorial identity

$$\sum_{k=1}^{\infty} k^n \sum_{l=0}^{k+1} \left[1+(-1)^{k+2-l}\right]\binom{k+1}{l} 3^l (2\sqrt{2})^{k+1-l} t^k$$

$$= \sum_{k=1}^{\infty} C_k(n) \left[\frac{1}{(3-2\sqrt{2}-t)^{k+1}} - \frac{1}{(3+2\sqrt{2}-t)^{k+1}}\right] t^k.$$

Proof: Because of $\phi(n, k) = \frac{1}{n!} C_k(n)$, by *Corollary* 4, then

$$\sum_{k=1}^{\infty} k^n \sum_{l=0}^{k+1} \left[1+(-1)^{k+2-l}\right]\binom{k+1}{l} 3^l (2\sqrt{2})^{k+1-l} t^k$$

$$= \sum_{k=1}^{\infty} C_k(n) \left[\frac{1}{(3-2\sqrt{2}-t)^{k+1}} - \frac{1}{(3+2\sqrt{2}-t)^{k+1}}\right] t^k.$$

Theorem 11: There exists the combinatorial identity

$$\sum_{k=1}^{\infty} C_k(n) \left[\binom{d+k-2}{k-1} + t\binom{d+k-2}{k}\right] \frac{t^k}{(1-t)^{d+k}} = \sum_{k=1}^{\infty} D(k-1;(a)) k^n t^k,$$

where $(a) = (1, 1, \cdots, 1)$, the number of 1 in (a) is d.

Proof: Because of $\phi(n, k) = \frac{1}{n!} C_k(n)$ and by *Theorem* 6, then

$$\sum_{k=1}^{\infty} C_k(n)\left[\binom{d+k-2}{k-1} + t\binom{d+k-2}{k}\right]\frac{t^k}{(1-t)^{d+k}} = \sum_{k=1}^{\infty} D(k-1;(a))k^n t^k,$$

where $(a) = (1, 1, \cdots, 1)$, the number of 1 in (a) is d.

Theorem 12: There exists the combinatorial identity

$$\sum_{k=1}^{\infty} C_k(n)\left[\frac{1}{(\gamma_1 - t)^{k+1}} - \frac{1}{(\gamma_2 - t)^{k+1}}\right]t^k = \sqrt{5}\sum_{k=1}^{\infty} F_k k^n t^k,$$

where F_k is the k-th Fibonacci number, $\gamma_1 = \frac{-1+\sqrt{5}}{2}$ and $\gamma_2 = \frac{-1-\sqrt{5}}{2}, k \in N$.

Proof: Because of $\phi(n, k) = \frac{1}{n!} C_k(n)$ and by *Theorem* 7, then

$$\sum_{k=1}^{\infty} C_k(n)\left[\frac{1}{(\gamma_1 - t)^{k+1}} - \frac{1}{(\gamma_2 - t)^{k+1}}\right]t^k = \sqrt{5}\sum_{k=1}^{\infty} F_k k^n t^k,$$

where F_k is the k-th Fibonacci number, $\gamma_1 = \frac{-1+\sqrt{5}}{2}$ and $\gamma_2 = \frac{-1-\sqrt{5}}{2}, k \in N$.

Theorem 13: There exists the combinatorial identity

$$\sum_{k=1}^{\infty} C_k(n)\left\{\frac{1}{\gamma_1 - \gamma_2}\left[\frac{1 - x\gamma_2}{(\gamma_2 - t)^{k+1}} + \frac{x\gamma_2 - 1}{(\gamma_1 - t)^{k+1}}\right] + \frac{x}{(\gamma_2 - t)^{k+1}}\right\} = \sum_{k=1}^{\infty} T_k(x) k^n t^k,$$

where $\gamma_1 = x + \sqrt{x^2 - 1}$, $\gamma_2 = x - \sqrt{x^2 - 1}$ and $T_k(x) = \frac{\gamma_1^k + \gamma_2^k}{2}$, $k \in N$.

