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A NOTE ON GENERALIZED APPROXIMATION OF SOLUTIONS OF FOUR POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT: We study some four point boundary value problems. We develope the generalized method of quasilinearization to obtain a monotone sequence of solutions of linear problems converging uniformly and rapidly to a solution of the problem in the C^1 norm. We improve some previously studied results, by allowing weaker hypotheses on the nonlinearity.

Keywords and Phrases: Four point problem, Generalized quasilinearization, Quadratic convergence.

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1. INTRODUCTION

In the present paper, we study approximation of solutions of second order nonlinear differential equations with four point boundary conditions (BCs) of the type,

$$x''(t) = f(t, x, x'),\ t \in I = [a, b],$$
$$x(a) = x(c),\ x(b) = x(d), \tag{1.1}$$

wehere $a < c \leq d < b$. We approximate our problem by a sequence of linear problems to obtain a monotone sequence of approximants. We show that under suitable conditions, the sequence converges quadratically to a solution of the original problem. We note that (1.1) is a problem at resonance since any constant function is a solution of the linear equation $x'' = 0$ with the four point BCs.

Existence theory for solutions of the four point boundary value problems has been presented in a number of papers by Rachunkova [3, 4, 5]. In theorem 1 of [4], Rachunkova, proved existence of solutions for the four point boundary value problem (1.1) under various combinations of sign conditions on the function $f(t, x, x')$. The results of [4] has been generalized by R.A. Khan and R.R. Lopez [2]. They established existence of solutions under more general conditions, of the existence of lower and upper solutions which are not necessarily constants. They also developed the generalized quasilinearization technique to approximate a solution of the boundary value problem (1.1). Under suitable conditions on f, they proved in [2] the existence of a monotone sequence of solutions of linear problems converging uniformly and quadratically to a solution of the problem (1.1) in the C^1 norm. In this paper, we generalize the results of [2] by allowing weaker hypotheses on f. We do not require $H(f) \le 0$ on $I \times \mathbb{R}^2$, as was assumed in theorem 4 of [2], where

$$\begin{aligned} H(f) &= (x-y)^2 f_{xx}(t, z_1, z_2) + 2(x-y)(x'-y')f_{xx'}(t, z_1, z_2) \\ &\quad + (x'-y')^2 f_{x'x'}(t, z_1, z_2), \end{aligned} \tag{1.2}$$

z_1 lies between x and y, and z_2 lies between x' and y'. The expression (1.2) is known as the quadratic form of f. Moreover, we replace the conditions

$$|f_{x'}(t, x, y_1) - f_{x'}(t, x, y_2)| \le L|y_1 - y_2|,\ y_1, y_2 \in \mathbb{R}.\ (t, x) \in I \times [\min \alpha, \max \beta],$$

$$f_{x'}(t, x, P) \ge 2LP, f_{x'}(t, x, -P) \le -2LP,\ (t, x) \in I \times [\min \alpha, \max \beta),$$

where $P > \max\{\|\alpha'\|, \|\beta'\|\}$, assumed in [1], by much weaker condition of the type

$$f_{x'}(t, x, P) \ge 0, f_{x'}(t, x, -P) \le 0 \text{ for } (t, x) \in I \times [\min \alpha, \max \beta],$$

and prove that the conclusion is still valid. Hence we enlarge the class of nonlinear four point problems to which the generalized method of quasilinearizations is applicable.

2. UPPER AND LOWER SOLUTIONS

We recall the concept of lower and upper solutions for the BVPs (1.1), [1].

Definition 2.1: Let $\alpha \in C^2(I)$. We say that α is a lower solution of (1) if

$$\alpha''(t) \ge f(t, \alpha(t), \alpha'(t)),\ t \in I$$

$$\alpha(a) \le \alpha(c),\ \alpha(b) \le \alpha(d).$$

An upper solution β of the BVP(1.1) is defined similarly by reversing the inequalities.

