International Journal of Combinatorial Graph Theory and Applications
Vol. 5, No. 1, (January- June, 2020)

DIRECTED ROMAN DOMINATION IN DIGRAPHS

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ABSTRACT: A directed Roman dominating function on a digraph D = (V, E) is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that for every vertex u for which f (u) = 0, there is at least one vertex n for which f (n) = 2 and $(n, u) \in E$. The weight of a directed Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a directed Roman dominating function of a directed graph G is called directed Roman dominating number of $\gamma_d(D)$. In this paper, we study the graph theoretic properties of this variant $\gamma_d(D)$ of the directed Roman dominating number for paths of a directed graph.

KEYWORDS AND PHRASES: Graph theory, Domination, Digraphs, Directed domination.

1. INTRODUCTION

Graph: A graph G is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices of G called edges. The vertex set and the edge set of G are denoted by V (G) and E (G) respectively. If e = {u, n} is an edge we write e = uν we say that e joints the vertices u and v, u and v are incident with e. If e_1 and e_2 are distinct edges of G incident with a common vertex then e_1 and e_2 are said to be adjacent edges. The number of vertices in G is called the order of G and the number of edges in G is called the size of G. A graph of order n and size m is called a (n, m) graph A graph is trivial if its vertex set is a singleton.

A vertex u is called a neighbour of a vertex v in G, if uv is an edge of G. The set of all neighbours of v is the open neighbourhood of ν and is denoted by N (v); the set $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v in G.

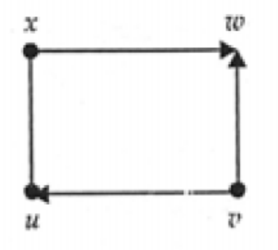
Let $S \subseteq V$, then define $N(S) = \cup_{u \in S} N(u)$ and $N[S] = \cup_{u \in S} N[u]$ if $(u, v) \in E$ then u is said to be adjacent to v and ν is said to be adjacent from u.

Digraph: A graph D = (V, E) is said to be digraph if E is subset of {(u, v); u, $\nu \in V, u \neq \nu$). Some times we done V (D) and E (D) instead of V and E respectively to stress the digraph D.

Representation. An edge $(u, \nu) \in E$ is represents as $u \to v$

If $(u, \nu) \in E$ and $(\nu, u) \in E$ then it is represent as $u \quad \to \nu$

Example 1:



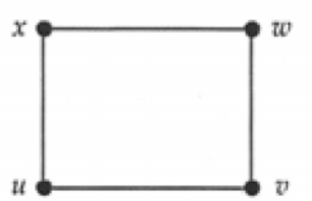
Here V (D) = {u, ν, w, u} and E (D) = {(u, x), (x, u), (ν, u), (v, w), (x, w)}.

Notations: $d_0(\nu)$ denotes the out degree of ν, $d_i(\nu)$ denotes the indegree of ν $d_{i0}(\nu)$ denotes the in-out degree of v. For example in the above example = 1. p and q denotes $|V|$ and $|E|$ respectively. $\delta_0(D)$ and $\Delta_0(D)$ denotes minimum and maximum out degree of D respectively.

We use the following notations.

$$N_0(v) = \{u \in V : (\nu, u) \in E\}$$
$$N_0[v] = \{\nu\} \cup N_0(\nu),$$
$$N_i(v) = \{u \in V : (u, \nu) \in E\},$$
$$N_i[v] = \{\nu\} \cup N_i(\nu)$$
$$N_{i0}(v) = \{u \in V : (u, \nu) \in E \text{ and } (\nu, u) \in E\},$$
$$N_{i0}[v] = \{\nu\} \cup N_{i0}(\nu).$$

Underlying Graph: Let D be a digraph. The underlying graph G (D) of D is the undirected graph obtained from D by removing the directions. For example the underlying graph of the digraph in Example 1 is



Proposition 1: $d_G(\nu) = d_i(\nu) + d_0(\nu) - d_{i0}(\nu)$ where G is the underlying graph of D. $d_0(\nu) = |N_0(\nu)|$ $d_i(\nu) = |N_i(\nu)|$ and $d_{i0}(\nu) = |N_{i0}(\nu)|$.

Proof: Proof is obvious.

Domination Number: The domination number of G is the minimum cardinality taken overall all dominating set in G and is denoted by $\gamma(G)$.

Independence Number: Independence number of a graph G is the maximum cardinality of an independent set of G and is denoted by $\beta(G)$.

