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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR NONLINEAR EIGENVALUE PROBLEMS WITH THE p-LAPLACIAN AND NONSMOOTH POTENTIAL

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ABSTRACT: In this paper we consider a nonlinear eigenvalue problem driven by the p-Laplacian dierential operator with a nonsmooth potential function (hemivariational inequality). Using a variational approach based on the nonsmooth critical point theory, we prove the existence of at least two nontrivial positive solutions as the positive parameter moves in a half-line. By strengthening the hypotheses we show that the solutions are strictly positive. Finally if the hypotheses are symmetric in R, then we have at least four solutions of constant sign.

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1. INTRODUCTION

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. In this paper we study the following nonlinear eigenvalue problem with a nonsmooth potential

(hemivariational inequality):

$$\left\{ \begin{array}{l} -div(\| Dx(z) \|^{p-2} Dx(z)) \in \lambda \partial j(z, x(z)) \ a.e. \ on \ Z, \\ x|_{\partial z} = 0, 1 < p < \infty. \end{array} \right\} \tag{1.1}$$

In problem (1.1), $\lambda \in \mathbb{R}$ is a positive parameter, $j(z, x)$ is a measurable function which is locally Lipschitz and in general nonsmooth in the x-variable and by $\partial j(z, \cdot)$ we denote the generalized subdifferential of $j(z, \cdot)$. We are interested in the existence of multiple nontrivial positive solutions for problem (1.1) as $\lambda > 0$ varies in a half-line.

Problem (1.1) has been investigated primarily in the context of semilinear equations (i.e. $p = 2$) with a smooth potential function (i.e. $j(z, \cdot) \in C^1(\mathbb{R})$ and so $\partial j(z, \cdot)$ is single-valued, see Section 2). We mention the works of Allegretto-Nistri-Zecca [1], Cac-Fink-Gatica [2], Castro-Shivaji [3], Dancer [7], Hai [14], Lions [17], Maya-Shivaji [18] and the references therein. Problems driven by the p-Laplacian differential operator, were investigated by Guo [12], Guo-Yang [13], Hai-Schmitt

[15], Hai-Shivaji [16] and Perera [21]. With the exception of Perera [21], the rest deal with problems defined on a ball or on an annulus and focus on existence and uniqueness questions. Perera [21] has the most general result in this direction, which substantially improves the multiplicity theorem of Maya-Shivaji [18] even in the context of semilinear problems. Our work here extends that of Perera [21].

Corresponding eigenvalue problems for hemivariational inequalities (i.e. problems with a nonsmooth potential), were studied by Chang [4], Filippakis-Gasinski- Papageorgiou [8], Goeleven-Motreanu [11] and Motreanu [19]. They all deal with semilinear problems (i.e. $p = 2$) and only Filippakis-Gasinski-Papageorgiou [8] address the question of existence of multiple positive solutions.

Hemivariational inequalities arise in the study of many complicated mechanical and engineering problems, where the relevant energy functionals are neither convex nor smooth (the so-called superpotentials). For example this is the case of nonmonotone multivalued interface laws or constitutive relations that occur in certain contact and friction processes, as well as of phenomena related to large displacements and deformations expressed by nonlinear strain-displacement laws. Moreover, eigenvalue problems such as (1.1) arise in the study of steady states of diffusion problems (see Cohen-Keller [6]) and in the stability analysis of mechanical systems (such as, for example, beam buckling). For a variety of applications of hemivariational inequalities, we refer to the book of Naniewicz-Panagiotopoulos [20].

Our approach is variational based on the nonsmooth critical point theory. The basis for this theory, is the subdifferential theory for locally Lischitz functions, due to Clarke [5]. In the next section, for easy reference, we recall the basic definitions and facts from these theories, which will be used in the analysis of problem (1.1). Our basic references are the books of Clarke [5] and Gasinski-Papageorgiou [9].