Definition 2.2: A continuous function $\omega : [0, \infty) \to [0, \infty)$ is called a Nagumo function if

$$\int_0^\infty \frac{sds}{\omega(s)} = +\infty.$$

We say that $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Bernstein-Nagumo condition on I relative to α, β, if there exists a Nagumo function ω such that

$$f(t, x, y)\, sgn(y) \le \omega(|y|) \text{ on } I \times [\alpha, \beta] \times \mathbb{R}, \tag{2.1}$$

$$f(t, x, y)\, sgn(y) \le -\omega(|y|) \text{ on } [a, b] \times [\alpha, \beta] \times \mathbb{R}, \tag{2.2}$$

For $u \in C(I)$ we write $\|u\| = \max\{|u(t)| : t \in (I)\}$ and for $v \in C^1(I)$ we write $\|v\|_1 = \|v\| + \|v'\|$. The following theorems is known, see [2].

Theorem 3: Assume that α and β are respectively lower and upper solutions of (1.1) such that $\alpha(t) \leq \beta(t)$, $t \in I$. If $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the Bernstein-Nagumo condition, then there exists a solution $x(t)$ of the boundary value problem (1) such that

$$\alpha(t) \leq x(t) \leq \beta(t),\ t \in I.$$

Moreover, there exists a constant $C > \max\{\|\alpha'\|, \|\beta'\|\}$ depends on α, β and ω such that $|x'(t)| < C$ on I.

3. GENERALIZED QUASILINEARIZATION TECHNIQUE

Now, we generalize the results of [2], by not demanding $H(f) \leq 0$ on $I \times \mathbb{R}^2$, and also impose less restrictive conditions on f. Since f is continuous and bounded on $I \times [\min \alpha, \max \beta] \times [-C, C]$, there always exists a function ϕ such that

$$H(f + \phi) \leq 0 \text{ on } I \times [\min \alpha, \max \beta] \times [-C, C], \tag{3.1}$$

where $\phi \in C^2(I \times \mathbb{R}^2)$ and is such that

$$H(\phi) \leq 0 \text{ on } I \times [\min \alpha, \max \beta] \times [-C, C], \tag{3.2}$$

where $C > \max\{\|\alpha'\|, \|\beta'\|\}$ is as given in Theorem 2.3. Define $q : \mathbb{R} \to \mathbb{R}$ by

$$q(x) = \begin{cases} C, & \text{if } x > C, \\ x, & \text{if } |x| \leq C, \\ -C, & \text{if } x < -C, \end{cases}$$

a retraction onto $[-C, C]$. Clearly, q is continuous and bounded. Also, define $q : \mathbb{R} \to \mathbb{R}$ by

$$z(x) = x - q(x),\ x \in \mathbb{R}, \tag{3.3}$$

then z is continuous and maps the interval $[-C, C]$ onto the point 0 (origin). Moreover, $z(x) \neq 0$ for $|x| > C$. Define $F : I \times \mathbb{R}^2 \to \mathbb{R}$ by $F(t, x, x') = f(t, x, x') + \psi(t, x, x')$ where $\psi(t, x, x') = \phi(t, x, z(x'))$. Then $F \in C^2(I \times \mathbb{R}^2)$ and

$$H(F) \leq 0 \text{ on } I \times [\min \alpha, \max \beta] \times [-C, C]. \tag{3.4}$$

We note that $\psi(t, x, x') = \phi(t, x, 0)$ for $|x'| \leq C$, for example, ψ may be of the form

$$\psi(t, x, x') = a(t, x) + b(t, x)(x' - q(x'))^r.$$

We state and prove our main result:

Theorem 3.1: Assume that

(B_1) $\alpha, \beta \in C^2(I)$ are lower and upper solutions of (1) such that $\alpha(t) \le \beta(t)$ on I.