Roman Dominating Number: Let G be an undirected graph. A function $f = (V_0, V_1, V_2)$ on G is a Roman dominating function (RDF) if $V_2 \succ V_0$ where $\succ$ means that the set V_2 dominates the set V_0 (i.e.) $V_0 \subseteq N(V_2)$. The weight of f is $f(v) = \sum_{\nu \in V} f(v) = 2n_2 + n_1$, where $n_i = |V_i|$ for i = 0, 1, 2. The Roman domination number denoted by $\gamma_R(G)$ equals the minimum weight of an RDF of G and we say that a function $f = (V_0, V_1, V_2)$ is a γ_R function if it is an RDF and $f(\nu) = \gamma_R(G)$.

Directed Dominating Number: Led D = (V, E) be a digraph. A set $S \subseteq V$ is called a directed dominating set in D if $N_0[S] = V$. The directed dominating number $\overline{\gamma(G)}$ is the minimum cardinality of a directed dominating set in D and a directed dominating set S of minimum cardinality is called a $\overline{\gamma}$ set of D.

2. DIRECTED ROMAN DOMINATING NUMBER

A directed Roman dominating function (abbreviated by dRDF) in a directed graph D = (V, E) is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that for every vertex u for which f (u) = 0 there is at least one vertex ν for which $f(\nu) = 2$ and $(\nu, u) \in E$. The weight of a directed Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a directed Roman dominating function of a directed graph D is called directed Roman dominating number and it is denoted by $\gamma_d(G)$.

Let $\nu \in S \subseteq V(D)$. Vertex u is called a diprivate neighbour of v with respect to S (denoted by u is an S – dpn of ν), if $(\nu, u) \in E(D)$ $(x, u) \notin E$ for all other $x \in S$. The set $dpn(\nu, S) = N_0[n] - N_0[S - \{\nu\}]$ of all S-dpns of ν is called the diprivate neighbourhood set of ν with respect to S. The set S is said to be di-irredundant it for every $\nu \in S$ $dpn(\nu, S) \neq \emptyset$. A S-dpn u of ν is said to be external if $u \notin S$.

Independent Vertex: A vertex ν is said to be independent with respect to a dRDF f if $f(\nu) \neq 0$.

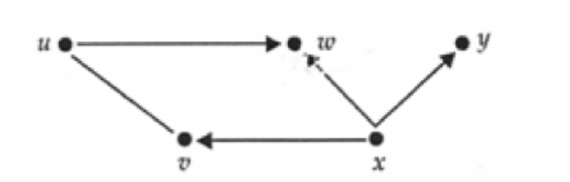
Uniformly Independent Vertex: A vertex ν is said to be uniformly independent if $f(\nu) \neq 0$ for all dRDF.

Proposition 2: If $d_i(\nu) = 0$ if and only if ν is uniformly independent vertex.

Proof: Proof is obvious.

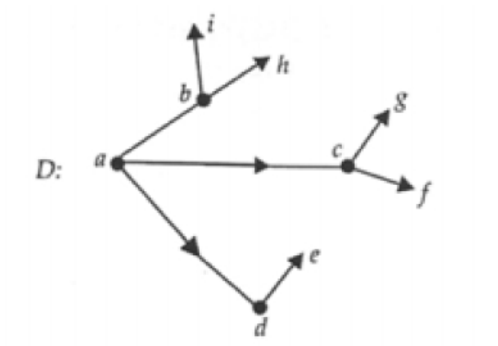
Definition: A dRDF f with $f(V) = \gamma_d(D)$ is called γ_d function.

Example 2: Define f (x) = 2, $f(\nu) = f(w) = f(y) = 0$, f (u) = 1, obviously this function f is a γ_d function. For if, f is not γ_d function, let g be a γ_d function. It is obvious that x is uniformly independent vertex. If g (x) = 1, then g (y) must be 1.



Now, $g(N_0[u]) \geq 2$. Therefore $g(V) \geq 4$. This is a contraction to minimality of g. Therefore f is the γ_d function.

Example 3: Here, $\gamma_d(D) = 7$. For a is uniformly independent vertex. Therefore $f(a) \neq 0$ for all dRDF. Let f be any arbitrary dRDF.



Case 1: $f(a) = 1$.

$f(\{b, i, h\} \geq 2$.

Similarly $f(\{c, g, f\}) \geq 2$. $f(\{d, e\}) \geq 2$.

Therefore $f(V) \geq 2 + 2 + 2 + 1 = 7$.

Case 2: $f(a) = 2$.

$f(\{b, i, h\}) \geq 2$.

Similarly $f(\{c, g, f\}) \geq 2$. $f(\{d, e\}) \geq 1$.

Therefore $f(V)) \geq 2 + 2 + 2 + 1 = 7$.

Define $g(a) = 2$, $g(b) = g(c) = g(d) = 0$ and $g(i) = g(h) = g(g) = g(e) = g(f) = 1$, $f(V) = 7$.