2. MATHEMATICAL BACKGROUND

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X,X^*). A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ we can find a neighborhood of U of $x \in X$ and a constant $k_U > 0$ (depending on U) such that

$$|\varphi(y) - \varphi(z)| \leq k_U \|y - z\| \text{ for all } y, z \in U.$$

Recall that if $\psi\, X \to \mathbb{R}$ is continuous convex, then it is locally Lipschitz. Similarly if $\psi \in C^1(X)$. If $\varphi : X \to \mathbb{R}$, is a locally Lipschitz function, the generalized directional derivative of ψ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easily seen that $h \to \varphi^0(x, h)$ is continuous, sublinear and so it is the support function of a nonempty, w^*-compact and convex set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x; h) \text{ for all } h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is called the generalized subdifferential of φ. If φ is also convex, then the generalized subdifferential coincides with the subdifferential $\partial_c\varphi$ in the sense of convex analysis, which is defined by

$$\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi(x + h) - \varphi(x) \text{ for all } h \in X\} \text{ for all } x \in X.$$

If $\varphi \in C^1(X)$, then

$$\partial\varphi(x) = \{\varphi'(x)\} \text{ for all } x \in X.$$

If $\varphi, \psi\ X \to \mathbb{R}$ are two locally Lipschitz functions, then

$$\partial(\varphi + \psi) \subseteq \partial\varphi + \partial\psi \text{ and } \partial(\lambda\varphi) = \lambda\partial\varphi \text{ for all } \lambda \in \mathbb{R}.$$

If $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function, then $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. In this case $c = \varphi(x)$ is a critical value of φ. If $x \in X$ is local extremum of φ (i.e. a local minimum or a local maximum), then $x \in X$ is a critical point of φ.

It is well-known that in the smooth critical point theory, crucial role plays a compactness type condition, known as the “Palais-Smale condition” (*PS*-condition for short). In the present nonsmooth setting, this condition takes the following form:

> A locally Lipschitz function $\varphi : X \to \mathbb{R}$, satisfies the nonsmooth Palais-Smale condition (the nonsmooth *PS*-condition for short), if any sequence $\{x_n\}_{n\geq 1}$ such that $\sup_{n\geq 1} |\varphi(x_n)| < +\infty$ and $m(x_n) = \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence.

Evidently, since for $\varphi \in C^1(X)$ we have $\partial\varphi(x) = \{\varphi'(x)\}$, we see that for smooth φ the above definition coincides with the classical one.

The following is a nonsmooth version of the well-known “Mountain Pass Theorem”.

Theorem 2.1. *If X is a reflexive Banach space, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmooth PS-condition and there exist x_0, $x_1 \in X$ and $\rho > 0$ such that*

(i) $\| x_1 - x_0\| > \rho$ and

(ii) $\max\{ \varphi (x_0), \varphi (x_1)\} < c_0 = \inf\{ \varphi(y) : \|y - x_0\| = \rho \}$

then φ *has a critical point* $x \in X$ *with critical value* $c = \varphi(x) \geq c_0$ *defined by the minimax relation*

$$c \overset{df}{=} \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)),$$

where

$$\Gamma \overset{df}{=} \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}.$$

Finally by λ_1 we denote the first (principal) eigenvalue of the negative p-Laplacian with Dirichlet boundary condition, i.e. of $(-\Delta_p, W_0^{1,p}(Z))$. We know that $\lambda_1 > 0$ and it has the following variational characterization

$$\lambda_1 = \inf\left\{\frac{\| Dx \|_p^p}{\| x \|_p^p} : x \in W_0^{1,p}(Z),\, x \neq 0\right\}$$

(see Gasinski-Papageorgiou [10], p.732).

3. MULTIPLE POSITIVE SOLUTIONS

The hypotheses on the nonsmooth potential function $j(z, x)$ are the following: $H(j)$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z, 0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to j(z, x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z, x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$, and all $u \in \partial j(z, x)$, we have

$$|u| \leq \alpha(z) + c|x|^{r-1}$$

with $\alpha \in L^\infty(Z)_+$, $c > 0$, $1 \leq r < p^* = \begin{cases} \dfrac{Np}{N-p} & if \quad N > p \\ +\infty & if \quad N \leq p \end{cases}$;

(iv) for almost all $z \in Z$ and all $x \geq 0$, we have

$$j(t,x) \le \frac{c_0}{p} x^p \text{ for some } c_0 > 0;$$

(v) $\limsup\limits_{x\to\infty} \dfrac{j(z,x)}{x^p} \le 0$ uniformly for almost all $z \in Z$;

(vi) there exists $v_0 > 0$ such that $\int_Z j(z, v_0)dz > 0$;

(vii) there exists $\delta > 0$ such that $j(z, x) \le 0$ for a.a. $z \in Z$ and all $0 \le x \le \delta$.

