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COMPARISON RESULTS AND APPROXIMATION OF EXTREMAL SOLUTIONS FOR SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Juan J. Nieto & Rosana Rodríguez-López

ABSTRACT: We improve some comparison results relative to the periodic boundary value problem for a second-order functional differential equation. The improvement consists on the consideration of different types of functional perturbations for a second-order linear equation. We also present an uniqueness result for a quasi-linear problem in presence of a couple of well-ordered upper and lower solutions. Finally, a monotone method is developed for a general second-order nonlinear functional problem.

Keywords: Functional differential equation, Periodic boundary value problem, Maximum principle, Upper and lower solutions, Monotone method.

1. PRELIMINARIES

We study the following nonlinear problem with periodic boundary value conditions

$$\begin{cases} -v''(t) = f(t, v(t), [p(v)](t)), \ \ a.e.\ t \in I \\ v(0) = v(T), \\ v'(0) = v'(T) \end{cases} \tag{1}$$

where $I = [0, T]$, $T > 0$, and $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$.

To this purpose, we first consider the periodic boundary value problem for a linear second-order ordinary differential equation subject to functional perturbation

$$\begin{cases} -v''(t) + Mv(t) + [p(v)](t) = \sigma(t), \ \ a.e.\ t \in I \\ v(0) = v(T), \\ v'(0) = v'(T) + \lambda. \end{cases} \tag{2}$$

Here $M > 0$, λ is a real constant, $\sigma \in L^1(I)$ and p is an operator not necessarily linear nor continuous. We consider solutions verifying that $v \in W^{2,1}(I)$, that is, v and v' are absolutely continuous on I and satisfy the equation and the boundary conditions.

In [1], it was considered the problem

$$\begin{cases} -y''(t) = f(t, y(t), y(w(t))), \ \ t \in I, \\ y(0) = y(T), \\ y'(0) = y'(T), \end{cases}$$

where $f \in C(I \times \mathbb{R}^2, \mathbb{R})$, $w \in C(I, [-r, T])$, $r > 0$, $t - r \leq w(t) \leq t$, $\forall t \in I$. The authors showed that the monotone method can be applied to this problem. In this context, comparison results are given.

In [2], maximum principles for functional-perturbed linear problems of the type (2) were presented. For example, the following results were proved.

Theorem 1.1: (*Theorem 5 [2]*) *Consider problem (2), where* $M > 0$, $\lambda \leq 0$, $\sigma \geq 0$ *a.e. on* I, *and the functional perturbation p satisfies*

$$\text{for every } w \in C(I),\ p(w) \in L^\infty(I), \tag{3}$$

there exists $N \geq 0$ *such that*

$$\text{for every } \tau \in I,\ w \in C(I),\ \ \mathrm{ess\,sup}_{t\in[0,\tau]}[p(w)](t) \leq N \max_{t\in[0,\tau]} w(t) \tag{4}$$

and

$$\frac{2N}{M}\left(\sinh\frac{T\sqrt{M}}{2}\right)^2 < 1. \tag{5}$$

Then, any solution $v \in W^{2,1}(I)$ *of problem (2) satisfies* $v \geq 0$ *on* I.

Here, for $w \in L^\infty(I)$, we define the essential supremum of w on I as the greatest lower bound of constants α such that $w(t) \leq \alpha$ a.e. on I and it is denoted by $\mathrm{ess\,sup}_{t\in I} w(t)$.

For the case where $[p(w)](t) = Nw(\theta(t))$, for $N \geq 0$ and $\theta : I \to I$, the authors obtained the following result

Corollary 1 *(Corollary 1 [2]) Consider problem*

$$\begin{cases} -v''(t) + Mv(t) + Nv(\theta(t)) = \sigma(t), \ \ a.e. \ \ t \in I \\ v(0) = v(T), \\ v'(0) = v'(T) + \lambda, \end{cases} \tag{6}$$

with $M > 0$, $N \geq 0$, $\lambda \leq 0$, $\sigma \geq 0$ *a.e. on I, and* $\theta : I \to I$ *such that*

$$\forall\, w \in C(I),\, p(w) \in L^1(I),$$

and $\theta(t) \leq t$, *for a.e.* $t \in I$. *Assume that estimate (5) holds. Then any solution* $v \in W^{2,1}(I)$ *of* (6) *satisfies* $v \geq 0$ *on I.*

