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EXISTENCE RESULTS FOR NONDENSELY DEFINED SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS

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ABSTRACT: In this paper, we shall establish sufficient conditions for the existence of integral solutions and extremal integral solutions for some nondensely defined semilinear perturbed functional differential inclusions in separable Banach spaces.

Keywords and Phrases: Nondensely defined operator, semilinear differential inclusion, fixed point, integrated semigroups, integral solution, extremal integral solution.

AMS (MOS) Subject Classifications: 34A60, 34G20, 34G25.

1. INTRODUCTION

Recently, in [3] we studied the existence of mild solutions and extremal mild solutions for first order semilinear functional differential inclusions in a separable Banach space $(E, |\cdot|)$ of the form

$$y'(t) - Ay(t) \in F(t, y_t) + G(t, y_t),\ t \in J := [0, T] \tag{1}$$

$$y(t) = \phi(t),\ t \in [-r, 0], \tag{2}$$

where $F, G : J \times C([-r, 0], E) \to P(E)$ are given multivalued maps satisfying some assumptions that will be specified later, $P(E)$ is the family of all nonempty subsets of E, $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on E, $\phi : [-r, 0] \to \overline{D(A)}$ a given continuous function. For any continuous function y defined on $[-r, T]$ and any $t \in J$ we denote by y_t the element of $C(-r, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta),\ \theta \in [-r, 0].$$

Here $y_t(.)$ represents the history of the state from $t - r$, up to the present time t.

In problem (1)-(2) the operator A was densely defined. However, as indicated in [6], we sometimes need to deal with nondensely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on [0, 1] and consider $A = \frac{\partial^2}{\partial x^2}$ in $C([0, 1], \mathbb{R})$ in order to measure the solutions in the sup-norm, then the domain,

$$D(A) = \{\phi \in C^2([0, 1], \mathbb{R}) : \phi(0) = \phi(1) = 0\},$$

is not dense in $C([0, 1], \mathbb{R})$ with the sup-norm. See [5] for more examples and remarks concerning the nondensely defined operators. Recently evolution functional differential equations with nondensely defined linear operators have received much attention (see for instance the papers by Adimy and Ezzinbi [1], Ezzinbi and Liu [11]).

In this paper, we extend the results of the problem (1)-(2) in the case when A is nondensely defined. We shall prove existence of integral solutions as well as existence of extremal integral solutions for the problems (1)-(2) under the mixed generalized Lipschitz and Carathéodory's conditions. Our approach will be based on the theory of integrated semigroups and fixed point theorems, for the existence of integral solutions, on a fixed point theorem for the sum of a contraction multivalued map and a completely continuous map and, for the extremal integral solutions, on the concept of upper and lower integral solutions combined with a similar version of the above cited fixed point theorem on ordered Banach spaces established very recently by Dhage [8]. The results of the present paper extend some one considered in Benchohra *et al*. [3] in the case of densely defined operators, and in Benchohra *et al.* [4] and Kamenskii *et al.* [16] in the case when $G \equiv 0$.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. $C(J, E)$ is the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

and $B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

$L^1(J, E)$ denotes the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^T |y(t)|\, dt.$$

Definition 2.1([2]): Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:

(i) $S(0) = 0$;

(ii) $t \to S(t)$ is strongly continuous;

(iii) $S(s)S(t) = \int_0^s \big(S(t+r) - S(r)\big)dr,$ for all $t, s \geq 0$.

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [2], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$. In particular $S'(t)x := \frac{d}{dt}S(t)x$ defines a bounded operator on the set $E_1 := \{x \in E : t \to S(t)x$ is continuously differentiable on $[0, \infty)\}$ and $S'(t))_{t\in 0}$ is a C_0 semigroup on $\overline{D(A)}$. Here and hereafter, we assume that A satisfies the Hille-Yosida condition (see [20]), that is, there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$,

$$\sup\{(\lambda - \omega)^n \,|(\lambda I - A)^{-n}| : \lambda\omega,\, n \in \mathbb{N}\} \leq M,$$

where $\rho(A)$ is the resolvent operator set of A and I is the identity operator in E.

Let $S(t))_{t\geq 0}$, be the integrated semigroup generated by A. We note that, since A satisfies the Hille-Yosida condition, $\|S'(t)\|_{B(E)} \leq Me^{\omega t}$, $t \geq 0$, where M and ω are the constants considered in the Hille-Yosida condition (see [18]).

In the sequel, we give some results for the existence of solutions of the following problem:

$$y'(t) = Ay(t) + g(t),\ t \geq 0, \tag{3}$$

$$y(0) = a \in E, \tag{4}$$

where A satisfies the Hille-Yosida condition, without being densely defined.