Remark 3.1. These hypotheses are more general than those used by Perera *[21], where $j(z, \cdot) \in C^1(\mathbb{R})$ and $f(z, x) = \partial j(z, x)$ satisfies $|f(z, x)| \le c|x|^{p-1}$ for a.a. $z \in Z$, all $x \in \mathbb{R}$ and with* $c > 0$. *The following nonsmooth locally Lispchitz functions j_1 and j_2, satisfy hypotheses $H(j)$ (for simplicity we drop the z-dependence):*

$$j_1(x) = \begin{cases} |x|^p \ln|x| + |x|^p & \text{if } |x| \le 1 \\ -\dfrac{1}{p}|x|^p + \dfrac{1}{p} + 1 & \text{if } |x| > 1 \end{cases}$$

and $j_2(x) = \dfrac{1}{p}|x|^p - c\max\{\dfrac{1}{r}|x|^r, \dfrac{1}{\theta}|x|^\theta\}$ *with* $1 \le r < p < \theta < p^*,\ c < \dfrac{1}{p}$.

Also the C^1-function $j_3(x)$ (again we drop the z-dependence) that follows, satisfies hypotheses $H(j)$, but not those of Perera [21]:

$$j_3(x) = \frac{1}{p}|x|^p - \max\{\frac{1}{r}|x|^r, \frac{1}{\theta}|x|^\theta\} \ \textit{with} \ 1 \le r < p < \theta < p^*.$$

We introduce the Lipschitz continuous truncation function $\mathcal{T}_+ : \mathbb{R} \to \mathbb{R}_+$ defined by

$$\mathcal{T}_+(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}.$$

We set $j_+(z, x) = j(z, \mathcal{T}_+(x))$. Clearly for every $x \in \mathbb{R}$, $z \to j_+(z, x)$ is measurable, while for almost all $z \in Z$, $x \to j_+(z, x)$ is locally Lipschitz. Moreover, from the nonsmooth chain rule (see Clarke [5], p.42), we have

$$\partial j_+(z,x) \subseteq \begin{cases} \{0\} & \text{if } \; x<0 \\ \{t\partial j(z,0)\}_{t\in[0,1]} & \text{if } \; x=0 \\ \partial j(z,x) & \text{if } \; x>0 \end{cases}. \tag{3.1}$$

Note that $j_+(z, x) = 0$ a.e. on Z, for all $x \leq 0$.

For $\lambda > 0$, we consider the functional $\varphi_+^\lambda : W_0^{1,p}(z) \to \mathbb{R}$ defined by

$$\varphi_+^\lambda(x) = \frac{1}{p}\| Dx \|_p^p - \lambda \int_Z j_+(z, x(z))dz \text{ for all } x \in W_0^{1,p}(z).$$

We know (see for example Gasinski-Papageorgiou [9], p.59), that φ_+^λ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz.

Proposition 3.2. *If hypotheses H(j) hold, then for every* $\lambda > 0$, φ_+^λ *is coercive on* $W_0^{1,p}(Z)$.