In [3] and for the continuous case, monotone iterative technique for problem (6) was developed under weaker hypotheses and thus improving results of [1] and debilitating hypothesis (5) to

$$\frac{2N}{M}\left(\sinh\frac{T\sqrt{M}}{4}\right)^2 < 1.$$

In section 2, we present some comparison results for problem (2), where p is of general type, improving some previous results. In section 3, the existence of solution for problem (2) is approached and, finally, in Section 4, monotone method for (1) is developed, generalizing some existing results. The interest of the results presented is the fact that the functional dependence in the equation is not necessarily Lipschitzian, and even in that general situation uniqueness results for nonlinear problems are obtained by using the method of upper and lower solutions.

2. MAXIMUM PRINCIPLE

Theorem 2.1 *Let* $v \in W^{2,1}(I)$ *a solution of (2), where* $M > 0$, $\lambda \leq 0$, $\sigma \geq 0$ *a.e. on I, and p such that*

$$\forall w \in C(I),\, p(w) \in L^1(I), \tag{7}$$

$$p(w) \leq 0 \text{ a.e. on } I, \text{ if } w \in C(I),\, w \leq 0 \text{ on } I \tag{8}$$

and for all $a < b \in I$ *and* $w \in C(I)$, *we have*

$$\int_a^b\int_s^b [p(w)](t)e^{\sqrt{M}(t+a-2S)}dt\, ds \leq \max_{s\in[0,b]} w(s), \text{ if } \max_{[0,b]} w \geq 0, \tag{9}$$

$$\int_a^b\int_a^s [p(w)](t)e^{-\sqrt{M}(t+b-2s)}dt\, ds \leq \max_{s\in[0,T]} w(s), \text{ if } \max_{[0,T]} w \geq 0. \tag{10}$$

Then, $v \geq 0$ *on I.*

Proof : If $v \geq 0$ on I is not true, then there exists $t_0 \in I$ such that $v(t_0) < 0$. If $v \leq 0$ on I, $v \not\equiv 0$, then, using (8), we get

$$-v''(t) = \sigma(t) - Mv(t) - [p(v)](t) \geq -Mv(t) - [p(v)](t)$$

for a.e. $t \in I$ and

$$v''(t) \leq Mv(t) + [p(v)](t) \leq 0, \text{ a.e. on } I.$$

Thus, v' is nonincreasing. Since $v'(0) \leq v'(T)$, then v' is a constant function. This implies that $v(t) = k < 0$. Then, by (8),

$$0 \leq Mk + [p(k)](t) \leq Mk,$$

which is a contradiction.

This shows that there exists at least one point $t_* \in I$ with $v(t_*) > 0$.

At this point, there are two possibilities:

Case 1: $v(0) = v(T) > 0$.

Then, there exists $t_1 \in (0, T)$ with

$$v(t_1) = \min_{[0,T]} v < 0, \text{ and } v'(t_1) = 0.$$

Let $\xi \in [0, t_1)$ such that $v(\xi) = \max_{[0,t_1]} v > 0.$ Then, we obtain that

$$-v''(t) + Mv(t) \geq -[p(v)](t), \text{ a.e. } t \in [0, t_1].$$

Let $\rho = \sqrt{M}$ and $v(t) = e^{\rho t}y(t)$ so that

$$v''(t) = \rho^2 e^{\rho t} y(t) + 2\rho e^{\rho t} y'(t) + e^{\rho t} y''(t),$$

and

$$-(2\rho e^{\rho t} y'(t) + e^{\rho t} y''(t)) \geq -[p(v)](t), \text{ a.e. } t \in [0, t_1],$$

$$-(2\rho e^{2\rho t} y'(t) + e^{2\rho t} y''(t)) \geq -[p(v)](t)e^{\rho t}, \text{ a.e. } t \in [0, t_1],$$

$$-(e^{2\rho t} y'(t))' \geq -[p(v)](t)e^{\rho t}, \text{ a.e. } t \in [0, t_1],$$

$$(e^{2\rho t} y'(t))' \leq [p(v)](t)e^{\rho t}, \text{ a.e. } t \in [0, t_1].$$

Integrating this inequality from s, $s \in [\xi, t_1)$, to t_1, we obtain

$$e^{2\rho t_1} y'(t_1) - e^{2\rho s} y'(s) \leq \int_s^{t_1} [p(v)](t)e^{\rho t}\, dt, \quad \text{for } s \in [\xi, t_1]$$

and

$$y'(t_1) = (v'(t_1) - \rho e^{\rho t_1} y(t_1))e^{-\rho t_1} = -\rho y(t_1) > 0.$$