Therefore $\gamma_d(D) = 7$.

Representation of a Digraph in Matrix Form: Let D be a (p, q) digraph. We define a square matrix $m(D) = m(m_{uv})$ of order p as follows:

$$m_{uv} = \begin{cases} 1 & \text{if } (u, \nu) \in E \\ 0 & \text{if } (u, \nu) \notin E \end{cases}$$

For example, m (D) of the Example 2 is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \end{pmatrix}$$

Properties of m (D).

Property 1: $D \cong G(D) \Leftrightarrow m(D)$ is a symmetric matrix.

Let N (D) and M (D) denotes the number of dRD functions and gd functions of D. Consider the set $S(D) = \{f : f : V(D) \to \{0, 1, 2\}\}$ obviously $|S(D)| = 3^p$, where $|S(D)|$ denotes the number of elements in S (D). Let $R(D) = \{f \in S(D) : f \text{ is dRDF}\}$. $N(D) = |R(D)|$. It is obvious that $R(D) \subseteq S(D)$. Therefore $N(D) \leq 3^p$.

Theorem 2: $N_d(D) \leq 3^p - 2^p + 1$.

Proof: Consider the set $A = \{f : V(D) \to \{0, 1\}\}$. It is obvious that $|A| = 2^p$. For $f \in A$ if f (u) = 0 for at least one u, then f is not a dRDF. Therefore f (u) = 1 for all $u \in V(D)$ is the only dRDF in A. Therefore, we find that there are $2^p - 1$ functions which are not dRDF. Therefore $N_d(D) \leq 3^p - 2^p + 1$.

Algorithm to define a dRDF

Step 1: Enter the matrix m (D)

Step 2: Choose the vertex ν with $d_0(\nu) = \Delta_0(D)$.

That is row with maximum number of 1's.

Step 3: Define $f(\nu) = 2$ and $f(u) = 0$ for all $u \in N_0(\nu)$.

Step 4: Delete all the rows and columns corresponding to the vertices at which f was defined. We get a reduced matrix.

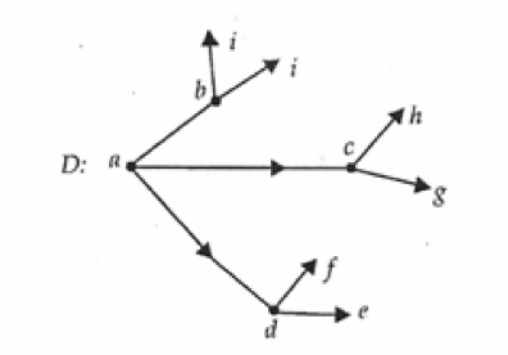
Step 5: Case (i) If f is defined for all the vertices. Go to Step 7.

Case (ii) $D = D - \{\nu : f(\nu) \text{ is defined}\}$

Step 6: Go to Step 1.

Step 7: End.

The dRDF defined using the above algorithm may not be γ_d functions, for example



The function defined using the above algorithm is

$$f(a) = 2,\ f(b) = 0,\ f(c) = 0,$$

$$f(d) = 0,\ f(e) = f(f) = f(g) = f(h) = f(i) = f(j) = 1,$$

$$f(V) = 8.$$

Now define

$$g(b) = g(c) = g(d) = 2,$$

$$g(a) = 1$$

$$g(x) = 0 \quad \text{if} \quad x \in \{a, b, c, d\}.$$

In the above graph $g(V) = 7$. Therefore, $f(V)$ is not a γ_d functions.

Theorem 3: For any digraph D, $\gamma_R(G(D)) \leq \gamma_d(D)$ £ $n + 1 - \Delta_0(D)$.

Proof: Let $f = (V_0, V_1, V_2)$ be a γ_d functions of D. Clearly, f is a Roman dominating function of G(D). Therefore, $\gamma_R(G(D)) < \gamma_d(D)$. Choose the vertex such that $\Delta_0(D) = d_0(\nu)$. Define $f(\nu) = 2$ and $f(u) = 0$ for all $u \in N_0(\nu)$. Define $f(x) = 1$ for all other vertices. Obviously f is a dRDF and $f(\nu) = n - \Delta_0(D) + 1$. Therefore, $\gamma_d(D) \leq n + 1 - \Delta_0(D)$.

Theorem 4: $\gamma_d(D) = \gamma_R$, $(G(D))$ if and only if every γ_d function of D is a γ_R function of G(D).

Proof: Proof is obvious.

Theorem 5: If f is dRDF in D, $S = \{u \in V(D) : f(u) = 2\}$ is a directed dominating set in D.