Proof. By virtue of hypothesis $H(j)(v)$, given $\varepsilon > 0$ we can find $M = M(\varepsilon, \lambda) > 0$ such that

$$\lambda j_+(z,x) = \lambda j(z,x) \leq \frac{\varepsilon}{p}|x|^p \text{ for a.a. } z \in Z \text{ and all } x \leq M. \tag{3.2}$$

On the other hand using the mean value theorem for locally Lipschitz functions (see Clarke [5], p.41) and hypothesis $H(j)(\text{iii})$, we can find $c_1 = c_1(\varepsilon, \lambda) > 0$ such that

$$\lambda j_+(z, x) \leq c_1 \text{ for a.a. } z \in Z \text{ and all } x < M \tag{3.3}$$

(recall that $j_+(z, x) = 0$ a.e. on Z, for all $x \leq 0$). Combining (3.2) and (3.3), we have

$$\lambda j_+(z,x) \leq \frac{\varepsilon}{p}|x|^p + c_1 \text{ for a.a. } z \in Z \text{ and all } x \in \mathbb{R}. \tag{3.4}$$

Then for every $x \in W_0^{1,p}(Z)$, we have

$$\varphi_+^\lambda(x) = \frac{1}{p}\| Dx \|_p^p - \lambda \int_Z j_+(z, x(z))dz$$

$$\geq \frac{1}{p}\| Dx \|_p^p - \frac{\varepsilon}{p}\| x \|_p^p - c_2 \text{ for some } c_2 = c_2(\varepsilon, \lambda) > 0 \text{ (see(3.4))}$$

$$\geq \frac{1}{p}\left(1-\frac{\varepsilon}{\lambda_1}\right)\| Dx \|_p^p - c_2 \tag{3.5}$$

(recall the variational characterization of $\lambda_1 > 0$).

Choose $0 < \varepsilon < \lambda_1$. Then from (3.5) and Poincare's inequality, it follows that φ_+^λ is coercive.

Proposition 3.3. *If hypotheses H(j) hold, then for every* $\lambda > 0$, φ_+^λ *is bounded below and satisfies the nonsmooth PS-condition.*

Proof. Since φ_+^λ is coercive (see Proposition 3.2), it is bounded below (see also (3.5)).

Now suppose that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is a sequence such that

$$|\varphi_+^\lambda(x_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1 \tag{3.6}$$

$$\text{and } m_\lambda(x_n) \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Recall that $m_\lambda(x_n) = \inf\{\| x^* \| : x^* \in \partial\varphi_+^\lambda(x_n)\}$.

Since $\partial\varphi_+^\lambda(x_n) \subseteq W^{-1,p'}(Z) = W_0^{1,p}(Z)^* \left(\frac{1}{p}+\frac{1}{p'}=1\right)$ is nonempty and weakly compact in $W^{-1,p'}(Z)$ and the norm functional in a Banach space is weakly lower semicontinuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi_+^\lambda(x_n)$ such that

$$m_\lambda(x_n) = \| x_n^* \| \text{ for all n} \geq 1.$$

Let $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_Z \| Dx \|^{p-2} (Dx, Dy)_{\mathbb{R}^N} dz \quad \text{for all } x, y \in W_0^{1,p}(Z).$$

Hereafter by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W^{-1,p'}(Z), W_0^{1,p}(Z))$. It is easy to see that A is monotone, continuous, hence maximal monotone. We know that for every $n \geq 1$

$$x_n^* = A(x_n) - \lambda u_n$$

with $u_n \in L^{r'}(Z)(\frac{1}{r}+\frac{1}{r'}=1), u_n(z) \in \partial j_+(z, x_n(z))$ a.e. on Z (see Gasinski-Papageorgiou [9], p.59). Because of (3.6) and the coercivity of φ_+^λ (Proposition 3.2), we have that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,}(Z)$ is bounded. So by passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,p}(Z) \text{ and } x_n \to x \text{ in } L^p(Z) \text{ and in } L^r(Z) \text{ (recall } r < p^*).$$

From (3.7), we have

$$|\langle A(x_n), x_n - x\rangle - \lambda \int_Z u_n(x_n - x)dz| \leq \varepsilon_n \| x_n - x \| \text{ with } \varepsilon_n \downarrow 0. \tag{3.8}$$

Hence $\varepsilon_n \|x_n - x\| \to 0$ as $n \to \infty$. Also because of hypothesis $H(j)$(iii), the sequence $\{u_n\}_{n\geq 1} \subseteq L^{r'}(Z)$ is bounded. Therefore

$$\int_Z u_n(x_n - x)dz \to 0 \text{ as } n \to \infty.$$

and so from (3.8) it follows that

$$\lim \langle A(x_n), x_n - x\rangle = 0 \tag{3.9}$$

But A being maximal monotone, it is generalized pseudomonotone (see Gasinski-Papageorgiou [10], p.330). Hence from (3.9), we infer that