(B_2) $f \in C^2[I \times \mathbb{R}^2]$ satisfies a Bernstein-Nagumo condition on I relative to α, β. Moreover,

$$f_x(t, x, x') \ge 0 \text{ on } I \times [\min \alpha, \max \beta] \times [-C, C], \text{ and} \tag{3.5}$$

$$f_{x'}(t, x, C) \ge 0, f_{x'}(t, x, -C) \le 0 \text{ for every } (t, x) \in I \times [\min \alpha, \max \beta]. \tag{3.6}$$

Further, assume that (3.4) holds for some function $\phi \in C^2(I \times \mathbb{R}^2)$.

Then, there exists a monotone sequence $\{w_n\}$ of solutions converging uniformly and quadratically to a solution of the problem.

Proof: We consider the following four point boundary value problems (BVP)

$$x''(t) = f(t, x, q(x')), \; t \in I,$$
$$x(a) = x(c), x(b) = x(d). \tag{3.7}$$

We note that any solution $x \in C^1(I)$ of the BVP (3.7) such that $|x'| \le C$ on I, is a solution of the original problem (1.1). But, as in the proof of the first part of theorem (3.1) of [1], any solution x of (3.7) with $\alpha \le x \le \beta$ on I, must satisfies $|x'| \le C$ on I, and hence is a solution of (1.1). Thus, it is enough to study (3.7). Using Taylor's theorem about $(t, y, q(y'))$, we obtain

$$f(t, x, q(x')) = f(t, y, q(y')) + F_{x'}(t, y, q(y'))(q(x') - q(y'))$$
$$- [\phi(t, z, 0) - \phi(t, y, 0)] + \frac{1}{2} H(F), \tag{3.8}$$

where

$$H(F) = (x - y)^2 F_{xx}(t, \xi_2, \xi_2) + 2(x - y)[q(x') - q(y')]$$
$$F_{xx'}(t, \xi_1, \xi_2) + [q(x') - q(y')]^2 F_{x'x'}(t, \xi_1, \xi_2), \tag{3.9}$$

ξ_1 lies between x and y, and ξ_2 lies between $q(x')$ and $q(y')$. We note that

$$q(x'), q(y') \in [-C, C] \text{ for all } x', y' \in \mathbb{R}.$$

Hence, in view of (8), we have

$$H(F) \le 0 \text{ for every } (t, x, x'), (t, y, y') \in I \times [\min \alpha, \max \beta] \times \mathbb{R},$$

and consequently, (1) takes the form

$$f(t, x, q(x')) \leq f(t, y, q(y')) + [F_x(t, y, q(y')) - \phi_x(t, y, 0)](x - y)$$
$$+ F_{x'}(t, y, q(y'))[q(x') - q(y')], \tag{3.10}$$

for every (t, x, x'), $(t, y, y') \in I \times [\min \alpha, \max \beta] \times \mathbb{R}$ and

$$-\phi(t, x, 0)(x - y) \leq -\phi_x(t, \beta(t), 0)(x - y) \text{ for } x \geq y. \tag{3.11}$$

where $y \leq \xi \leq x$. Substituting in (14), we get

$$f(t, x, q(x')) \leq f(t, y, q(y')) + [F_x(t, y, q(y')) - \phi_x(t, \beta(t), 0)](x - y)$$
$$+ F_{x'}(t, y, q(y'))(q(x') - q(y')),\ x \geq y \tag{3.12}$$

on $I \times [\min \alpha, \max \beta] \times \mathbb{R}$. Define k on $I \times \mathbb{R}^4$ by

$$k(t, x, x'; y, y') = f(t, y, q(y')) + [F_x(t, y, q(y') - \phi_x(t, \beta(t), 0)]\ (x - y)$$
$$+ F_{x'}(t, y, q(y'))[q(x') - q(y')]. \tag{3.13}$$