Hence,

$$-e^{2\rho s}y'(s) < \int_s^{t_1}[p(v)](t)\, e^{\rho t}dt, \text{ for } s \in [\xi, t_1]$$

and

$$-y'(s) < \int_s^{t_1}[p(v)](t)\, e^{\rho t}dt\, e^{-2\rho s}, \text{ for } s \in [\xi, t_1].$$

Now, integrate from ξ to t_1 to obtain

$$y(\xi) < -y(t_1) + y(\xi) \le \int_\xi^{t_1}\int_s^{t_1}[p(v)](t)e^{\rho t}dt\, e^{-2\rho s}ds,$$

and

$$v(\xi) = e^{\rho\xi}y(\xi) < e^{\rho\xi}\int_\xi^{t_1}\int_s^{t_1}[p(v)](t)e^{\rho t}dt\, e^{-2\rho s}ds.$$

Using condition (9), we get

$$v(\xi) < \int_\xi^{t_1}\int_s^{t_1}[p(v)](t)e^{\rho(t+\xi-2s)}dt\, ds \le \max_{[0,t_1]} v = v(\xi),$$

which is a contradiction.

Case 2: $v(0) = v(T) \le 0$.

Let $t_2 \in [0, T]$ with

$$v(t_2) = \min_{s\in[0,T]} v(s) < 0 \text{ and } v'(t_2) = 0.$$

If $t_2 \in (0, T)$, then $v'(t_2) = 0$ and if $t_2 = 0$ or $t_2 = T$, then $v(0) = v(T) = v(t_2)$ and $v'(T) \ge v'(0) \ge 0 \ge v'(T)$, therefore $v'(0) = v'(T) = 0$. Let $\xi \in (0, T)$ with $v(\xi) = \max_{[0,T]} v > 0.$ If $\xi < t_2$, we repeat the procedure of the first case and we reach a contradiction. If $t_2 < \xi$, let $\rho = -\sqrt{M}$ and $v(t) = e^{\rho t}y(t)$. Then, we obtain

$$(e^{2\rho t}y'(t))' \le [p(v)](t)\, e^{\rho t}, \text{ a.e. } t \in [0, T].$$

Integrating this inequality from t_2 to s ($s \in (t_2, \xi]$), we obtain

$$e^{2\rho s}y'(s) < e^{2\rho s}y'(s) - e^{2\rho t_2}y'(t_2) \le \int_{t_2}^{s}[p(v)](t)e^{\rho t}dt, \text{ for } s \in [t_2, \xi],$$

where

$$y'(t_2) = (-\rho e^{\rho t_2} y(t_2))e^{-\rho t_2} = -\rho y(t_2) < 0,$$

so that

$$y'(s) < \int_{t_2}^{s} [p(v)](t)e^{\rho t} dt\ e^{-2\rho s}, \text{ for } s \in [t_2, \xi].$$

We integrate from t_2 to ξ,

$$y(\xi) < y(\xi) - y(t_2) \le \int_{t_2}^{\xi}\int_{t_2}^{s} [p(v)](t)e^{\rho(t-2s)} dt\ ds,$$

and, using condition (10), we obtain

$$v(\xi) = e^{\rho\xi} y(\xi) < \int_{t_2}^{\xi}\int_{t_2}^{s} [p(v)](t)e^{\rho(t+\xi-2s)} dt\ ds \le \max_{[0,T]} v = v(\xi),$$

reaching again a contradiction.