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle,$$
$$\Rightarrow \|Dx_n\|_p \to \|Dx\|_p \text{ as } n \to \infty.$$

We also have $Dx_n \xrightarrow{w} Dx$ in $L^p(Z,\mathbb{R}^N)$. Since the Lebesgue space $L^p(Z,\mathbb{R}^N)$ is uniformly convex it has the Kadec-Klee property (see Gasinski-Papageorgiou [10], p. 911). Therefore it follows that

$$Dx_n \to Dx \text{ in } L^p(Z,\mathbb{R}^N) \text{ as n} \to \infty.$$

From this and Poincare's inequality, we conclude that

$$x_n \to x \text{ in } W_0^{1,p}(Z),$$

$\Rightarrow \varphi_+^\lambda$ satisfies the nonsmooth PS-condition.

Using this Proposition and Theorem 2.1.6, p.144, of Gasinski-Papageorgiou [9], we obtain the following corollary.

Corollary 3.4. *If hypotheses $H(j)$ hold, then for every $\lambda > 0$ we can find $x_0 \in W_0^{1,p}(Z)$ such that*

$$\varphi_+^\lambda(x_0) = \inf\left[\varphi_+^\lambda(x) : x \in W_0^{1,p}(Z)\right]$$

Proposition 3.5. *If hypotheses $H(j)$ hold, then we can find $\lambda^* > 0$ such that for all we have $\lambda \geq \lambda^*$* we have

$$\varphi_+^\lambda(x_0) < 0$$

with $x_0 \in W_0^{1,p}(Z)$ *as in Corollary 3.4.*

Proof. We consider the integral functional $\eta_+ : L^p(Z) \to \mathbb{R}$ defined by

$$\eta_+(x) = \int_Z j_+(z, x(z))dz$$

Clearly η_+ is continuous and because of hypothesis $H(j)(vi)$, for the constant function $v_0 \in L^p(Z)$, we have

$$\eta_+(v_0) > 0.$$

Since the Sobolev space $W_0^{1,p}(Z)$ is embedded densely in $L^p(Z)$, we can find $u_0 \in W_0^{1,p}(Z)$ such that

$$\eta_+(u_0) > 0.$$

Recall that $j_+(z, x) = 0$ a.e. on Z, for all $x \leq 0$. So $\eta_+(u_0^+) = \eta_+(u_0) > 0$ and so we may assume without any loss of generality that $u_0 \geq 0$, $u_0 \neq 0$ (because $\eta_+(0) = 0 < \eta_+(u_0)$). Then

$$\varphi_+^\lambda(u_0) = \frac{1}{p}\| Du_0 \|_p^p - \lambda\int_Z j_+(z, u_0(z))dz$$

$$= \frac{1}{p}\| Du_0 \|_p^p - \lambda\eta_+(u_0).$$

Since $\eta_+(u_0) > 0$, we see that if we choose $\lambda^* > 0$ large, then for $\lambda \geq \lambda^*$, we have

$$\varphi_+^\lambda(u_0) = \frac{1}{p}\| Du_0 \|_p^p - \lambda\int_Z j_+(z, u_0(z))dz < 0,$$

$$\Rightarrow \varphi_+^\lambda(x_0) \le \varphi_+^\lambda(u_0) < 0 .$$

Proposition 3.6. *If hypotheses* $H(j)$ *hold, then for every* $\lambda > 0$ *the origin is a strict local minimizer of* φ_+^λ .