Then, k is continuous and bounded on $I \times [\min \alpha, \max \beta] \times \mathbb{R} \times [\min \alpha, \max \beta] \times \mathbb{R}$ and satisfies the following relations

$$\begin{cases} f\left(t, x, q\left(x'\right)\right) \leq k\left(t, x, x'; y, y'\right), \\ f\left(t, x, q\left(x'\right)\right) = k\left(t, x, x'; x, x'\right), \end{cases} \tag{3.14}$$

for $x \geq y$ and for every (t, x, x'), $(t, y, y') \in I \times [\min \alpha; \max \beta] \times \mathbb{R}$. By the mean value theorem, there exist c_1; c_2 depending on y, y' respectively, such that

$$f(t, y, q(y')) - f(t, \alpha(t), \alpha'(t)) = f_x(t, c_1, c_2)(y - \alpha(t))$$
$$+ f_{x'}(t, c_1, c_2)[q(y') - \alpha'(t)],\ t \in I, \tag{3.15}$$

where $\alpha(t) \leq c_1 \leq y$ and c_2 lies between $q(y')$ and $\alpha'(t)$ on I. Define

$$l(t, x, x', y, y') = f(t, \alpha(t), \alpha'(t)) + f_x(t, c_1, c_2)(x - \alpha(t))$$
$$+ f_{x'}(t, c_1, c_2)[q(x') - \alpha'(t)]. \tag{3.16}$$

Then, l is continuous and bounded on $I \times [\min \alpha, \max \beta] \times \mathbb{R} \times [\min \alpha; \max \beta] \times \mathbb{R}$ and in view of (3.15), satisfies the following relations

$$\begin{cases} l\left(t, y, y'; y, y'\right) = f\left(t, y, q\left(y'\right)\right), \\ l\left(t, \alpha\left(t\right), \alpha'\left(t\right); y, y'\right) = f\left(t, \alpha\left(t\right), \alpha'\left(t\right)\right). \end{cases} \tag{3.17}$$

Now, we define

$$g(t, x, x'; y, y') = \begin{cases} k(t, x, x'; y, y'), \text{ if } x \geq y, \\ l(t, x, x'; y, y'), \text{ if } x \leq y. \end{cases} \tag{3.18}$$

Clearly, g is continuous and bounded $I \times [\min \alpha, \max \beta] \times \mathbb{R} \times [\min \alpha, \max \beta] \times \mathbb{R}$ and therefore satisfies a Bernstein-Nagumo condition on I. For every $(t, y, y') \in I \times [\min \alpha, \max \beta] \times \mathbb{R}$, we consider the four point BVP

$$x''(t) = g(t, x, x', y, y'),\ t \in I,$$
$$x(a) = x(c),\ x(b) = x(d), \tag{3.19}$$

Using (3.14), (3.17), and (3.18) the choice of C, we have the following relations

$$g(t, \alpha(t), \alpha'(t); y, y') = l(t, \alpha(t), \alpha'); y, y' = f(t, \alpha(t), \alpha'(t)) \leq \alpha''(t),\ t \in I,$$
$$g(t, \beta(t), \beta'(t); y, y') = k(t, \beta(t), \beta'); y, y' \geq f(t, \beta(t), \beta'(t)) \leq \beta''(t),\ t \in I,$$

which imply that α, β are lower and upper solutions respectively of (23) for every $(t, y, y') \in I \times [\min \alpha, \max \beta] \times \mathbb{R}$. Since g satisfies the Nagumo condition, hence there exists a constant $C_1 > \max\{\|\alpha'\|, \|\beta'\|\}$ depends on α, β, and a Nagumo function, such that any solution x of (3.19) with the property $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in I$ must satisfies $|x'(t)| \leq C_1$ on I.