Proof: Let $u \in V$. Suppose that $f(u) = 0$ then by definition there is a vertex $\nu \in V$ such that $(\nu, u) \in E$ and $f(u) = 2$. Therefore $N_0[S] = V$.

Let f be a dRDF function of D and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{\nu \in V / f(\nu) = i\}$ and $|V_i| = n_i$, for i = 0, 1, 2. There exists a one to one correspondence between the functions $f : V \rightarrow \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Therefore one can write $f = (V_0, V_1, V_2)$.

A function $f = (V_0, V_1, V_2)$ is a directed Roman dominating function (dRDF) if $V_0 \subseteq N_0(\nu_2)$. The weight of f is $f(\nu) = \sum_{\nu \in V} f(\nu) = 2n_2 + n_1$.

Proposition 6: For any digraph D of order n, $\gamma(G(D)) = \gamma_R(G(D))) = \gamma_d(D)$ if and only if $D = \overline{K_n}$.

Proof: It is obvious that if $D = \overline{K_n}$ then G(D) = D and $\gamma(D) = \gamma_R(D) = \gamma_d(D) = n$. Conversely, let $f = (V_0, V_1, V_2)$ be a γ_d function $\gamma_d(D) = |V_1| + 2|V_2|$, $\gamma_R(G(D)) \leq |V_1| + 2|V_2|$. But given that $\gamma_R(G(D)) = |V_1| + 2|V_2| = \gamma(G(D))$, $\gamma(G(D)) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_d(D)$.

Therefore,

$$|V_1| + |V_2| = |V_1| + 2|V_2|$$

$$|V_2| = 0$$

$$|V_0| = 0$$

$$g_d(D) = |V_1| = |V| = n,$$

$$g_d(G(D)) = n,$$

$$G(D) = \overline{K_n}.$$

Therefore

$$D = \overline{K_n}.$$

Proposition 7: Let $f = (V_0, V_1, V_2)$ be any g_d function. Then

(a) $\Delta_0(D(V_1)) = 1$, where $D(V_1)$ is the subgraph induced by V_1.

(b) $(V_2 \times V_1) \cap E(D) = Ø$.

(c) For all $u \in V_0$, $|N_0(u) \cap V_1| \leq 2$.

(d) V_2 is a directed dominating set of $D(V_0 \cup V_1)$.

(e) Let $D_1 = D_1(V_0 \cup V_2)$ the digraph generated by $V_0 \cup V_2$ from D. Let $\nu \in V_2$ and $N_i(\nu) \cap V_2 \neq Ø$. Then ν has at least two V_2 – diprivate neighbourhood in V_0.

(f) Let $\nu \in V_2$ and has precisely one external V_2-dpn, say $w \in V_0$ and $(w, \nu) \in E(D)$ then $N_0(w) \cap V_1 = Ø$.

(g) $k_1 = |\{u \in V_2 : |N_i(\nu) \cap V_2| \neq \emptyset\}|$ and $c = |\{w \in V_0 : |N_i(w) \cap V_2| \geq 2\}|$. Then $n_0 \geq n_2 + k_1 + c$.

Proof:

(a) Suppose $\Delta_0(D(V_1)) > 1$, there is at least one vertex $\nu \in V_1$ such that $\{\nu_1, \nu_2, \nu_3, \ldots, \nu_m\} \subseteq N_0(\nu) \cap V_1$ with $m > 1$. Now, define $g(\nu) = 2$, $g(\nu i) = 0$ for all $i = 1, 2, \ldots, m$ and $g(\nu) < f(\nu)$, which is a contradiction. Then for $\Delta_0(D(V_1)) = 1$.

(b) Suppose $(V_2 \times V_1) \cap E(D) \neq \emptyset$. Let $(\nu_2, \nu_1) \in E(D)$ with $\nu_2 \in V_2$ and $\nu_1 \in V_1$. Now define $g(\nu_1) = 0$, $g(u) = f(u)$ for all $u \neq \nu_1$ certainly g is a γ_d function $g(V) = f(V) - 1$, which is a contradiction.

(c) Suppose there exists some $u \in V_0$ and $|N_0(u) \cap V_1| \geq 3$. Then there exists $\{u_1, u_2, u_3, \ldots, u_m\} \subseteq N_0(u) \cap V_1$, where $m \geq 3$. Now define $g(u) = 2$, $g(u_i) = 0$. Certainly g is a γ_d function and $g(V) < f(V)$, which is a contradiction. Therefore, $|N_0(u) \cap V_1| \leq 2$.

(d) Any vertex $\nu \in D[V_0 \cup V_2]$ is either in V_2 or it is adjacent from a vertex in V_2. Therefore V_2 is a directed dominating set of $D(V_0 \cup V_1)$.