Theorem 2.1[18]: Let $g : J \to E$ be a continuous function. Then for $y_0 \in \overline{D(A)}$, there exists a unique continuous function $y : J \to E$ such that

(i) $\int_0^t y(s)ds \in D(A)$ for $t \in J$,

(ii) $y(t) = a + A\int_0^t y(s)ds + \int_0^t g(s)ds,\ t \in J,$

(iii) $|y(t)| \leq Me^{\omega t}\left(|a| + \int_0^t e^{-\omega s}|g(s)|ds\right),\ t \in J.$

Moreover, y is given by the following variation of constants formula:

$$y(t) = S'(t)a + \frac{d}{dt}\int_0^t S(t-s)g(s)ds, \quad t \geq 0. \tag{5}$$

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then ([17]) for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition (with $n = 1$) it easy to see that $\lim_{\lambda\to\infty} |B_\lambda x| \le M |x|$, since

$$\left|B_\lambda\right| = \left|\lambda(\lambda I - A)^{-1}\right| \le \frac{M\lambda}{\lambda - \omega}.$$

Thus $\lim_{\lambda\to\infty} |B_\lambda| \le M$. Also if y satisfies (5), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda g(s)ds, t \ge 0. \tag{6}$$

Let (X, d) be a metric space. We use the notations:

$$P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\},$$
$$P_{bd}(X) = \{Y \in P(X) : Y \text{ bounded}\},$$
$$P_{cv}(X) = \{Y \in P(X) : Y \text{ convex}\},$$
$$P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}.$$

Consider: $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by:

$$H_d\left(A, B\right) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b), d(a,B) = \inf_{b\in B} d(a,b).$ Then $(P_{bd,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space [17].

A multivalued map $N : J \to P_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $Y : J \to \mathbb{R}^+$ defined by

$$Y(t) = d(x, N(t)) = \inf\{d(x, z) : z \in N(t)\},$$

is measurable.

Definition 2.2: A measurable multivalued function $F : J \to P_{bd,cl}(X)$ is said to be integrably bounded if there exists a function $w \in L^1(J, \mathbb{R}^+)$ such that $\|v\| \le w(t)$ a.e. $t \in J$ for all $v \in F(t)$.

A multivalued map $G : X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ i.e. $\sup_{x\in B}\left\{\sup\left\{\left|y\right| : y \in G(x)\right\}\right\} < \infty$. G is called upper sem-continuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $G(x_0)$ is nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $G(X_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $G(\mathcal{V}) \subseteq \mathcal{U}$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_{bd}(X)$. If the multivalued map G is completely continuous with nonempty compact valued, then G is u.s.c. if and only if G has closed graph i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_*)$ imply $y_* \in G(x_*)$.

Definition 2.3: A multivalued operator $N : J \to P_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \text{ for each } x, y \in X,$$

(b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

(c) N has a fixed point if there exists $x \in X$ such that $x \in N(x)$.

For more details on multivalued maps and the proof of known results cited in this section we refer interested reader to the books of Deimling [7], Gorniewicz [12], Hu and Papageorgiou [15] and Kamenskii *et al.* [16].

Our main result is based upon the following form of the fixed point theorem of Dhage [8, 9].

Theorem 2.2: Let X be a Banach space, $\mathcal{A} : X \to P_{cl,cv,bd}(X)$ and $\mathcal{B} : X \to P_{cp,cv}(X)$ two multivalued operators satisfying

(a) $\mathcal{A}$ is contraction, and

(b) $\mathcal{B}$ is completely continuous.

Then either

(i) The operator inclusion $\lambda x \in \mathcal{A}x + \mathcal{B}x$ has a solution for $\lambda = 1$, or

(ii) the set $\mathcal{E} = \{u \in X | u \in \lambda \mathcal{A}u + \lambda \mathcal{B}u, 0 \leq \lambda \leq 1\}$ is unbounded.

We also need the following definitions in the sequel.

Definition 2.4: A multivalued map $\beta : J \times E \to P(E)$ is said to be Carathéodory if

(i) $t \mapsto \beta(t, x)$ is measurable for each $x \in E$, and

(ii) $x \mapsto \beta(t, x)$ is u.s.c. for almost all $t \in J$.

Furthermore, a Carathéodory map β is said to be L^1-Carathéodory map if

(iii) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(J, R_+)$ such that

$$\|\beta(t, x)\| := \sup\{|v| : v \in \beta(t, x)\} \leq h_\rho(t),$$

for a.e. $t \in J$, and for all $|x| \leq \rho$.

For each $y \in C([-r, T], E)$ let the set $S_{F,y}$ known as the set of selectors from F defined by

$$S_{F,y} = \{v \in L^1(J, E) : v(t) \in F(t, y_t) \text{ a.e. } t \in J\}.$$

Then we have the following lemma due to Lasota and Opial [19].