Corollary 2: *Consider the particular case of*

$$[p(v)](t) = Nv(\theta(t)),$$

where $N \ge 0$, and $\theta : I \to I$, $\theta(t) \le t$, *for a.e.* $t \in I$ *and p satisfies (7). If* $v \in W^{2,1}(I)$ *is a solution of (2), where* $M > 0$, $\lambda \le 0$, $\sigma \ge 0$ *a.e. on I and*

$$\frac{2N}{M}\left(\sinh \frac{T\sqrt{M}}{2}\right)^2 \le 1, \tag{11}$$

then $v \ge 0$ *on I.*

Proof: It is evident that condition (8) is valid. Let $a < b \in I$ and $w \in C(I)$ with $\max_{[0,b]} w \ge 0,$ then

$$\int_a^b\int_s^b Nw(\theta(t))e^{\sqrt{M}(t+a-2s)} dt\ ds$$

$$\le\ N\left(\max_{[0,b]} w\right)\int_a^b\int_s^b e^{\sqrt{M}(t+a-2s)} dt\ ds$$

$$= \frac{N}{\sqrt{M}}\left(\max_{[0,b]} w\right)\int_a^b (e^{\sqrt{M}(b+a-2s)} - e^{\sqrt{M}(a-s)})\,ds$$

$$= \frac{N}{\sqrt{M}}\left(\max_{[0,b]} w\right)\left(\frac{e^{\sqrt{M}(b+a-2b)}}{-2\sqrt{M}} + \frac{e^{\sqrt{M}(a-b)}}{\sqrt{M}} + \frac{e^{\sqrt{M}(b+a-2a)}}{2\sqrt{M}} - \frac{1}{\sqrt{M}}\right)$$

$$= \frac{N}{M}\left(\max_{[0,b]} w\right)\left(\frac{-1}{2}e^{\sqrt{M}(a-b)} + e^{\sqrt{M}(a-b)} + \frac{1}{2}e^{\sqrt{M}(b-a)} - 1\right)$$

$$= \frac{N}{M}\left(\max_{[0,b]} w\right)\left(\frac{1}{2}e^{\sqrt{M}(a-b)} + \frac{1}{2}e^{\sqrt{M}(b-a)} - 1\right)$$

$$= \frac{N}{2M}\left(\max_{[0,b]} w\right)\left(e^{\sqrt{M}(a-b)} + e^{\sqrt{M}(b-a)} - 2\right)$$

$$= \frac{2N}{M}\left(\max_{[0,b]} w\right)\left(\sinh\frac{\sqrt{M}(b-a)}{2}\right)^2$$

$$\leq \frac{2N}{M}\left(\sinh\frac{T\sqrt{M}}{2}\right)^2\left(\max_{[0,b]} w\right) \leq \max_{s\in[0,b]} w(s),$$

and, if $\max\limits_{[0,T]} w \geq 0$,

$$\int_a^b\int_a^s Nw(\theta(t))e^{-\sqrt{M}(t+b-2s)}dt\,ds$$

$$\leq \frac{N}{\sqrt{M}}\left(\max_{[0,T]} w\right)\int_a^b (-e^{-\sqrt{M}(b-s)} + e^{-\sqrt{M}(a+b-2s)})\,ds$$

$$= \frac{N}{M}\left(\max_{[0,T]} w\right)\left(-1 + \frac{1}{2}e^{-\sqrt{M}(a-b)} + e^{-\sqrt{M}(b-a)} - \frac{1}{2}e^{-\sqrt{M}(b-a)}\right)$$

$$= \frac{N}{M}\left(\max_{[0,T]} w\right)\left(\frac{1}{2}e^{-\sqrt{M}(b-a)} + \frac{1}{2}e^{-\sqrt{M}(a-b)} - 1\right)$$

$$= \frac{N}{2M}\left(\max_{[0,T]} w\right)\left(e^{-\sqrt{M}(b-a)} + e^{\sqrt{M}(b-a)} - 2\right)$$

$$= \frac{2N}{M}\left(\max_{[0,T]} w\right)\left(\sinh\frac{\sqrt{M}(b-a)}{2}\right)^2$$

$$\le \frac{2N}{M}\left(\sinh\frac{T\sqrt{M}}{2}\right)^2\left(\max_{[0,T]} w\right) \le \max_{s\in[0,T]} w(s).$$