(e) Suppose there is only one V_2-diprivate neighbourhood in V_0, say w. Let $u \in N_i(\nu) \cap V_2$. Now from a new function g such that $g(\nu) = 0$ and $g(w) = 1$, for all other vertices the value of g is equal to the value of f then g is a dRDF with smaller weight than f, which is a contradiction.

(f) Suppose $N_0(w) \cap V_1 \neq \emptyset$. Define $g(\nu) = 0$, $g(y) = 0$ for every $y \in N_0(w) \cap V_1$, $g(w) = 2$ and $g(x) = 0$. For any other $x \in V(D)$, weight of g is smaller than f, which is contradiction.

(g) Let $k_0 = |\{\nu \in V_2 : |N_i(\nu) \cap V_2| = \emptyset\}|$. Then $k_0 + k_1 = n_2$ by (e) $n_0 \geq k_0 + 2k_1 + c = n_2 + k_1 + c$.

Theorem 8:

$$\gamma_d(D) = |V(D)| \text{ if and only if } \Delta_0(D) = 1.$$

Proof: Let $g_d(D) = n = |V|$. Using Theorem 4, $\gamma_d(D) \leq n + 1 - \Delta_0(D)$ that is $n \leq n + 1 - \Delta_0(D)$, that is $\Delta_0(D) \leq 1$. Therefore $\Delta_0(D) = 1$. Conversely, assume $\Delta_0(D) = 1$. Then there is no reducing vertex in V. Therefore $\gamma_d(D) = n$.

3. DIRECTED ROMAN DOMINATING NUMBER FOR PATHS

To find the $\gamma_d(D)$, where D is a dipath. Let D be a dipath. Define $T(D) = \{\nu \in V(D) : d_0(\nu) = 2\}$.

Definition: A vertex $u \in T(D)$ is said to be independent from ν if $d(u, \nu) \geq 3$.

Proposition 9: Let $\nu_1, \nu_2, \ldots, \nu_r$ be the vertices in $T(D)$ and ν_i is independent from ν_j for every $i \neq j$. Then $\gamma_d(D) \leq n - r$.

Proof: Define $f(\nu_i) = 2$ for every $i = 1, 2, 3, \ldots, r$ and $f(u) = 0$ for every vertex u, which is adjacent to any one of ν_i. And also define, $f(x) = 1$ for all other vertices in V. Now, $f(V) = n - r$ and f is a dRDF. Therefore $\gamma_d(D) \leq n - r$.

Let $\hat{P}_n$ be the collection of dipaths of length n. Obviously $|\hat{P}_n| = 3^n$.

Proposition 10: If $D \in \hat{P}_1$ then $\gamma_d(D) = 2$.

Proof: The proof is obvious.

Proposition 11: Let $D \in \hat{P}_2$ and $G(D) = \nu_0, \nu_1, \nu_2$.

Then

$$\gamma_d(D) = \begin{cases} 2 & \text{if } \ d_0(\nu_1) = 2, \\ 3 & \text{otherwise.} \end{cases}$$

Proof: The proof is obvious.

Definition: Let $\hat{Q}_n \in \hat{P}_n$, then define extension of $\hat{Q}_n = \{D \in \hat{P}_n + 1 : D - \{\nu_{n+1}\} \in \hat{Q}_n\}$. It is denoted by $T(\hat{Q}_n)$. Obviously $|T(\hat{Q}_n)| = 3|(\hat{Q}_n)|$.

Define

$$A_1(\hat{Q}_n) = \{D \in \hat{Q}_n : d_i(\nu_n) = 0\}$$
$$A_2(\hat{Q}_n) = \{D \in \hat{Q}_n : d_{i0}(\nu_n) = 1\}$$
$$A_3(\hat{Q}_n) = \{D \in \hat{Q}_n : d_0(\nu_n) = 0\}$$
$$|A_i(\hat{Q}_n)| = a_i(\hat{Q}_n) \quad \text{for all } \ i = 1, 2, 3$$
$$B_1(\hat{Q}_n)| = \{D \in \hat{Q}_n : d_0(\nu_{n-1})| \neq 2\}$$
$$B_2(\hat{Q}_n)| = \{D \in \hat{Q}_n : d_0(\nu_{n-1})| = 2\}$$
$$|B_i(\hat{Q}_n)| = a_i(\hat{Q}_n) \quad \text{for } \ i = 1,2$$

We can obviously observe the following $\hat{Q}_n = B_1(\hat{Q}_n) \cup B_2(\hat{Q}_n)$

$$B_1(\hat{Q}_n) \cap B_2(\hat{Q}_n) = \varnothing,$$
$$\hat{Q}_n = \cup_{i=1}^{3} A_i(\hat{Q}_n)$$
$$A_i(\hat{Q}_n) \cap A_j(\hat{Q}_n) = \varnothing \quad \text{for all } \ i \neq j,$$
$$\Sigma_{i=1}^{3} a_i(\hat{Q}_n) = \Sigma_{i=1}^{2} b_i(\hat{Q}_n) = |\hat{Q}_n|.$$

Theorem 12:

(i) $a_i(B_1(T(\hat{Q}_n))) = \sum_{i=1}^{3} a_i(\hat{Q}_n) = |\hat{Q}_n|$.