Proof. We argue indirectly. So suppose that the proposition is not true. Then we can find $\{x_n\}_{n\ge 1} \subseteq W_0^{1,p}(Z)$, $x_n \ne 0$ for all $n \ge 1$ such that

$$\|x_n\| \to 0 \text{ as } n \to \infty \text{ and } \varphi_+^\lambda(x_n) \le \varphi_+^\lambda = (0) = 0 \text{ for all } n \ge 1. \tag{3.10}$$

Since $j_+(z, x) = 0$ for a.a. $z \in Z$, all $x \le 0$, from the inequality in (3.10), we deduce that $x_n^+ \ne 0$ for all $n \ge 1$. Also because $\|x_n\| \to 0$, by passing to a suitable subsequence if necessary, we may assume that

$$x_n(z) \to 0 \text{ a.e. on } Z.$$

Be Egorov's theorem, we know that given $\varepsilon > 0$, we can find $\overline{Z}_\varepsilon$ a closed subset of Z such that $| Z \setminus \overline{Z}_\varepsilon |_N < \varepsilon$ (by $| \cdot |_N$ we denote the Lebesgue measure on $\mathbb{R}^N$) and

$$x_n(z) \to 0 \text{ uniformly for } z \in \overline{Z}_\varepsilon .$$

Therefore, if $\delta > 0$ is as in hypothesis $H(j)$ (vii), we can find $n_0 = n_0(\delta) \ge 1$ such, that

$$|x_n(z)| \le \delta \text{ for all } z \in \overline{Z}_\varepsilon \text{ and all } n \ge n_0.$$

So by virtue of hypothesis $H(j)$(vii), we have

$$j_+(z, x_n(z)) \le 0 \text{ a.e. on } \overline{Z}_\varepsilon \text{ for all } n \ge n_0. \tag{3.11}$$

Then, if $n \ge n_0$, we have

$$\varphi_+^\lambda(x_n) = \frac{1}{p}\| Dx_n \|_p^p - \lambda\int_Z j_+(z, x_n(z))dz$$

$$= \frac{1}{p}\| Dx_n \|_p^p - \lambda\int_{\overline{Z}_\varepsilon} j_+(z, x_n(z))dz - \lambda\int_{Z\setminus\overline{Z}_\varepsilon} j_+(z, x_n(z))dz$$

$$\geq \frac{1}{p}\| Dx_n \|_p^p - \lambda \int_{Z \setminus \overline{Z}_\varepsilon} j_+(z, x_n(z))dz \quad \text{(see (3.11))}$$

$$\geq \frac{1}{p}\| Dx_n^+ \|_p^p - \lambda \int_{Z \setminus \overline{Z}_\varepsilon} j_+(z, x_n^+(z))dz$$

(recall that $j_+(z, x) = 0$ a.e. on Z, for all $x \leq 0$)

$$\geq \frac{1}{p}\| Dx_n^+ \|_p^p - \frac{\lambda c_0}{p}\int_{Z \setminus \overline{Z}_\varepsilon} | x_n^+(z) |^p \, dz \quad \text{(see hypothesis } H(j)(iv)) \quad (3.12)$$

From the Sobolev embedding theorem, we know that $W_0^{1,p}(Z)$ is embedded continuously in $L^{p}(Z)$. So $(x_n^+)^p \in L^{\frac{p^*}{p}}(Z)$. Set $\eta = \frac{p^*}{p} > 1$. Using Hölder's inequality, we have

$$\int_{Z \setminus \overline{Z}_\varepsilon} (x_n^+)^p \, dz = \int_Z \chi_{Z \setminus \overline{Z}_\varepsilon}^{(Z)}(z)(x_\eta^+(z))^p \, dz$$

$$\leq \| \chi_{Z \setminus \overline{Z}_\varepsilon} \|_{\eta'} \| x_\eta^+ \|_p^p \quad \left(\frac{1}{\eta} + \frac{1}{\eta'} = 1\right)$$

$$\leq | Z \setminus \overline{Z}_\varepsilon |_N^{\frac{1}{\eta'}} \frac{c_3}{p} \| Dx_n^+ \|_p^p \quad \text{for some } c_3 > 0$$

$$\leq \frac{\varepsilon^{\frac{1}{\eta'}} c_3}{p} \| Dx_n^+ \|_p^p \quad \text{for all } n \geq 1. \tag{3.13}$$