This result improves Corollary 1 (Corollary 1 [2]), since we allow the identity on estimate (11), and it is also an extension of Theorems 2.2 and 2.2* in [1], taking the delay function

$$\theta(t) = \max\{w(t), 0\} = \begin{cases} w(t), & \text{if } \ w(t) > 0, \\ 0, & \text{if } \ w(t) \le 0, \end{cases} \quad t \in I,$$

instead of $t - r \le w(t) \le t$, $t \in I$, since solutions are constant in $[-r, 0]$. In our formulation, it is not required that the delay satisfies $t - r \le w(t)$, $t \in I$. Note that condition (7) trivially holds if θ is continuous. Next, we give an extension of Theorem 5 in [2].

Corollary 3 *Suppose that p is such that hypotheses (3) and (4) hold. If* $v \in W^{2,1}(I)$ *is a solution of (2), with* $M > 0$, $\lambda \le 0$, $\sigma \ge 0$ *a.e. on I, and (11) holds, then* $v \ge 0$ *on I.*

Proof: It is obvious the validity of condition (8). Indeed, if $w \in C(I)$, $w \le 0$ on I, then

$$[p(w)](t) \le ess\sup_{[0,t]}[p(w)] \le N \max_{[0,t]} w \le 0, \text{ for a.e. } t \in I.$$

Also, for $a < b \in I$, and a.e. $t \in [a,b]$, $[p(w)](t) \le N \max_{[0,b]} w$, and, following a procedure similar to the one in the previous corollary, we obtain that (9) and (10) are true.

Remark 1: *In fact, Corollary 2 is a particular case of Corollary 3, since*

$$\theta(t) \le t, \quad \text{a.e. } t \in I$$

implies that

$$ess\sup_{t\in[0,\tau]} Nw(\theta(t)) \le N \max_{[0,\tau]} w, \text{ for all } \tau \in I, w \in C(I).$$

Compare Corollary 3 with Theorem 1.1 (Theorem 5 [2]).

Now, we present a first result for the case of integral type dependence.

Corollary 4 *Suppose that p verifies (7) and there exists $N \geq 0$ with*

$$[p(w)](t) \leq N\int_0^t w(r)dr, \textit{ for a.e. } t \in I.$$

If $v \in W^{2,1}(I)$ *is a solution of (2), where* $M > 0$, $\lambda \leq 0$, $\sigma \geq 0$ *a.e. on I, and the following estimate holds*

$$\frac{2NT}{M}\left(\sinh\frac{T\sqrt{M}}{2}\right)^2 \leq 1, \tag{12}$$

then $v \geq 0$ *on I*.

Proof: It is easy to check that, for $w \in C(I)$ with $w \leq 0$ on I,

$$[p(w)](t) \leq N\int_0^t w(r)dr \leq 0, \text{ for a.e. } t \in I.$$

Now, let $a < b \in I$ and $w \in C(I)$ with $\max\limits_{[0,b]} w \geq 0$,

$$\begin{aligned}
&\int_a^b\int_s^b [p(w)](t)e^{\sqrt{M}(t+a-2s)}dt\,ds \\
&\quad \leq N\int_a^b\int_s^b\int_0^t w(r)dr\, e^{\sqrt{M}(t+a-2s)}dt\,ds \\
&\quad \leq N\int_a^b\int_s^b\left(\max_{[0,b]} w\right)\int_0^t dr\, e^{\sqrt{M}(t+a-2s)}dt\,ds \\
&\quad \leq N\left(\max_{[0,b]} w\right)\int_a^b\int_s^b te^{\sqrt{M}(t+a-2s)}dt\,ds \\
&\quad \leq NT\left(\max_{[0,b]} w\right)\int_a^b\int_s^b e^{\sqrt{M}(t+a-2s)}dt\,ds \\
&\quad = \frac{2NT}{M}\left(\max_{[0,b]} w\right)\left(\sinh\frac{\sqrt{M}(b-a)}{2}\right)^2
\end{aligned}$$

$$\leq \frac{2NT}{M}\left(\sinh \frac{T\sqrt{M}}{2}\right)^2 \left(\max_{[0,b]} w\right) \leq \max_{s\in[0,b]} w(s).$$

