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REFINED HYERS-ULAM SUPERSTABILITY OF APPROXIMATELY ADDITIVE MAPPINGS

Matina John Rassias & John Michael Rassias

ABSTRACT: In 1940 (and 1964) S.M. Ulam proposed the well-known Ulam stability problem. In 1941 D.H. Hyers solved the Hyers-Ulam problem for linear mappings. In 1951 D.G. Bourgin has been the second author treating the Ulam problem for general additive mappings. In 1978 according to P.M. Gruber this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1982-2004 we established the Hyers-Ulam stability for the Ulam problem for different mappings. In 1992-2000 we investigated the Ulam stability for Euler-Lagrange mappings. In this article we solve the Ulam superstability problem for approximately additive functional equations being exactly additive.

Keywords and Phrases: Ulam superstability, Eustathy problem, Exactly additive equation

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1. INTRODUCTION

In 1940 (and 1964) S. M. Ulam [28] proposed the Ulam stability problem:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true ?"

In particular he stated the stability question:

"Let G_1 be a group and G_2 a metric group with the metric $\rho(\cdot,\cdot)$. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then a unique homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941 D. H. Hyers [13] solved this problem for linear mappings. In 1951 D.G. Bourgin [3] was the second author to treat the Ulam problem for general additive mappings. In 1978, according to P. M. Gruber [12], this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1980 and in 1987, I. Fenyö [7, 8] established the stability of the Ulam problem for quadratic and other mappings. In 1987 Z. Gajda and R. Ger [10] showed that one can get analogous stability results for subadditive multifunctions. Other interesting stability results have been achieved also by the following authors: J. Aczél [1], C. Borelli and G. L. Forti [2, 9], P. W. Cholewa [4], St. Czerwik [5], and H. Drljevic [6], and Pl. Kannappan [15]. In 1982-2004 we [16-27] solved the above Ulam problem for different mappings. In 1999 P. Gavruta [11] answered a question of ours [18] concerning the stability of the Cauchy equation. In 1998 S.-M. Jung [14] and in 2002-2003 we [25,26] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. In 1992-2000 we ([19], [21-24]) investigated the Ulam stability for Euler-Lagrange mappings.

We introduce the name "*eustathy*" derived from the Greek word *eustathia*: stability [Greek *eu* : well, (*h*)*estamae* : stand]. In this paper we impose the following problem which we call it *eustathy problem*:

"Are there any correspondences satisfying a certain property approximately and having the property exactly?"

Here we provide an answer to our above problem by proving a theorem that approximately additive mappings can be exactly additive. In this article we solve the Ulam (*eustathy)* problem for approximately additive functional equations being exactly additive (that is, *superstability*).

Definition 1.1: Let X and Y be real linear spaces. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_p) \in R^p - \{(0, 0, ..., 0)\}$. Then a mapping $A : X \to Y$ is called *additive*, if the additive functional equation

$$A\left(\sum_{i=1}^{p} a_i x_i\right) = \sum_{i=1}^{p} a_i A(x_i) \tag{*}$$

holds for every $x_i \in X$ ($i = 1, 2, ..., p$), where p is arbitrary but fixed and equals to 2, 3, ... and any fixed $\alpha(\neq 0) : 0 < m = \sum_{i=1}^{p} a_i \neq 1$.

Definition 1.2: Let X and Y be real normed linear spaces. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_p) \in R^p - \{(0, 0, ..., 0)\}$. Then a mapping $f : X \to Y$ is called *approximately additive*, if the approximately additive functional inequality

$$\left\| f\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i f(x_i) \right\| \leq cK_r\left(\|x_1\|, \|x_2\|, ..., \|x_p\|\right), \tag{**}$$

holds for every $(x_1, x_2, ..., x_p) \in X^p$, where p is arbitrary but fixed and equals to 2, 3, ..., with a real constant $c \geq 0$ (independent of $x_1, x_2, ..., x_p \in X$), any fixed α ($\neq 0$): $0 < m = \sum_{i=1}^{p} a_i \neq 1$ and any fixed real ($1 \neq$) $r \geq 0$:

$$K_r = K_r\left(\|x_1\|, \|x_2\|, ..., \|x_p\|\right) = \begin{cases} p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right) - \left\|\sum_{i=1}^{p} x_i\right\|^r, \text{ if } r \rangle 1 \\ \left(\sum_{i=1}^{p}\|x_i\|\right)^r - p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right), \text{ if } 0 \le r \langle 1 \end{cases}, \tag{1.1}$$

holds for every $(x_1, x_2, ..., x_p) \in X^p$.

Lemma 1.1: If K_r is given via (1.1), then $K_r \ge 0$ for any fixed real $0 \le r \ne 1$.

Proof: In fact, take a function $F = F(t) = t^r (t \ge 0$ and $r \in R)$. It is clear that for $F''(t) = r(r-1)t^{r-2} \ge 0$ for $r \in R : r \ge 1$. Thus F is convex for $r \ge 1$. Therefore

$$F\left(\frac{1}{p}\sum_{i=1}^{p} t_i\right) \le \frac{1}{p}\left(\sum_{i=1}^{p} F(t_i)\right), \text{ or}$$

$\left(\frac{1}{p}\sum_{i=1}^{p} t_i\right)^r \le \frac{1}{p}\left(\sum_{i=1}^{p} t_i^r\right)$ for $r \in R : r \ge 1$, and $t_i \ge 0$ $(i = 1, 2, ..., p)$, where p is arbitrary but fixed and equals to 2, 3 , ... Taking $t_i = \|x_i\| \ge 0$ for $x_i \in X (i = 1, 2, ..., p)$ and $r \in R : r \ge 1$, we get $\left(\frac{1}{p}\sum_{i=1}^{p}\|x_i\|\right)^r \le \frac{1}{p}\left(\sum_{i=1}^{p}\|x_i\|^r\right)$, or $p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right) \ge \left(\sum_{i=1}^{p}\|x_i\|\right)^r$ for $r \ge 1$.

But it is clear that $\left\|\sum_{i=1}^{p} x_i\right\|^r \le \left(\sum_{i=1}^{p}\|x_i\|\right)^r$ for $r \ge 0$ .Therefore we have that

$$K_r = p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right) - \left\|\sum_{i=1}^{p} x_i\right\|^r \ge 0$$

for $r > 1$. Similarly $F''(t) = r(r-1)t^{r-2} \le 0$ for $0 \le r < 1$. Thus F is *concave* for $r \in R$: $0 \le r < 1$. Therefore $\left(\frac{1}{p}\sum_{i=1}^{p} t_i\right)^r \ge \frac{1}{p}\left(\sum_{i=1}^{p} t_i^r\right)$. Taking $t_i = \|x_i\| \ge 0$ $(i = 1, 2, ..., p)$, we get $K_r = \left(\sum_{i=1}^{p}\|x_i\|\right)^r - p^{r-1}\left(\sum_{i=1}^{p}\|x_i\|^r\right) \ge 0$ for $0 \le r < 1$, completing the proof of Lemma 1.1.

Let us denote $I_1 = \{(r, m) \in R^2 : 0 \le r < 1, m > 1 \text{ and } r > 1, 0 < m < 1\}$, and $I_2 = \{(r, m) \in R^2 : 0 \le r < 1, 0 < m < 1 \text{ and } r > 1, m > 1\}$, such that $m^{r-1} < 1$ for any $(r, m) \in I_1$, and $m^{1-r} < 1$ for any $(r, m) \in I_2$. Note that approximately additive mappings are not additive in case $K_r = 1$ and $m > 0$. In this case Y is assumed to be complete. Also $K_0 = 0$ and the singular case $K_1 = \sum_{i=1}^{p} \|x_i\| - \left\| \sum_{i=1}^{p} x_i \right\|$ (≥0).