Returning to (3.12) and using (3.13), we have

$$\varphi_+^\lambda(x_n) \geq \frac{1}{p}(1 - \lambda \varepsilon^{\frac{1}{\eta'}} c_4) \| Dx_n^+ \|_p^p \quad \text{for some } c_4 > 0, \text{ all } n \geq n_0 \tag{3.14}$$

For given $\lambda > 0$, we choose $\varepsilon = \varepsilon(\lambda) > 0$, such that $(1 - \lambda \varepsilon^{\frac{1}{\eta'}} c_4) > 0$ and so from (3.14), we see that

$$\varphi_+^\lambda(x_n) > 0 = \varphi_+^\lambda(0) \quad \text{for all } n \geq n_0,$$

a contradiction to the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$.

Theorem 3.7. *If hypotheses H(j) hold, then there exists* $\lambda^* > 0$ *such that for every* $\lambda \geq \lambda^*$ *problem (1.1) has at least two nontrivial solutions* $x_0, x_1 \varepsilon\, C_0^1(\overline{Z})_+$.

Proof. From Proposition 3.4 and 3.5, we know that we can find $\lambda^* > 0$ and $r > 0$ such that for every $\lambda \geq \lambda^*$ we have

$$\varphi_+^\lambda(x_0) < \varphi_+^\lambda(0) = 0 < \inf[\varphi_+^\lambda(x) : \| x \| = r] = c_0 \tag{3.15}$$

Here $x_0 \in W_0^{1,p}(Z)$ is as in Corollary 3.4. Because of (3.15) and Proposition 3.3 we can apply Theorem 2.1 and obtain $x_1 \in W_0^{1,p}(Z)$ such that

$$c_0 \leq \varphi_+^\lambda(x_1) \tag{3.16}$$

$$\text{and} \quad 0 \in \partial\varphi_+^\lambda(x_1) \tag{3.17}$$

From (3.15) and (3.16), we see that $x_0 \neq x_1$, $x_0 \neq 0$, $x_1 \neq 0$. Moreover, Corollary 3.4 implies that

$$0 \in \partial\varphi_+^\lambda(x_0)\,. \tag{3.18}$$

From (3.17) and (3.18) we see that for $i = 0, 1$, we have

$$A(x_i) = u_i \text{ with } u_i \in L^{r'}(Z),\ u_i(z) \in \partial j_+(z, x_i(z)) \text{ a.e. on } Z. \tag{3.19}$$

On (3.19), we act with the test function $-x_i^- \in W_0^{1,p}(Z)$ and we obtain

$$\| Dx_i^- \|_p^p = \int_Z u_i(-x_i^-)dz = 0 \ \text{(see (3.1))},$$

$$\Rightarrow x_i^- = 0, \text{ i.e. } x_i \geq 0,\ x_i \neq 0,\ i = 0, 1.$$

From (3.19) it follows that x_0, x_1 are nontrivial positive solutions of problem (1.1). Moreover, from the nonlinear regularity theory (see Gasinski-Papageorgiou [10], p.738), we have that $x_0, x_1 \in C_0^1(\overline{Z})_+$.

Recall that the Banach space $C_0^1(\overline{Z})$ is an ordered Banach space with order cone given by

$$C_0^1(\overline{Z})_+ = \{x \in C_0^1(\overline{Z}) : x(z) \geq 0 \text{ for all } z \in \overline{Z}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_0^1(\overline{Z})_+ = \{x \in C_0^1(\overline{Z})_+ : x(z) > 0 \text{ for all } z \in Z, \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z\}.$$

If we strengthen hypotheses $H(j)$, we can conclude that the two solutions of problem (1.1) belong in $\operatorname{int} C_0^1(\overline{Z})_+$. So we strengthen hypotheses $H(j)$ as follows:

$\underline{H(j)'}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z, 0) = 0$ a.e. on Z, it satisfies hypotheses $H(j)(i) \to (vii)$ and