For $a < b \in I$ and $w \in C(I)$ with $\max_{[0,T]} w \geq 0$,

$$\int_a^b \int_a^s [p(w)](t) e^{-\sqrt{M}(t+b-2s)} dt\ ds$$

$$\leq N\int_a^b \int_a^s \int_0^t w(r) dr\ e^{-\sqrt{M}(t+b-2s)} dt\ ds$$

$$\leq N\int_a^b \int_a^s \left(\max_{[0,T]} w\right) \int_0^t dr\ e^{-\sqrt{M}(t+b-2s)} dt\ ds$$

$$= N\left(\max_{[0,T]} w\right) \int_a^b \int_a^s t e^{-\sqrt{M}(t+b-2s)} dt\ ds$$

$$\leq NT\left(\max_{[0,T]} w\right) \int_a^b \int_a^s e^{-\sqrt{M}(t+b-2s)} dt\ ds$$

$$= \frac{2NT}{M}\left(\max_{[0,T]} w\right)\left(\sinh \frac{\sqrt{M}(b-a)}{2}\right)^2$$

$$\leq \frac{2NT}{M}\left(\sinh \frac{T\sqrt{M}}{2}\right)^2 \left(\max_{[0,T]} w\right) \leq \max_{s\in[0,T]} w(s).$$

We can find a weaker estimation on the constants.

Corollary 5: *In Corollary 4, we can replace condition (12) by the following estimate*

$$\frac{N}{M}\left(T\cosh(T\sqrt{M}) - \frac{1}{\sqrt{M}}\sinh(T\sqrt{M})\right) \leq 1, \tag{13}$$

and the conclusion is still valid.

Proof: Let $a < b \in I$ and $w \in C(I)$ with $\max_{[0,b]} w \geq 0,$ then following the proof of Corollary 4, we obtain that

$$\int_a^b \int_s^b [p(w)](t) e^{\sqrt{M}(t+a-2s)} dt\, ds \leq N\left(\max_{[0,b]} w\right) \int_a^b \int_s^b t e^{\sqrt{M}(t+a-2s)} dt\, ds.$$

Now, using integration by parts,

$$\int_s^b t\, e^{\sqrt{M}(t+a-2s)} dt =$$

$$= \left. t \frac{e^{\sqrt{M}(t+a-2s)}}{\sqrt{M}} \right]_s^b - \int_s^b \frac{e^{\sqrt{M}(t+a-2s)}}{\sqrt{M}} dt$$

$$= \frac{b}{\sqrt{M}} e^{\sqrt{M}(b+a-2s)} - \frac{s}{\sqrt{M}} e^{\sqrt{M}(a-s)} - \frac{1}{M} e^{\sqrt{M}(b+a-2s)} + \frac{1}{M} e^{\sqrt{M}(a-s)}$$

$$= \left(\frac{b}{\sqrt{M}} - \frac{1}{M}\right) e^{\sqrt{M}(b+a-2s)} + \left(\frac{1}{M} - \frac{s}{\sqrt{M}}\right) e^{\sqrt{M}(a-s)},$$

and, thus,

$$\int_a^b \int_s^b t\, e^{\sqrt{M}(t+a-2s)} dt\, ds =$$

$$= \int_a^b \left\{ \left(\frac{b}{\sqrt{M}} - \frac{1}{M}\right) e^{\sqrt{M}(b+a-2s)} + \left(\frac{1}{M} - \frac{s}{\sqrt{M}}\right) e^{\sqrt{M}(a-s)} \right\} ds$$

$$= \left. \left(\frac{b}{\sqrt{M}} - \frac{1}{M}\right) \frac{e^{\sqrt{M}(b+a-2b)} - e^{\sqrt{M}(b+a-2a)}}{-2\sqrt{M}} + \left(\frac{1}{M} - \frac{s}{\sqrt{M}}\right) \frac{e^{\sqrt{M}(a-s)}}{-\sqrt{M}} \right]_a^b -$$

$$- \int_a^b \frac{e^{\sqrt{M}(a-s)}}{-\sqrt{M}} \left(-\frac{1}{\sqrt{M}}\right) ds$$