Lemma 2.1: Let E be a Banach space. β is L^1-Carathéodory, with compact convex values, and let $\Gamma : L^1(J, E) \to C(J, E)$ be a linear continuous mapping. Then the operator

$$\Gamma \circ S_\beta : C(J, E) \to P_{cp,cv}(C(J, E))$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

3. EXISTENCE OF INTEGRAL SOLUTIONS

In this section we give our main existence result for problem (1)-(2). We assume that F and G are compact and convex valued multivalued maps. Before stating and proving this one, we give the definition of its integral solution.

Definition 1: We say that $y : [-r, T] \to E$ is an integral solution of (1)-(2) if

(i) $y \in C([-r, T], E)$,

(ii) $\int_0^t y(s)ds \in D(A)$ for $t \in J$,

(iii) there exist functions $v, w \in L^1(J, E)$, such that $v(t) \in F(t, y_t)$, $w(t) \in G(t, y_t)$ a.e. in J, $y(t) = \phi(t)$, $t \in [-r, 0]$ and

$$y(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)\left[v(s) + w(s)\right]ds.$$

Let us introduce the following hypotheses:

(H1) A satisfies the Hille-Yosida condition.

(H2) The operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$.

(H3) The multifunction $t \mapsto G(t, x)$ is measurable, and integrably bounded for each $x \in C([-r, 0)], E)$, and $G(t, x)$ is convex for each $(t, x) \in J \times C([-r, 0], E)$.

(H4) There exists a function $k \in L^1(J, R_+)$ such that the multifunction G satisfies:

(i) $H_d(G(t, x), G(t, y)) \leq k(t)\, \|x - y\|_\infty$ a.e. $t \in J$, for all $x, y \in C([-r, 0], E)$, and

(ii) $M^* \int_0^T e^{-\omega s} k(s) ds < 1$, where $M^* = Me^{\omega T}$ if $\omega > 0$ and $M^* = 1$ if $\omega \leq 0$.

(H5) F is L^1-Carathéodory with compact convex values.

(H6) There exists a function $l \in L^1(J, \mathbb{R})$ with $l(t) > 0$ for a.e. $t \in J$, and nondecreasing function $\psi : \mathbb{R}_+ \to (0, \infty)$ such that

$$\|F(t, x)\|_{P(E)} \leq l(t)\, \psi(\|x\|), \text{ a.e. } t \in J, \text{ for all } x \in C([-r, 0], E).$$

Theorem 3.1: Assume that (H1)-(H6) hold and $\phi(0) \in \overline{D(A)}$. Suppose that

$$\int_{c_0}^{\infty} \frac{ds}{s + \psi(s)} > \int_0^T \gamma(s) ds, \tag{7}$$

where

$$c_0 = M^* \left|\phi(0)\right| + M^* \int_0^T e^{-\omega s} \left|G(s,0)\right| ds$$

and

$$\gamma(t) = \max\{Me^{-\omega t} k(t), Me^{-\omega t} l(t)\} \text{ for } t \in J.$$

Then the problem (1)-(2) has at least one integral solution.

Proof: Transform the problem (1)-(2) into a fixed point problem. Consider the multivalued operators:

$$\mathcal{A}, \mathcal{B} : C([-r, T], E) \to P(C([-r, T], E))$$

defined by

$$\mathcal{A}(y) := \left\{ h \in C\left(\left[-r, T\right], E\right) : h(t) = \begin{cases} \phi(t) & \text{if } t \in \left[-r, 0\right]; \\ S'(t)\phi(0) & \\ + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s) f(s) ds, & \text{if } t \in J. \end{cases} \right\}$$

and

$$\mathcal{B}(y) := \left\{ h \in C([-r,T],E) : h(t) = \begin{cases} 0 & \text{if } t \in [-r,0]; \\ \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)g(s)ds, & \text{if } t \in J, \end{cases} \right\}$$

where $f \in S_{F,y}$ and $g \in S_{G,y}$.

Then the problem of finding the integral solution of (1)-(2) is reduced to finding the solution of the operator inclusion $y(t) \in \mathcal{A}(y)(t) + \mathcal{B}(y)(t)$, $t \in [-r, T]$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfies all the conditions of Theorem 2. The proof will be given in several steps.

Step 1: We shall show that the operator $\mathcal{B}$ is closed, convex and bounded valued and it is a contraction. This will be given in several claims.

Claim 1: $\mathcal{B}(y)$ is closed for each $y \in C([-r, T], E)$.