Now, we develop the iterative scheme of linear problems. As a first approximation, we choose $w_0 = \alpha$ and consider the linear four point boundary value problem

$$x''(t) = g(t, x, x'; w_0, w_0'),\ t \in I$$
$$x(a) = x(c),\ x(b) = x(d). \tag{3.20}$$

Using (B_1), (3.14) and (3.18), we obtain,

$$g(t, w_0(t), w_0'(t); w_0(t), w_0'(t)) = f(t, w_0(t), w_0'(t),\ t \in I,$$
$$g(t, \beta(t), \beta'(t); w_0(t), w_0'(t)) = k(t, \beta(t), \beta'(t); w_0(t), w_0'(t)) \geq f(t, \beta'(t)) \geq \beta'(t),\ t \in I,$$

which imply that w_0 and β are lower and upper solutions of (3.20) respectively. Hence, by Theorem 2.3, there exists a solution w_1 of (3.20) such that

$$w_0 \leq w_1 \leq \beta \text{ and } |w_1'| \leq C_1 \text{ on } I.$$

Using (3.14), (3.18) and the fact that w_1 is a solution of (3.20) and $w_1 \geq w_0$ on I, we obtain

$$w_1''(t) = g(t, w_1(t), w_1'(t); w_0(t), w'(t))$$
$$= k(t, w_1(t), w_1'(t); w_0(t), w_0'(t)) \geq f(t, w_1(t), q(w_1'(t))), \tag{3.21}$$

which implies that w_1 is a lower solution of (3.7).

In view of (3.14), (3.18), (3.21) and (B_1), we can show that w_1 and β are lower and upper solutions of the problem

$$x''(t) = g(t, x, x'; w_1, w_1');\ t \in I$$
$$x(a) = x(c),\ x(b) = x(d). \tag{3.22}$$

Hence, by Theorem 2.3, there exists a solution w_2 of (3.22) such that

$$w_1 \leq w_2 \leq \beta \text{ and } |w_2'| \leq C_1 \text{ on } I.$$

Again, w_2 can be shown to be a lower solution of (3.7). Continuing this process, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$w_0 \leq w_1 \leq w_2 \leq w_3 \leq \ldots \leq w_{n-1} \leq w_{n-1} \leq w_n \leq \beta,\ t \in I.$$

That is,

$$w_0(t) \leq w_n(t) \leq \beta(t) \text{ and } |w_n'(t)| \leq C_1,\ n \in \mathbb{N},\ t \in I, \tag{3.23}$$

where w_n is a solution of the problem

$$x''(t) = g(t, x, x'; w_{n-1}, w_{n-1}'),\ t \in I$$
$$x(a) = x(c),\ x(b) = x(d).$$

Since $g(t, w_n, w_n'; w_{n-1}, w_{n-1}')$ is bounded, we can find a constant $A > 0$ (independent on n) such that

$$|g(t, w_n, w_n'; w_{n-1}, w_{n-1}')| \leq A \text{ on } I.$$

Using the relation $x'(t) = x'(a) + \int_a^t x''(s)\,ds$, we obtain

$$\left|w_n'(t) - w_n'(s)\right| \leq \int_s^t \left|g(u, w_n, w_n'; w_{n-1}, w'_{n-1})\right| du \leq A|t - s|, \tag{3.24}$$

for any $s, t \in I$, ($s \leq t$). The inequalities (3.23) and (3.24) imply that the sequences $\{w_n^{(j)}\}$($j = 0, 1$) are uniformly bounded and equi-continuous on I and hence the Arzelà-Ascoli theorem guarantees the existence of subsequences and a function $x \in C^1(I)$ with $w_n^{(j)}$($j = 0, 1$) converging uniformly to $x^{(j)}$ on I as $n \to \infty$. Passing to the limit, we obtain $g(t, w_n, w_n'; w_{n-1}, w_{n-1}') \to f(t, x, q(x'))$. Thus, x is a solution of the boundary value problem (3.7).