Proof: Because of $\phi(n, k) = \frac{1}{n!} C_k(n)$ and by *Theorem* 8, then

$$\sum_{k=1}^{\infty} C_k(n)\left\{\frac{1}{\gamma_1 - \gamma_2}\left[\frac{1 - x\gamma_2}{(\gamma_2 - t)^{k+1}} + \frac{x\gamma_2 - 1}{(\gamma_1 - t)^{k+1}}\right] + \frac{x}{(\gamma_2 - t)^{k+1}}\right\} = \sum_{k=1}^{\infty} T_k(x) k^n t^k,$$

where $\gamma_1 = x + \sqrt{x^2-1}$, $\gamma_2 = x - \sqrt{x^2-1}$ and $T_k(x) = \frac{\gamma_1^k+\gamma_2^k}{2}, k \in N$.

Corollary 12: There exists the combinatorial identity

$$\sum_{k=1}^{\infty}\left\{k^n \sum_{l=1}^{[\frac{k}{2}]}\binom{k}{2l}2^{3l+1}3^l5^{k-2l}\right\}t^k = \sum_{k=1}^{\infty} C_k(n)\left[\frac{-5+2\sqrt{6}}{(5+2\sqrt{6}-t)^{k+1}} + \frac{15-2\sqrt{6}}{(5-2\sqrt{6}-t)^{k+1}}\right]t^k.$$

Proof: Let $x = 5$, then $\gamma_1 = 5 + 2\sqrt{6}$, $\gamma_2 = 5 - 2\sqrt{6}$. For $x = 5$,

$$T_k(5) = \frac{(5+2\sqrt{6})^k + (5-2\sqrt{6})^k}{2} = \frac{1}{2}\left[\sum_{l=0}^{k}\binom{k}{l}(2\sqrt{6})^l 5^{k-l} + \sum_{l=0}^{k}\binom{k}{l}(-2\sqrt{6})^l 5^{k-l}\right]$$

$$\frac{1}{2}\sum_{l=0}^{k}[1+(-1)^l]\binom{k}{l}(2\sqrt{6})^l 5^{k-l} = \frac{1}{2}\sum_{l=0}^{[\frac{k}{2}]}2\binom{k}{2l}(2\sqrt{6})^{2l}5^{k-2l}$$

$$\sum_{l=0}^{[\frac{k}{2}]}\binom{k}{2l}2^{2l}6^l5^{k-2l} = \sum_{l=0}^{\left[\frac{k}{2}\right]}\binom{k}{2l}2^{3l}3^l5^{k-2l}$$

$$\frac{1}{\gamma_1-\gamma 2}\left[\frac{1-x\gamma_2}{(\gamma_2-t)^{k+1}} + \frac{x\gamma_2-1}{(\gamma_1-t)^{k+1}}\right] + \frac{x}{(\gamma_2-t)^{k+1}}$$

$$\frac{1}{4\sqrt{6}}\left[\frac{10\sqrt{6}-24}{(5-2\sqrt{6}-t)^{k+1}} + \frac{24-10\sqrt{6}}{(5+2\sqrt{6}-t)^{k+1}}\right] + \frac{5}{(5-2\sqrt{6}-t)^{k+1}}$$

$$\frac{1}{2}\left[\frac{-5+2\sqrt{6}}{(5+2\sqrt{6}-t)^{k+1}} + \frac{15-2\sqrt{6}}{(5-2\sqrt{6}-t)^{k+1}}\right].$$

Then by *Theorem* 13, there exists the combinatorial identity

$$\frac{1}{2}\sum_{k=1}^{\infty} C_k(n)\left[\frac{-5+2\sqrt{6}}{(5+2\sqrt{6}-t)^{k+1}} + \frac{15-2\sqrt{6}}{(5-2\sqrt{6}-t)^{k+1}}\right]t^k = \sum_{k=1}^{\infty}k^n\sum_{l=0}^{[\frac{k}{2}]}\binom{k}{2l}2^{3l}3^l5^{k-2l}t^k,$$

$$\sum_{k=1}^{\infty}\left\{k^n\sum_{l=0}^{[\frac{k}{2}]}\binom{k}{2l}2^{3l+1}3^l5^{k-2l}t^k\right\}=\sum_{k=1}^{\infty}C_k(n)\left[\frac{-5+2\sqrt{6}}{(5+2\sqrt{6}-t)^{k+1}}+\frac{15-2\sqrt{6}}{(5-2\sqrt{6}-t)^{k+1}}\right]t^k.$$