Let $(y_n)_{n\geq 0} \in \mathcal{B}(y)$ such that $y_n \to \tilde{y}$ in $C(J, E)$. Then $\tilde{y} \in C(J, E)$ and there exists $g_n \in S_{G,y}$ such that for each $t \in J$

$$y_n(t) = \frac{d}{dt}\int_0^t S(t-s)g(s)ds.$$

Using the fact that G has compact values and from hypotheses (H3), (H4), we may pass to a subsequence if necessary to get that g_n converges to $g \in L^1(J, E)$ and hence $g \in S_{G,y}$. Then for each $t \in J$,

$$y_n(t) \to \tilde{y}(t) = \frac{d}{dt}\int_0^t S(t-s)g(s)ds.$$

So, $\tilde{y} \in B(y)$.

Claim 2: $\mathcal{B}(y)$ is convex for each $y \in C([-r, T], E)$.

Let $h_1, h_2 \in \mathcal{B}(y)$, then there exists $g_1, g_2 \in S_{G,y}$ such that, for each $t \in J$ we have

$$h_i(t) = \frac{d}{dt}\int_0^t S(t-s)g_i(s)ds, i = 1, 2.$$

Let $0 \leq \delta \leq 1$. Then, for each $t \in J$, we have

$$(\delta h_1 + (1-\delta)h_2)(t) = \frac{d}{dt}\int_0^t S(t-s)[\delta g_1(s) + (1-\delta)g_2(s)]ds.$$

Since $G(t, y)$ has convex values, one has

$$\delta h_1 + (1-\delta)h_2 \in B(y).$$

Claim 3: $\mathcal{B}(y)$ is bounded for each $y \in C([-r, T], E)$.

Let $h \in \mathcal{B}(y)$. Then, there exists $g \in S_{G,y}$ such that

$$h(t) = \frac{d}{dt}\int_0^t S(t-s)g(s)ds, \quad t \in J.$$

By (H3) we have for all $t \in J$

$$|h(t)| \leq Me^{\omega T}\int_0^T e^{-\omega s}|g(s)|ds$$

$$\leq M^*\int_0^T e^{-\omega s}g(s)ds.$$

Then $\|h\| \leq M^*\int_0^T e^{-\omega s}g(s)ds$ for all $h \in \mathcal{B}(y)$. Hence $\mathcal{B}(y)$ is a bounded subset of $C([-r, T], E)$ for all $y \in C([-r, T], E)$.

Claim 4: $\mathcal{B}$ is a contraction.

Let $y_1, y_2 \in C(-r, T)], E)$, and $h_1 \in \mathcal{B}(y_1)$. Then, there exists $g_1(t) \in G(t, y_{1t})$ such that

$$h_1(t) = \frac{d}{dt}\int_0^t S(t-s)g_1(s)ds, t \in J.$$

From (H4) it follows that

$$H_d(G(t, y_{1t}), G(t, y_{2t}) \leq k(t)\,\|y_{1t} - y_{2t}\|.$$

Hence there is $v \in G(t, y_{2t})$ such that

$$|g_1(t) - v| \leq k(t)\,\|y_{1t} - y_{2t}\|, t \in J.$$

Consider $\mathcal{U} : J \to P(E)$ given by

$$\mathcal{U}(t) = \{v \in E : |g_1(t) - v| \leq k(t)\,\|y_{1t} - y_{2t}\|\}.$$

Since the multivalued operator $\mathcal{V}(t) = \mathcal{U}(t) \cap G(t, y_{2t})$ is measurable (see Proposition III.4 in [4]) there exists $g_2(t)$ a measurable selection for $\mathcal{V}$. So, $g_2(t) \in G(t, y_{2t})$ and for each $t \in J$ we have

$$|g_1(t) - g_2(t)| \leq k(t)\,\|y_{1t} - y_{2t}\|.$$

Let us define for each $t \in J$

$$h_2(t) = \frac{d}{dt}\int_0^t S(t-s)g_2(s)ds.$$

It follows that $h_2 \in \mathcal{B}(y_2)$ and

$$\begin{aligned}|h_1(t) - h_2(t)| &\leq M^* \int_0^T e^{-\omega s} \left|g_1(s) - g_2(s)\right| ds \\ &\leq M^* \int_0^T e^{-\omega s} k(s) \left\|y_{1s} - y_{2s}\right\| ds \\ &\leq \left(M^* \int_0^T e^{-\omega s} k(s) ds\right) \left\|y_1 - y_2\right\|_\infty .\end{aligned}$$

Taking the supremum over t, we obtain

$$\left\|h_1 - h_2\right\|_\infty \leq \left(M^* \int_0^T e^{-\omega s} k(s) ds\right) \left\|y_1 - y_2\right\|_\infty .$$

From this and the analogous inequality obtained by interchanging the roles of y_1 and y_2 it follows that

$$H_d\left(\mathcal{B}(y_1), B(y_2)\right) \leq \left(M^* \int_0^T e^{-\omega s} k(s) ds\right) \left\|y_1 - y_2\right\|_\infty .$$

This shows that $\mathcal{B}$ is a contraction, since $M^* \int_0^T e^{-\omega s} k(s) ds < 1$ by (H4).