(ii) $a_2(B_1(T(\hat{Q}_n))) = a_3(B_1(T(\hat{Q}_n))) = a_3(\hat{Q}_n)$.

(iii) $a_1(B_2(T(\hat{Q}_n))) = 0$.

(iv) $a_2(B_2(T(\hat{Q}_n))) = a_3(B_2(T(\hat{Q}_n))) = a_1(\hat{Q}_n) + a_2(\hat{Q}_n)$.

Proof: Let $D \in \hat{Q}_n$ and $G(D) = \nu_0\nu_1\nu_2\nu_3, \ldots, \nu_n$. Form a new digraph D_1, by adjoining a new vertex ν_{n+1} such that $G(D_1) = \nu_0\nu_1\nu_2\nu_3\ldots\ldots\nu_n, \nu_{n+1}$ and $E(D_1) = E(D) \cup \{(\nu_{n+1}, \nu_n)\}$. Certainly $D_1 \in A_1(B_1(T(\hat{Q}_n)))$ therefore there is a one to one correspondence between $D_1 \in A_1(B_1(T(\hat{Q}_n)))$ and $D \in \hat{Q}_n$. Therefore,

$$a_i(B_1(T(\hat{Q}_n))) = |\hat{Q}_n|.$$

(ii) Let $D \in A_3(\hat{Q}_n)$ and $G(D) = \nu_0\nu_1\nu_3 \ldots \nu_n$. Form new digraphs D_1 and D_2, by adjoining a new vertex ν_{n+1} such that $G(D_1) = G(D_2) = \nu_0\nu_1\nu_2\ldots\nu_n\nu_{n+1}$ and $E(D_1) = E(D) \cup \{(\nu_{n+1}, \nu_n), (\nu_n, \nu_{n+1})\}$ and $E(D_2) = E(D) \cup \{(\nu_n, \nu_{n+1})\}$.

Certainly $D_1 \in A_2(B_1(T(\hat{Q}_n)))$ and $D_2 \in A_3(B_1(T(\hat{Q}_n)))$. Therefore there is a one to one correspondence between $D_1 \in A_2(B_1(T(\hat{Q}_n)))$ and $D \in A_3(\hat{Q}_n)$ also between $D_2 \in A_3(B_1(T(\hat{Q}_n))))$ and $D \in A_3(\hat{Q}_n)$. Therefore $a_2(B_1(T(\hat{Q}_n))) = a_3(B_1(T(\hat{Q}_n))) = a_3(\hat{Q}_n)$.

(iii) There is no paths in $B_2(T(\hat{Q}_n))$ with $d_i(\nu_{n+1}) = 0$. Therefore

$$a_1(B_2(\hat{Q}_{n+1})) = 0.$$

(iv) Let $D \in A_1(\hat{Q}_n) \cup A_2(\hat{Q}_n)$. Form new digraphs D_1 and D_2, by adjoining a new vertex ν_{n+1} such that $G(D_1) = G(D_2) = \nu_0\nu_1\nu_2\nu_3 \ldots \nu_n\nu_{n+1}$ and $E(D_1) = E(D) \cup \{(\nu_{n+1}, \nu_n)\ (\nu_n, \nu_{n+1})\}$ and $E(D_2) = E(D) \cup \{(\nu_n, \nu_{n+1})\}$. Certainly $D_1 \in A_2(B_2(T(\hat{Q}_n)))$ and $D_2 \in A_3(B_2(T(\hat{Q}_n)))$. There is a one to one correspondence between $D_2 \in A_3(B_2(T(\hat{Q}_n)))$ and $D \in A_1(\hat{Q}_n) \cup A_2(\hat{Q}_n)$. Hence, $a_2(B_2(T(\hat{Q}_n))) = a_3(B_2(T(\hat{Q}_n))) = a_1(\hat{Q}_n) + a_2(\hat{Q}_n)$.

Lemma 13: Let $\hat{Q}_n = \{D \in \hat{P}_n\ ; \Delta_0(D) = 1\}$. Then $a_2(\hat{Q}_n) = a_3(\hat{Q}_n) = 1$.