2. APPROXIMATELY ADDITIVE MAPPINGS BEING EXACTLY ADDITIVE

Theorem 2.1: Let X and Y be normed linear spaces. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_p) \in R^p - \{(0, 0, ..., 0)\}$: $0 < m = \sum_{i=1}^{p} a_i \neq 1$, where p is arbitrary but fixed and equals to 2, 3, ... Assume in addition that $f : X \to Y$ is an approximately additive mapping satisfying (**) with $1 \neq r \geq 0$. Define

$$f_n(x) = \begin{cases} m^{-n} f(m^n x), \text{ if } (r, m) \in I_1 \\ m^n f(m^{-n} x), \text{ if } (r, m) \in I_2 \end{cases}$$

for all $x \in X$ and $n \in N_o = \{0, 1, 2, ...\}$, where

$I_1 = \{(r, m) \in R^2 : 0 \le r < 1, m > 1 \text{ and } r > 1, 0 < m < 1\}$, and

$I_2 = \{(r, m) \in R^2 : 0 \le r < 1, 0 < m < 1 \text{ and } r > 1, m > 1\}$.

Then the formula

$$A(x) = f_n(x) \tag{1.2}$$

exists for all $x \in X$ and $n \in N_o$ and $A : X \to Y$ is the unique additive mapping satisfying

$$f(x) = A(x) \tag{1.2a}$$

for all $x \in X$.

Proof: It is useful for the following to observe that, from (**) with $x_i = 0$ ($i = 1, 2, ..., p$) and $0 < m \neq 1$, we get $|m - 1| \, \|f(0)\| \le 0$, or

$$f(0) = 0. \tag{1.3}$$

Now claim for $n \in N_0 = \{0, 1, 2, ...\}$ that

$$f(x) = f_n(x) \tag{1.3a}$$

holds for all $x \in X$. For $n = 0$ it is trivial. From (1.1) with $x_i = x$ ($i \in N_p = \{1, 2, ..., p\}$), we obtain

$$K_r = \|x\|^r \begin{cases} p^{r-1} \cdot p - p^r = 0, \text{ if } r \rangle 1 \\ p^r - p^{r-1} \cdot p = 0, \text{ if } 0 \le r \langle 1 \end{cases} \text{ or}$$

$$k_r = k_r(\|x\|, \|x\|, ..., \|x\|) = \|x\|^r \cdot 0 = 0, \tag{1.4}$$

for every $x \in X$ and any fixed real $r \in R : 0 \le r \ne 1$ with $p = 2, 3, ...$Similarly from (1.1) with $x_i = m^{-1}x$ ($i \in N_p$), we get

$$K_r = \|x\|^r m^{-r} \begin{cases} p^{r-1} \cdot p - p^r = 0, \text{ if } r \rangle 1 \\ p^r - p^{r-1} \cdot p = 0, \text{if } 0 \le r \langle 1 \end{cases}$$

or

$$K_r = K_r(m^{-1}\|x\|, m^{-1}\|x\|, ..., m^{-1}\|x\|) = \|x\|^r m^{-r} \cdot 0 = 0, \tag{1.5}$$

for every $x \in X$ and any fixed real $r \in R : 0 \le r \ne 1$ with $p = 2, 3, ...$From (1.4) and (**), with $x_i = x(i \in N_p)$, we get $\|f(mx) - mf(x)\| \le cK_r(\|x\|, \|x\|, ..., \|x\|) = 0$, or

$$f(x) = m^{-1}f(mx), \tag{1.6}$$

which is (1.3a) for $n = 1$, if I_1 holds. Similarly, from (1.5) and (**), with $x_i = m^{-1}x$ ($m \ne 0$) ($i \in N_p$), we obtain $\|f(x) - mf(m^{-1}x)\| \le cK_r(m^{-1}\|x\|, m^{-1}\|x\|, ..., m^{-1}\|x\|) = 0$ or

$$f(x) = mf(m^{-1}x), \tag{1.7}$$

which is (1.3a) for $n = 1$, if I_2 holds.