Step 2: We shall show that the operator $\mathcal{A}$ is compact and convex valued and it is completely continuous. This will be given again in several claims.

Claim 1: $\mathcal{A}(y)$ is compact for each $y \in C([-r, T], E)$.

Observe that the operator $\mathcal{A}$ is equivalent to the composition $\mathcal{L} \circ S_F$ of two operators on $L^1(J, E)$, where $\mathcal{L} : L^1(J, E) \to C(J, E)$ is the continuous operator defined by

$$\mathcal{L}v(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)v(s)ds, \; t \in J.$$

It then suffices to show that the composition operator $\mathcal{L} \circ S_F$ has compact values on $C(J, E)$. Let $y \in C(J, E)$ be arbitrary and let v_n be a sequence in $S_F(y)$. Then, by the definition of S_F, $v_n(\mathrm{t}) \in F(t, y_t)$ a.e. $t \in J$. Since $F(t, y_t)$ is compact, we may pass to a subsequence if necessary to get that $v_n(t)$ converge to $v \in L^1(J, E)$, where $v(t) \in F(t, y_t)$ a.e. for $t \in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on J as $n \mapsto \infty$. In order to show that the convergence is uniform, we first show that $\mathcal{L}v_n$ is an equicontinuous sequence. Let $\tau_1, \tau_2 \in J$, then we have

$$|\mathcal{L}v_n(\tau v_n) - \mathcal{L}v_n(\tau_2)| \leq |S'(\tau_1) - S'(\tau_1) - S'(\tau_2))\phi(0)|$$

$$+ \left|\lim_{\lambda\to\infty}\int_0^{\tau_1} [S'(\tau_1 - s) - S'(\tau_2 - s)]B_\lambda v_n(s)ds\right|$$

$$+ \left|\lim_{\lambda\to\infty}\int_{\tau_1}^{\tau_2} S'(\tau_2 - s)B_\lambda v_n(s)ds\right|.$$

As $\tau_1 \mapsto \tau_2$, the right hand-side of the above inequality tends to zero. Since $S'(t)$ is a strongly continuous operator and the compactness of $S'(t)$ for $t > 0$ implies the continuity in uniform topology. Hence $\{\mathcal{L}v_n\}$ is equicontinuous, and an application of Arzelá-Ascoli theorem implies that there is a uniformly convergent subsequence. Then we have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S_F)(y)$ as $j \mapsto \infty$, and so $(\mathcal{L} \circ S_F)(y)$ is compact. Therefore $\mathcal{A}$ is a compact valued multivalued operator on $C(J, \overline{D(A)})$.

Claim 2: $\mathcal{A}(y)$ is convex for each $y \in C([-r, T], E)$.

As in Step 1, Claim 2 it is easily to show that $\mathcal{A}$ is a convex valued multivalued operator.

Claim 3: $\mathcal{A}$ maps bounded sets into bounded sets in $C([-r, T], E)$.

Let B a bounded set in $C([-r, T], E)$. There exists a real number $q > 0$ such that $\|y\| \leq q$ for any $y \in B$. Now for each $h \in \mathcal{A}(y)$, there exists $f \in S_{F,y}$ such that

$$h(t) = S'(t)\phi_0 + \frac{d}{dt}\int_0^t S(t-s)f(s)ds, t \in J.$$

Then for each $t \in J$ we get

$$|h(t)| \leq M^*|\phi(0)| + M^*\int_0^T e^{-\omega s}h_q(s)ds.$$

This further implies that

$$\|h\|_\infty \leq M^*|\phi(0)| + M^*\int_0^T e^{-\omega s}h_q(s)ds,$$

for all $h \in \mathcal{A}(y) \subset \mathcal{A}(B) = \cup_{y\in B}\mathcal{A}(y)$. Hence $\mathcal{A}(B)$ is bounded.

Claim 4: $\mathcal{A}$ maps bounded sets into equicontinuous sets.

Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ and B be, as above, a bounded set and $h \in \mathcal{A}(y)$ for some $y \in B$. Then, there exists $f \in S_{F,y}$ such that

$$h(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \; t \in J.$$

As in Step 2, Claim 1, we can easily show that $|h(\tau_2) - h(\tau_1)| \to 0$ as $\tau_2 \to \tau_1$. As a consequence of Claims 1 to 3 together with Arzelá-Ascoli theorem it suffices to show that $\mathcal{A}$ maps B into a precompact set in $\overline{D(A)}$. Let $0 < t < T$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B$, and $f \in S_{F,y}$ we define

$$h_\epsilon(t) = S'(t)\phi(0) + S'(\epsilon)\lim_{\lambda\to\infty}\int_0^{t-\epsilon} S'(t-s-\epsilon)B_\lambda f(s)ds.$$