Proof: We will prove by induction, when n = 1, the lemma is obviously true. Assume the induction hypothesis for $\hat{Q}_{n-1}$. By induction hypotheses for $a_2(\hat{Q}_{n-1}) = a_3(\hat{Q}_{n-1}) = 1$ by the Theorem 12.

$$a_2(B_1(T(\hat{Q}_{n-1}))) = a_3(B_1(T(\hat{Q}_{n-1}))) = a_3(\hat{Q}_{n-1}) = 1. \quad (1)$$

Claim. $B_1(T(\hat{Q}_{n-1})) = \hat{Q}_n$.

$B_1(T(\hat{Q}_{n-1})) \subseteq \hat{Q}_n$ is obvious. Let $D \in \hat{Q}_n$, and $G(D) = \nu_0\nu_1\nu_2 \dots \nu_{n-1}\nu_n$. Let $D_1 = D - \{\nu_n\}$. Obviously $D_1 \in \hat{Q}_{n-1}$. Therefore $D \in T(\hat{Q}_{n-1})$ and $D \in B_1(T(\hat{Q}_{n-1}))$ since $\Delta_0(D) = 1$.

Therefore, Equation (1) becomes $a_2(\hat{Q}_n) = a_3(\hat{Q}_n) = 1$.

Theorem 14: There are $2n + 1$ dipaths with length n and $\Delta_0(D) = 1$.

Proof: Let $D \in \hat{P}_n$ and $G(D) = \nu_0\nu_1\nu_2 \dots \nu_{n-1}\nu_n$. Let $\hat{Q}_n = \{D \in \hat{P}_n : \Delta_0(D) = 1\}$. We will prove the theorem by induction on n. It is obvious that there are three dipaths of length 1 and $\Delta_0(D) = 1$. Assume the induction hypothesis for $\hat{Q}_{n-1}$. By induction hypothesis there are $2n - 1$ dipaths in $\hat{Q}_{n-1}$.

That is $|\hat{Q}_{n-1}| = 2n - 1$. By the above theorem $a_1(B_1(T(\hat{Q}_{n-1}))) = |\hat{Q}_{n-1}| = 2n - 1$, $a_2(B_1(T(\hat{Q}_{n-1}))) = a_3(B_1(T(\hat{Q}_{n-1}))) = a_3(\hat{Q}_{n-1})$. By the Lemma 13 $a_2(B_1(T(\hat{Q}_{n-1}))) = a_3(B_1(T(\hat{Q}_{n-1}))) = 1$, $|\hat{Q}_{n-1}| = 2n - 1 + 1 + 1 = 2n + 1$. Hence proving the theorem.

Proposition 15: Let $n \geq 2$, fix r so that, $1 \leq r \leq n - 1$, and define $F_{r,n} = \{D \in \hat{P}_n : d_0(\nu_r) = 2$ and $d_0(\nu_i) = 1$ for all $i \neq r\}$. Then $a_3(F_{r,n}) = 2r$.

Proof: We will prove by induction on n. Let n = 2, then the following is true.

$A_3((F_{r,2})) =$ 

Therefore $a_3((F_{r,2})) = 2 = 2r$ since $r = 1$.

Assume induction hypothesis for $n - 1$. Let $D \in F_{r,n}$.

Case 1: Let $r = n - 1$. Then clearly $D - \{\nu_n\} \in \{x \in \hat{P}_{n-1} : \Delta_0(x) = 1\}$.

Let $\hat{Q}_{n-1} = \{x \in \hat{P}_{n-1} : \Delta_0(x) = 1\}$. Using the Theorem 14, we have $|\hat{Q}_{n-1}| = 2n - 1$. By Lemma 13 $a_2(\hat{Q}_{n-1}) = a_3(\hat{Q}_{n-1}) = 1$. Therefore

$$a_3(F_{r,n}) = a_3(B_2(T(_{n-1})))$$
$$= a_3(\hat{Q}_{n-1}) + a_2(\hat{Q}_{n-1}) \quad \text{(using theorem 14)}$$
$$= 2n - 3 + 1$$
$$= 2n - 2$$
$$= 2(n - 1)$$
$$= 2r.$$

Case 2: Let $1 < r < n - 1$. By induction hypotheses $a_3(F_{r,n-1}) = 2r$. But $a_3(F_{r,n}) = a_3(F_{r,n-1}) = a_3(\hat{Q}_{n-1})$. Therefore $a_3(F_{r,n}) = 2r$. Hence proving the proposition.

Proposition 16: Let $n \geq 2$ and r is fixed, $1 \leq r \leq n-1$ define $F_{r,n} = \{D \in \hat{P}_n : d_0(\nu_r) = 2$ and $d_0(\nu_i) \neq 2$ for all $i \neq r\}$, $|F_{r,n}| = 4r(n-r)$.