Quadratic Convergence: Now, we show that the convergence of the sequence of solutions is quadratic. For this, we set $e_n(t) = x(t) - w_n(t)$, $t \in I$, where x is a solution of the problem (1.1). Then, $e_n \in C^2(I)$ and $e_n(t) \geq 0$, $t \in I$. Moreover, the BCs imply that

$$e_n(a) = e_n(c),\ e_n(d) = e_n(b). \tag{3.25}$$

Hence, that there exist $t_1 \in (a, c)$ and $t_2 \in (d, b)$ such that

$$e_n'(t_1) = 0,\ e_n'(t_2) = 0. \tag{3.26}$$

Moreover, we have,

$$e_n(t) = x''(t) - w_n(t) = [F(t, x, x') - \phi(t, x, 0)] - g(t, w_n, w_n; w_{n-1}, w_{n-1}), t \in I. \tag{3.27}$$

Applying Taylor's theorem about $(t,\ w_{n-1},\ q(w_{n-1}'))$ and using (3.18) and the definition (3.13) of k, we obtain

$$\begin{aligned}
e_n''(t) &= f\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right) + F_x\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right)e_{n-1} \\
&\quad + F_{x'}\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right)\left(x' - q\left(w'_{n-1}\right)\right) + \frac{1}{2}H(F) - \left[f\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right)\right. \\
&\quad + \left[F_x\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right) - \phi_x(t, \beta, 0)\right]a_n + F_{x'}\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right)\left(q(w'_n)\right. \\
&\quad \left.\left. - q(w'_{n-1})\right)\right] - \left[\phi(t, x, 0) - \phi(t, w_{n-1}, 0)\right] \\
&= F_x\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right)e_n(t) + F_{x'}\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right)\left(x' - q\left(w'_n\right)\right) \\
&\quad - \frac{1}{2}\left|H(F)\right| - \left[\phi(t, x, 0) - \phi(t, w_{n-1}, 0)\right] + \phi_x(t, \beta, 0)\,a_n \\
&= f_x\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right)e_n(t) + F_{x'}\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right)\left(x' - q\left(w'_n\right)\right) \\
&\quad - \frac{1}{2}\left|H(F)\right| + \phi_x(t, w_{n-1}, 0)\,e_n - \left[\phi(t, x, 0) - \phi(t, w_{n-1}, 0)\right] + \phi_x(t, \beta, 0)\,a_n,
\end{aligned}$$

where

$H(F) = e_{n-1}^2 F_{xx}(t, \xi_1, \xi_2) + 2e_{n-1}(x' - q(w_{n-1}'))\, F_{xx'}(t, \xi_1, \xi_2) + (x' - q(w_{n-1}'))^2 F_{x'x'}(t, \xi_1, \xi_2)$, $w_{n-1}(t) \leq \xi_1 \leq x(t)$, ξ_2 lies between $q(w_{n-1}'(t))$ and $x'(t)$. In (3.28), we used the notation $a_n = w_n - 2_{n-1}$ and the fact that $e_{n-1} - a_n = e_n$. In view of (6), we have

$$\phi(t, x, 0) - \phi(t, w_{n-1}, 0) \le \phi_x(t, w_{n-1}, 0)e_{n-1}.$$

Consequently,

$$\phi_x(t, w_{n-1}, 0)e_n - [\phi(t, x, 0) - \phi(t, w_{n-1}, 0)] + \phi_x(t, \beta, 0)a_n$$
$$\ge [\phi_x(t, \beta, 0) - \phi_x(t, w_{n-1}, 0)]a_n \ge \phi_{xx}(t, \eta, 0)(\beta - w_{n-1})e_{n-1},$$

where $w_{n-1} \le \eta \le \beta$. We can choose $r > 1$ such that $\beta - w_{n-1} \le re_{n-1}$ Then, we have,

$$\phi_x(t, w_{n-1}, 0)e_n - [\phi(t, x, 0) - \phi(t, w_{n-1}, 0)] + \phi_x(t, \beta, 0)a_n \ge -de_{n-1}^2, \qquad (3.29)$$

where $d = r\max\{\phi_{xx}(t, x, 0) : x \in [\min \alpha, \max \beta]$. Using (3.29) and the assumption (3.5), we can rewrite (3.28) as follows

$$e_n(t) \ge F_{x'}(t, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) - \frac{1}{2}|H(F)| - d\,|e_{n-1}|^2. \qquad (3.30)$$