Since $S'(t)$ is a compact operator, the set $H_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in \mathcal{A}(y)\}$ is precompact in $\overline{D(A)}$ for every ϵ, $0 < \epsilon < t$. Moreover, for every $h \in \mathcal{A}(y)$ we have

$$\left|h(t) - h_\varepsilon(t)\right| \le M^*\int_{t-\varepsilon}^t e^{-\omega s}h_q(s)ds.$$

Therefore there are precompact sets arbitrarily close to the set $H(t) = \{h(t) : h \in \mathcal{A}(y)\}$. Hence the set $\{h(t) : h \in \mathcal{A}(B)\}$ is precompact in E. Thus we can conclude that $\mathcal{A}$ is a completely continuous operator.

Claim 5: $\mathcal{A}$ has closed graph.

Let $y_n \to y_*$, $h \in \mathcal{A}(y_n)$, and $h_n \to h_*$. We shall show that $h_* \in \mathcal{A}(y_*)$. Now $h_n \in \mathcal{A}(y_n)$ means that there exists $f_n \in S_{F,y_n}$ such that

$$h_n(t) = S'(t)\phi(0) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f_n(s)ds, \; t \in J.$$

We must prove that there exists $f_* \in S_{F,y_*}$ such that

$$h_*(t) = S'(t)\phi(0) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)f_*(s)ds, \; t \in J.$$

Consider the linear and continuous operator $\mathcal{K} : L^1(J, E) \to C(J, E)$ defined by

$$\mathcal{K}v(t) = \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda f(s)ds, \; t \in J.$$

We have

$$|(h_n(t) - S'(t)\phi(0)) - (h_*(t) - S'(t)\phi(0))| = |(h_n(t) - h_*(t)|$$
$$\leq \|h_n(t) - h_*\|_\infty \to 0, \text{ as } n \mapsto \infty.$$

From Lemma 2.1 it follows that $\mathcal{K} \circ S_F$ is a closed graph operator and from the definition of $\mathcal{K}$ one has

$$h_n(t) - S'(t)\phi(0) \in \mathcal{K}\, S_F(y_n)$$

As $y_n \to y_*$ and $h_n \to h_*$, there is a $f_* \in S_F(y_*)$ such that

$$h_*(t) = S'(t)\phi(0) + \lim_{\lambda\to\infty}\int_0^t S'(t-s)f_*(s)ds.$$

Hence the multivalued operator $\mathcal{A}$ is an upper semi-continuous operator on $C([-r, T], E)$.

Step 3: A priori bounds

Now it remains to show that the set

$$\mathcal{E} = \{y \in C([-r, T], E) : y \in \sigma\mathcal{A}(y) + \sigma\mathcal{B}(y) \text{ for some } 0 < \sigma < 1\}$$

is bounded.

Let $y \in \mathcal{E}$ be any element. Then, there exists $f \in S_{F,y}$ and $g \in S_{G,y}$ such that

$$y(t) = \sigma S'(t)\phi(0) + \sigma\lim_{\lambda\to\infty}\int_0^t S'(t-s)f(s)ds + \sigma\lim_{\lambda\to\infty}\int_0^t S'(t-s)g(s)ds, \ t \in J,$$

for some $0 < \sigma < 1$. Then for each $t \in J$ we have

$$|y(t)| \leq M^*|\phi(0)| + M^*\int_0^t e^{-\omega s}l(s)\psi(\|y_s\|)ds$$

$$+M^*\int_0^t e^{-\omega s}k(s)\|y_s\|ds + M^*\int_0^t e^{-\omega s}|G(s,0)|ds.$$

Consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\},\ 0 \leq t \leq T.$$

Let $t^* \in [-r, t]$ such that $\mu(t) = |y(t^*)|$. If $t^* \in J$ then by the previous inequality we have

$$\mu(t) \leq M^*|\phi(0)| + M^*\int_0^t e^{-\omega s}l(s)\psi(\mu(s))ds$$

$$+M^*\int_0^t e^{-\omega s}k(s)\mu(s)ds+M^*\int_0^T e^{-\omega s}\left|G(s,0)\right|ds$$

$$\leq\ c_0+\int_0^t \gamma(s)\left[\mu(s)+\psi(\mu(s))\right]ds.$$

It $t^* \in[-r, 0]$ then $|\mu(t) \leq \|\phi\|$ and the previous inequality holds.