Proof: We will prove by induction on n.

$F_{r,2} =$ 

Therefore,

$$|F_{r,2}| = 4$$
$$= 4.1$$
$$= 4r(n-r), \text{ where } r = 1, n = 2.$$

Assume induction hypothesis for $n-1$. Let $D \in F_{r,n}$.

Case 1: $r = n-1$.

Then $D - \{\nu_n\} \in \{x \in \hat{P}_{n-1} : \Delta_0(x) = 1\}$. Let $\hat{Q}_{n-1} = \{x \in \hat{P}_{n-1} : \Delta_0(x) = 1\}$, then it is obvious that $F_{r,n} = B_2(T(\hat{Q}_{n-1}))$.

Now,

$$|F_{r,n}| = b_2(T(\hat{Q}_{n-1}))$$
$$= a_1(B_2(T(\hat{Q}_{n-1}))) + a_2(B_2(T(\hat{Q}_{n-1}))) + a_3(B_2(T(\hat{Q}_{n-1})))$$
$$= 0 + a_1(\hat{Q}_{n-1}) + a_2(\hat{Q}_{n-1}) + a_1(\hat{Q}_{n-1}) + a_2(\hat{Q}_{n-1}) \quad \text{(using theorem 12)}$$
$$= 2(a_1(\hat{Q}_{n-1}) + a_2(\hat{Q}_{n-1}))$$
$$= 2|\hat{Q}_{n-1}| - a_3(\hat{Q}_{n-1}))$$
$$= 2(2n-1-1) \quad \text{(by theorem 14)}$$
$$= 2(2n-2)$$
$$= 4(n-1)$$
$$= 4r(n-r), \text{ since } r = n-1.$$

Case 2: $r < n-1$.

Now, clearly, $D - \{\nu_n\} \in F_{r,n-1}$. By induction hypothesis.

Let $\hat{Q}_{n-1} = F_{r,n-1}$, $|F_{r,n-1}| = 4r(n-1-r)$. It is obvious that $F_{r,n} = B_1(T(\hat{Q}_{n-1}))$. Therefore,

$$
\begin{aligned}
|F_{r,n}| &= b_1(T(\hat{Q}_{n-1})) \\
&= |B_1(T(\hat{Q}_{n-1})| \\
&= a_1(B_1(T(\hat{Q}_{n-1}))) + a_2(B_1(T(\hat{Q}_{n-1}))) + a_3(B_1(T(\hat{Q}_{n-1}))) \\
&= |\hat{Q}_{n-1}| + a_3(\hat{Q}_{n-1}) + a_3(\hat{Q}_{n-1}) \quad \text{(by theorem 12)} \\
&= |F_{r,n-1}| + 2a_3(F_{r,n-1}) \\
&= 4r(n-1-r) + 2.2r \\
&= 4r(n-1-r) + 4r \\
&= 4r(n-1-r+1) \\
&= 4r(n-r).
\end{aligned}
$$

Hence proving the proposition.

Proposition 17: $\hat{Q}_n = \{D \in \hat{P}_n : \gamma_d(D) = n-1\}$ $|\hat{Q}_n| = 2(n+1)n(n-1)/3$.

Proof: Let $n \geq 2$ and r is fixed, $1 \leq r \leq n-1$, define $F_{r,n} = \{D \in \hat{P}_n : d_0(\nu_r) = 2$ and $d_0(\nu_i) \neq 2$ for all $i \neq r\}$. Clearly,

$$
\begin{aligned}
|\hat{Q}_n| &= \sum_{r=1}^{n-1} F_{r,n} \\
&= \sum_{r=1}^{n-1} 4r(n-r) \\
&= \sum_{r=1}^{n-1} 4rn - 4r^2 \\
&= 4\left\{\sum_{r=1}^{n-1} nr - \sum_{r=1}^{n-1} r^2\right\} \\
&= 4\{(nn(n-1)/2) - (n(n-1)(2n-1)/6)\} \\
&= (2nn(n-1)) - (2n(n-1)(2n-1)/3) \\
&= 2n(n = 1)\{(n-(2n-1)/3)\} \\
&= 2n(n-1)\{(3n-2n+1)/3\} \\
&= 2n(n-1)((n+1)/3) \\
|\hat{Q}_n| &= 2(n+1)n(n-1)/3.
\end{aligned}
$$

Hence proving the proposition.

4. OPEN PROBLEMS

1. How many γ_d functions for a digraph D?
2. How will you check a given dRD function is whether γ_d function or not?
3. How many $D \in \hat{P}_n$ with $\gamma_d(D) = n - r$.

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