Theorem 14: For any $C_k(n)$, $k, n \in N$, if $G(t) = \sum_{k=1}^{\infty} g(k)t^k$, $g(k)$ is one complex coefficient, $k \geq 1$, then there exists the combinatorial formula

$$\sum_{k=1}^{\infty}C_k(n)G^{(k)}(t)\frac{t^k}{k!}=\sum_{k=1}^{\infty}g(k)k^nt^k.$$

Proof: Because of $C_k(n) = k!S(n, k) = n!\phi(n, k)$, then $\phi(n, k) = \frac{1}{n!}C_k(n)$. By *Lemma* 2, there exists the equality

$$\sum_{k=1}^{\infty}\phi(n,k)G^{(k)}(t)\frac{t^k}{k!}=\sum_{k=1}^{\infty}\frac{g(k)}{n!}k^nt^k.$$

Then

$$\sum_{k=1}^{\infty}\frac{1}{n!}C_k(n)G^{(k)}(t)\frac{t^k}{k!}=\sum_{k=1}^{\infty}\frac{g(k)}{n!}k^nt^k,\sum_{k=1}^{\infty}C_k(n)G^{(k)}(t)\frac{t^k}{k!}=\sum_{k=1}^{\infty}g(k)k^nt^k.$$

The proof is completed.

Corollary 13: For any $C_k(n)$, $k, n \in N$, there exists the equality

$$\sum_{k=1}^{\infty}C_k(n)\frac{t^k}{(1-t)^{k+1}}=\sum_{k=1}^{\infty}k^nt^k.$$

Proof: Let $G(t) = \frac{1}{1-t}$, then $G^{(k)}(t) = \frac{k!}{(1-t)^{k+1}}$ and $G(t) = \frac{1}{1-t} = \sum_{k=0}^{\infty}$, here $g(k) = 1$, $k \geq 1$. By *Theorem* 13 we have the equality

$$\sum_{k=1}^{\infty}C_k(n)\frac{t^k}{(1-t)^{k+1}}=\sum_{k=1}^{\infty}k^nt^k.$$

Corollary 14: Let $C(n)$ be the total number of chains, then

$$C(n)=\sum_{k=1}^{\infty}\frac{k^n}{2^{k+1}}. \quad \textit{(see Corollary 6)}$$

Proof: We give another method to prove *Corollary* 14.

Let $t = \frac{1}{2}$ and by *Corollary* 13. Then we derive the explicit formula as follows

$$\sum_{k=1}^{\infty} C_k(n) \frac{(\frac{1}{2})^k}{(1-\frac{1}{2})^{k+1}} = \sum_{k=1}^{\infty} k^n \left(\frac{1}{2}\right)^k, C(n) = \sum_{k=1}^{n} C_k(n) = \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}.$$

6. THE REPRESENTING FORMULA OF $\phi(n)$

Theorem 15: If $\phi(n)$ is the number of all associated numbers, then

$$\phi(n) = \frac{1}{n!} \sum_{k=1}^{n} \Delta^k O^n$$

, where Δ is the difference operator.

Proof: By *Corollary* 5, then $\phi(n) = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}$. On the other hand, by *Corollary* 7, then $\sum_{k=1}^{\infty} \Delta^k O^n = \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}$. Finally, $\phi(n) = \frac{1}{n!} \sum_{k=1}^{n} \Delta^k O^n$.

Here we solve the representing formula of $\phi(n)$.

Corollary 15: There exists the combinatorial identity

$$\sum_{k=1}^{n} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n = \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}.$$

Proof: Because of $\Delta^k O^n = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$ (see[6]) and by the course of Theorem 15 $\sum_{k=0}^{n} \Delta^k O^n = \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}$, then

$$\sum_{k=1}^{n} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n = \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}}.$$

7. CONCLUSIONS AND FUTURE WORK

In this paper, we solve the generating function of associated numbers $\phi(n, k)$, obtain the explicit formulas of $C(n)$ and $\phi(n)$, discuss series of combinatorial formulas involving Lucas number, Pell number, Fibonacci number and Chebishev numbers, finally, present the representing formula of $\phi(n)$ on the difference operator. In future work, we will give some other results on associated numbers $\phi(n, k)$.

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