Let us take the right hand-side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq v(t) \text{ for all } t \in J,$$
$$v(0) = c_0$$

and

$$v' \in(t)\ = \gamma(y)[\mu(t) + \psi[\mu(t))]$$
$$\leq \gamma(t)[v(t) + \psi(v(t))], \text{ a.e. } t \in J.$$

Thus

$$\int_{c_0}^{v(t)}\frac{du}{u+\psi(u)}\leq\int_0^T \gamma(s)ds<\int_{c_0}^{\infty}\frac{ds}{s+\psi(s)}.$$

Consequently, by condition (7), there exists a constant d such that $v(t) \leq d$, $t \in J$ and hence $\|y\|_\infty \in d$ where d depends only on the constants M, ω and the functions l, k and ω. This shows that the set $\mathcal{E}$ is bounded. As a result the conclusion (ii) of Theorem 2.2 does not hold. Hence the conclusion (i) holds and consequently $A(y) + B(y)$ has a fixed point which is an integral solution of problem (1)-(2).

4. EXISTENCE OF EXTREMAL INTEGRAL SOLUTIONS

In this section we shall prove the existence of maximal and minimal integral solutions of problem (1)-(2) under suitable monotonicity conditions on the functions involved in it.

Definition 4.1: A nonempty closed subset C of a Banach space X is said to be a cone if

(i) $C + C \subset C$,

(ii) $\lambda C \subset C$

(iii) $\{-C\} \cap \{C\} = \{0\}$

A cone C is called normal if the norm $\|\cdot\|$ is semi-monotone on C, i.e., there exists a constant $N > 0$ such that $\|x\| \leq N\,\|y\|$, whenever $x \leq y$. We equip the space X

= $C(J, E)$ with the order relation $\leq$ induced by a cone $\mathcal{C}$ in E, that is for all $y, \bar{y} \in X : y \leq \bar{y}$ if and only if $\bar{y}(t) - y(t) \in C$, $\forall t \in J$. In what follows will assume that the cone C is normal. Cones and their properties are detailed in [12,13]. Let a, $b \in X$ be such that $a \leq b$. Then, by an order interval $[a, b]$ we mean a set of points in X given by

$$[a, b] = \{x \in X | a \leq x \leq b\}.$$

Let $D, Q \in P_{cl}(X)$. Then by $D \leq Q$ we mean $a \leq b$ for all $a \in D$ and $b \in Q$. Thus $a \leq D$ implies that $a \leq b$ for all $b \in Q$; in particular, if $D \leq D$, then, it follows that D is a singleton set.

Definition 4.2: Let X be an ordered Banach space. *A* mapping $T : X \to P(X)$ is called isotone increasing if $T(x) \leq T(y)$ for any $x, y \in X$ with $x < y$. Similarly, T is called isotone decreasing if $T(x) \geq T(y)$ whenever $x < y$.

Definition 4.3 [14]: We say that $x \in X$ is the least fixed point of G in X if $x \in Gx$ and $x \leq y$ whenever $y \in X$ and $y \in Gy$. The greatest fixed point of G in X is defined similarly by reversing the inequality. If both least and greatest fixed point of G in X exist, we call them extremal fixed point of G in X.

In the sequel we use the following fixed point theorem.

Theorem 4.1 [10]: Let $[a, b]$ be an order interval in a Banach space and let B_1, B_2 : $[a, b] \to X$ be two functions satisfying

(a) B_1 is a contraction,

(b) B_2 is completely continuous,

(c) B_1 and B_2 are strictly monotone increasing, and

(d) $B_1(x) + B_2(x) \in [a, b]$, $\forall x \in [a, b]$.

Further if the cone C in X is normal, then the equation $x \in B_1(x) + B_1(x)$ has a least fixed point x_* and a greatest fixed point $x^* \in [a, b]$. Moreover $x_* = \lim_{n\to\infty} x_n$ and $x^* = \lim_{n\to\infty} y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a, b]$ defined by

$$x_{n+1} \in B_1(x_n) + B_2(x_n),\ x_0 = a \text{ and } y_{n+1} \in B_1(y_n) + B_2(y_n),\ y_0 = b.$$

We need the following definitions in the sequel.

Definition 4.4: We say that a continuous function $v : [-r, T] \to E$ is a lower integral solution of problem (1)-(2) if there exists functions $f, g \in L^1(J, E)$ such that $f(t) \in F(t, y_t)$, $g(t) \in G(t, y_t)$, a.e. on J, $y(t) \leq \phi(t)$, $t \in [-r, 0]$, and

$$y(t) \le S'(t)\phi(0) + \frac{d}{dt}\int_0^t S\left(t-s\right)\left[f(s)+g(s)\right]ds,\, t \in J.$$

Similarly an upper integral solution w of IVP (1)-(2) is defined by reversing the order.

Definition 4.5: An integral solution x_M of IVP (1)-(2) is said to be maximal if for any other integral solution x of IVP (1)-(2) on J, we have that $x(t) \in x_M(t)$ for each $t \in J$.

Similarly a minimal integral solution of IVP (1)-(2) is defined by reversing the order of the inequalities.