Let

$$P_1 = \max\{|F_{xx}(t, z_1, z_2)|, |F_{xx'}(t, z_1, z_2)|, |F_{x'x'}(t, z_1, z_2)| : t \in I,$$
$$z_1 \in [\min w_0, \max \beta], z_2 \in [-C, C]\},$$

then

$$|H(F)| \le P_1(|e_{n-1}| + |x' - q(w'_{n-1})|^2) \le P_1(|e_{n-1}| + |e'_{n-1}|)^2 \le P_1\|e_{n-1}\|_1^2. \quad (3.31)$$

Further, since $\psi_{x'}(t, x, x') = 0$ for $|x'| \le C$, hence (3.30) takes the form

$$e''_n(t) \ge f_{x'}(t, w_{n-1}, q(w'_{n-1}))(x' - q(w'_n)) - \left(\frac{P_1}{2} + d\right)\|e_{n-1}\|_1^2$$

$$= f_{x'}(t, w_{n-1}, q(w'_{n-1}))e'_n + f_{x'}(t, w_{n-1}, q(w'_{n-1}))(w'_n - q(w'_n)) - \left(\frac{P_1}{2} + d\right)\|e_{n-1}\|_1^2.$$

(3.32)

Applying the mean value theorem, we obtain

$$f_{x'}(t, w_{n-1}(t), q(w'_{n-1}(t))) = f_{x'}(t, w_{n-1}(t), q(w'_n(t)))$$
$$+ f_{x'x'}(t, w_{n-1}(t), \eta)\,[q(w'_{n-1}(t)) - q(w'_n(t))], \qquad (3.33)$$

where η lies between $q(w'_{n-1}(t))$ and $q(w'_{n-1}(t)$. We discuss various cases:

If $w'_n(t) > C$ for some $t \in I$, then

$$w'_n(t) - q(w'_n(t)) > 0,\ q(w'_n(t)) = C \text{ and } q(w'_{n-1}(t)) - q(w'_n(t)) \le 0.$$

Using the above relations, (3.4), (3.6) and (3.33), we obtain,

$$[w_n'(t) - q(w_n'(t))]f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) \geq [w_n'(t) - q(w_n'(t))]f_{x'}(t, w_{n-1}(t), C) \geq 0.$$

If $w_n'(t) < -C$ for some $t \in I$, then

$$w_n'(t) - q(w_n'(t)) < 0,\ q(w_{n-1}'(t)) = -C \ \text{ and } q(w_{n-1}'(t)) - q(w_n'(t)) \geq 0.$$

Using the above relations, (3.4), (3.6) and (3.33), we obtain,

$$[w_{n+1}'(t) - q(w_{n+1}'(t))]f_{x'}(t, w_n(t), q(w_n'(t))) \geq [w_{n+1}'(t) - q(w_{n+1}'(t))]f_{x'}(t, w_n(t), -C) \geq 0.$$

If $|w_{n+1}'(t)| \leq C$, for some $t \in I$, then $q(w_{n+1}'(t)) = w_{n+1}'(t)$, and hence for such values of t, we get,

$$[w_n'(t) - q(w_n'(t))]f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) = 0.$$

Thus for every $t \in I$, we have,

$$[w_n'(t) - q(w_n'(t))]f_{x'}(t, w_{n-1}(t), q(w_{n-1}'(t))) \geq 0,$$

and consequently, (3.32) can be rewritten as,

$$e_n(t) \geq f_{x'}\left(t, w_{n-1}, q\left(w'_{n-1}\right)\right)e'_n(t) - \left(\frac{P_1}{2} + d\right)\|e_{n-1}\|_1^2, \tag{3.34}$$

which is the same as (3.16) of [2, p. 1110]. Hence following the same procedure as was done in [2], we can get quadratic convergence of the sequence of iterates.

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