Assume (1.3a) is true and from (1.6), with $m^n x$ on place of x, we get:

$$f(m^{n+1}x) = mf(m^n x) = mm^n f(x) = m^{n+1} f(x). \tag{1.8}$$

Similarly, from (1.7) with $m^{-n} x$ on place of x, we obtain:

$$f(m^{-(n+1)}x) = m^{-1}f(m^{-n}x) = m^{-1}m^{-n} f(x) = m^{-(n+1)} f(x). \tag{1.9}$$

These formulas (1.8) and (1.9) by induction, prove formula (1.3a). It is obvious from (1.3a) that A *defines* a mapping $A : X \to Y$, given by (1.2). Finally, claim from (**) and (1.3a) we can get that $A : X \to Y$ is additive.

In fact, it is clear from the functional inequality (**), the Lemma 1.1 and the formula (1.3a) that the following functional inequality

$$m^{-n} \left\| f\left(\sum_{i=1}^{p} a_i m^n x_i \right) - \sum_{i=1}^{p} a_i f\left(m^n x_i \right) \right\| \le m^{-n} c K_r \left(\|m^n x_1\|, \|m^n x_2\|, ..., \|m^n x_p\| \right)$$

holds for all $(x_1, x_2, ..., x_p) \in X^p$, and all $n \in N_0$, with $f_n(x) = m^{-n}f(m^n x)$: I_1 holds.

Therefore $\left\| f_n\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i f_n(x_i) \right\| \le m^{n(r-1)} c K_r\left(\|x_1\|, \|x_2\|, ..., \|x_p\|\right)$ or

$$\left\| A\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i A(x_i) \right\| \le m^{n(r-1)} c K_r \xrightarrow[n\to\infty]{} 0,$$

because $m^{r-1} < 1$ for any $(r, m) \in I_1$, or

$$A\left(\sum_{i=1}^{p} a_i x_i\right) = \sum_{i=1}^{p} a_i A(x_i), \tag{1.10}$$

yielding that mapping $A : X \to Y$ satisfies the additive functional equation (*). Similarly, from (**), the Lemma 1.1 and (1.3a) we get that

$$m^n \left\| f\left(\sum_{i=1}^{p} a_i m^{-n} x_i\right) - \sum_{i=1}^{p} a_i f(m^{-n} x_i) \right\| \le m^n c K_r\left(\|m^{-n}x_1\|, \|m^{-n}x_2\|, ..., \|m^{-n}x_p\|\right)$$

holds for all $(x_1, x_2, ..., x_p) \in X^p$, and all $n \in N_0$, with $f_n(x) = m^n f(m^{-n} x)$: I_2 holds.

Therefore $\left\| f_n\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i f_n(x_i) \right\| \le m^{n(1-r)} c K_r\left(\|x_1\|, \|x_2\|, ..., \|x_p\|\right)$ or

$$\left\| A\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i A(x_i) \right\| \le m^{n(1-r)} c K_r \xrightarrow[n\to\infty]{} 0$$

because $m^{1-r} < 1$ for $(r, m) \in I_2$, implying that $A : X \to Y$ satisfies (*), completing the proof that A can be an additive mapping in X. This completes *the existence proof* of the above Theorem 2.1. *The Uniqueness proof* of Theorem 2.1 is clear, because if $A : X \to Y$ and $A' : X \to Y$ are two additive mappings satisfying (1.2a) then A and A' satisfy $A(x) - A'(x) = f(x) - f(x) = 0$, or $A(x) = A'(x)$ for all $x \in X$.

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Matina John Rassias
Statistics and Modelling Science
University of Strathclyde
Livingstone Tower, 26 Richmond Str.
Glasgow, Scotland UK G1 1XH.
E-mail: matina@stams.strath.ac.uk

John Michael Rassias
National and Capodistrian University of Athens,
4, Agamemnonos Str. Aghia Paraskevi
Attikis 15342, Athens, Greece.
E-mail: jrassias@primedu.uoa.gr
URL: http://www.primedu.uoa.gr/~jrassias/