(viii) for almost all $z \in Z$, all $x \geq 0$ and all $u \in \partial j(z, x)$, we have

$$-\widehat{c}x^{p-1} \leq u \quad \textit{with } \widehat{c} > 0.$$

Theorem 3.8. *If hypotheses $H(j)'$ hold, then there exists $\lambda^* > 0$ such that for every $\lambda \geq \lambda^*$ problem (1.1) has at least two solutions*

$$x_0, x_1 \in \operatorname{int} C_0^1(\overline{Z})_+.$$

Proof. Let $x_0, x_1 \in C_0^1(\overline{Z})_+$ be the nontrivial solutions obtained in Theorem 3.7. We have

$$A(x_i) = u_i \; i = 0, 1 \text{ (see the proof of Theorem 3.7).}$$

Recall $u_i \in L^{r'}(Z)$, $u_i(z) \in \partial j_+(z, x_i(z))$ a.e. on Z. We have

$$-\operatorname{div}(\|Dx_i(z)\|^{p-2}Dx_i(z)) = u_i(z) \text{ a.e. on } Z,$$

$$\Rightarrow \operatorname{div}(\|Dx_i(z)\|^{p-2}Dx_i(z)) \leq \widehat{c}x_i(z)^{p-1} \text{ a.e. on } Z,\ i = 0, 1$$

(see hypothesis $H(j)'(viii)$)

We can apply the nonlinear strong maximum principle of Vazquez [22] (see also Gasinski-Papageorgiou [10], p.738), to conclude that $x_0, x_1 \in \operatorname{int} C_0^1(\overline{Z})_+$.

If the hypotheses in $H(j)$ are symmetric in the positive and negative semiaxis, we can have a multiplicity result for solutions of constant sign. So the new hypotheses on the nonsmooth potential $j(z, x)$ are the following:

$H(j)_s$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z, 0) = 0$ a.e. on Z and

(i) for all x $\in \mathbb{R}$, $z \to j(z, x)$ is measurable;

(ii) for almost all $z \in$ Z, $x \to j(z, x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$, and all $u \in \partial j(z, x)$, we have

$$|u| \leq \alpha(z) + c|x|^{r-1}$$

with $\alpha \in L^\infty(Z)_+$, $c > 0$, $1 \leq r < p^* = \begin{cases} \frac{N_P}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \leq p \end{cases}$;

(iv) for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$j(t, x) \leq \frac{c_0}{p} x^p \text{ with } c_0 > 0;$$

(v) $\limsup\limits_{|x|\to\infty} \frac{j(z,x)}{x^p} \leq 0$ uniformly for almost all $z \in Z$;

(vi) there exist $v_0 > 0 > v_1$ such that $\int_Z j(z, v_0)dz > 0$, $\int_Z j(z, v_1)dz > 0$;

(vii) there exists $\delta > 0$ such that for almost all $z \in Z$ and all $|x| \leq \delta$, $j(z, x) \leq 0$.

Then working as above, separately on the positive semiaxis and on the negative semiaxis, we obtain the following multiplicity result.

Theorem 3.9. *If hypotheses $H(j)_s$ hold, then there exists $\lambda^* > 0$ such that $\lambda \geq \lambda^*$ problem (1.1) has at least four nontrivial solutions, two positive $x_0, x_1 \in C_0^1(\overline{Z})_+$ and two negative $y_0, y_1 \in -C_0^1(\overline{Z})_+$.*

Again if we strengthen hypotheses $H(j)_s$, we can improve the conclusion of Theorem 3.9. So we assume the following for the nonsmooth potential $j(z, x)$:

$H(j)'_s$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z, 0) = 0$ a.e. on Z, it satisfies hypotheses $H(j)_s(i) \to (vii)$ and

(viii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z, x)$, we have

$$-\widehat{c}\,|\,x\,|^{p-1} \leq u \quad \text{with } \widehat{c} > 0.$$

Then arguing as in the Theorem 3.8, this time on both semiaxes, we obtain the following multiplicity result.

Theorem 3.10. *If hypotheses $H(j)'_s$ hold, then there exists $\lambda^* > 0$ such that* for all every $\lambda \geq \lambda^*$*problem* (1.1) *has at least four solutions* $x_0, x_1 \in \operatorname{int} C_0^1(\overline{Z})_+$ and $y_0, y_1 \in -\operatorname{int} C_0^1(\overline{Z})_+$.

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