Definition 4.6: A multivalued function $F(t, x)$ is called strictly monotone increasing in x almost everywhere for $t \in J$, if $F(t, x) \le F(t, y)$ a.e. $t \in J$ for all $x, y \in E$ with $x < y$. Similarly $F(t, x)$ is called strictly monotone decreasing in x almost everywhere for $t \in J$, if $F(t, x) \ge F(t, y)$ a.e. $t \in J$ for all $x, y \in E$ with $x < y$.

We consider the following assumptions in the sequel.

(H7) The multivalued functions $F(t, y)$ and $G(t, y)$ are strictly monotone nondecreasing in y for almost each $t \in J$.

(H8) $S'(t)$ is order-preserving, that is, $S'(t)(v) \ge 0$ whenever $v \ge 0$.

(H9) The IVP (1)-(2) has a lower integral solution v and an upper integral solution w with $v \le w$.

Theorem 2: Assume that (H1)-(H9) hold. Then IVP (1)-(2) has minimal and maximal integral solutions on $[-r, T]$.

Proof: It can be shown, as in the proof of Theorem 3.1 that $\mathcal{A}$ is completely continuous and $\mathcal{B}$ is a contraction on $[v, w]$. We shall show that $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on $[v, w]$. Let $y, \overline{y} \in [a, b]$ be such that $y \le \overline{y},\ y \ne \overline{y}$. Then by (H7), (H8), we have for each $t \in J$

$$\begin{aligned}\mathcal{A}(\mathrm{y}) &= \left\{h \in C(J, E) : h(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)f(s)ds,\, f \in S_{F,y}\right\}\\ &\le \left\{h \in C(J, E) : h(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)f(s)ds, f \in S_{F,\bar{y}}\right\}\\ &= A\left(\overline{y}\right).\end{aligned}$$

Similarly, $\mathcal{B}(y) \leq \mathcal{B}(\bar{y})$. Therefore $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on $[v, w]$. Finally, let $x \in [v, w]$ be any element. By (H9), we deduce that

$$v \leq \mathcal{A}(v) + \mathcal{B}(v) \leq \mathcal{A}(x) + \mathcal{B}(x) \leq \mathcal{A}(w) + \mathcal{B}(w) \leq w,$$

which shows that $\mathcal{A}((x) + \mathcal{B}(x) \in [v, w]$ for all $x \in [v, w]$. Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all conditions of Theorem 4.1. Hence IVP (1)-(2) has maximal and minimal integral solutions on J. This completes the proof.

5. AN EXAMPLE

Consider the system:

$$\frac{\partial}{\partial t}u(t, x) \in \frac{\partial^2}{\partial x^2}u(t, x) + \tilde{F}(u_t(\cdot, x)) + \tilde{G}(u_t(\cdot, x)), t \in [0, T], x \in [0, \pi] \quad (8)$$

$$u(t, 0) = u(t, \pi), t \in [0, T] \quad (9)$$

$$u(\theta, x) = \varphi(\theta, x), \theta \in [-r, 0], x \in [0, \pi], \quad (10)$$

where $r > 0$, $\varphi \in C([-r, 0] \times [0, \pi], \mathbb{R})$ and $\tilde{F}, \tilde{G} : C([-r, 0], \mathbb{R}) \to \mathcal{P}(\mathbb{R})$.

To write (8)-(10) in the form (1)-(2) we choose $X = C([0, \pi], \mathbb{R})$, and for each $\theta \in [-r, 0]$, $t \in [0, T]$, $x \in [0, \pi]$

$$y(t)(x) = u(t, x)\phi(\theta)(x) = \varphi(\theta, x),$$

$$F(t, y_t)(x) = \tilde{F}(u_t(\cdot, x)), G(t, y_t)(x) = \tilde{G}(u_t(\cdot, x)),$$

and denote by $Ay := y''$ with domain

$$D(A) = \{u \in C^2([0, \pi], \mathbb{R}) : u(0) = u(\pi)\}.$$

We have

$$\overline{D(A)} = \{u \in C([0, \pi], R) : u(0) = u(\pi)\} \neq X.$$

It well known (see [5]) that satisfies the following properties

(i) $(0, \infty) \subset \rho(A)$

(ii) $\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$

This implies that the operator A satisfies the Hille-Yosida condition (with $M = 1$ and $\omega = 0$) on X. Then problem (8)-(10) can be written as

$$y'(t) \in Ay(t) + F(t, y_t) + G(t, y_t),\ t \in [0, T],$$

$$u(t) = \phi(t),\ t \in [-r, 0].$$

Thus under appropriate conditions on the functions $\widetilde{F}$ and $\widetilde{G}$ as those in (H1)-(H9), the problem (8)-(10) has an integral solution as well as extremal integral solutions.

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