$$= \left(\frac{b}{\sqrt{M}} - \frac{1}{M}\right) \frac{e^{\sqrt{M}(b-a)} - e^{\sqrt{M}(a-b)}}{2\sqrt{M}} + \left(\frac{1}{M} - \frac{b}{\sqrt{M}}\right) \frac{e^{\sqrt{M}(a-b)}}{-\sqrt{M}} +$$

$$+\frac{1}{\sqrt{M}}\left(\frac{1}{M}-\frac{a}{\sqrt{M}}\right)+\frac{1}{M\sqrt{M}}(e^{\sqrt{M}(a-b)}-1)$$

$$=\frac{b}{2M}e^{\sqrt{M}(b-a)}-\frac{b}{2M}e^{\sqrt{M}(a-b)}-\frac{1}{2M\sqrt{M}}e^{\sqrt{M}(b-a)}+$$

$$+\frac{1}{2M\sqrt{M}}e^{\sqrt{M}(a-b)}+\frac{b}{M}e^{\sqrt{M}(a-b)}-\frac{a}{M}$$

$$=\frac{b}{M}\frac{e^{\sqrt{M}(b-a)}+e^{-\sqrt{M}(b-a)}}{2}-\frac{1}{M\sqrt{M}}\frac{e^{\sqrt{M}(b-a)}-e^{-\sqrt{M}(b-a)}}{2}-\frac{a}{M}$$

$$=\frac{1}{M}\left\{b\cosh(\sqrt{M}(b-a))-\frac{1}{\sqrt{M}}\sinh(\sqrt{M}(b-a))-a\right\}.$$

We give an upper bound for the expression

$$b\cosh(\sqrt{M}(b-a))-\frac{1}{\sqrt{M}}\sinh(\sqrt{M}(b-a))-a,$$

for $0 \le a < b \le T$. Take $s = b - a \in (0, T]$, $(b = a + s)$, and define

$$\phi(a,s)=(a+s)\cosh(\sqrt{M}s)-\frac{1}{\sqrt{M}}\sinh(\sqrt{M}s)-a,$$

for $s \in (0, T]$ and $0 \le a = b - s \le T -$ s. We take a fixed s and differentiate ϕ with respect to the variable a, obtaining

$$\cosh(\sqrt{M}s)-1>0, \text{ if } s>0,$$

so that ϕ is nondecreasing in a for 2 fixed $s \in (0, T]$. Then, for $s \in (0, T]$ fixed, the maximum is attained at $a = T - s$:

$$\phi(a,s)\le(T-s+s)\cosh(\sqrt{M}s)-\frac{1}{\sqrt{M}}\sinh(\sqrt{M}s)-(T-s)$$

$$=T\cosh(\sqrt{M}s)-\frac{1}{\sqrt{M}}\sinh(\sqrt{M}s)-(T-s)=\varphi(s),\ s\in(0,T].$$

Function φ is a nondecreasing function in $[0, T]$, then

$$\phi(a,s) \le \varphi(T) = T\cosh(T\sqrt{M}) - \frac{1}{\sqrt{M}}\sinh(T\sqrt{M}),$$

and, by estimate (13),

$$\int_a^b \int_s^b [p(w)](t) e^{\sqrt{M}(t+a-2s)} dt\, ds$$

$$\le \left(\max_{[0,b]} w\right)\frac{N}{M}\left(T\cosh(T\sqrt{M}) - \frac{1}{\sqrt{M}}\sinh(T\sqrt{M})\right)$$

$$\le \left(\max_{[0,b]} w\right).$$

This provides the validity of (9).

For $a < b \in I$ and $w \in C(I)$ with $\max\limits_{[0,T]} w \ge 0,$ we obtain

$$\int_a^b \int_a^s [p(w)](t) e^{-\sqrt{M}(t+b-2s)} dt\, ds \le N\left(\max_{[0,T]} w\right)\int_a^b \int_a^s t\, e^{-\sqrt{M}(t+b-2s)} dt\, ds.$$

An analogous calculus leads to

$$\int_a^s t\, e^{-\sqrt{M}(t+b-2s)} dt$$

$$= t\left.\frac{e^{-\sqrt{M}(t+b-2s)}}{-\sqrt{M}}\right]_a^s - \int_a^s \frac{e^{-\sqrt{M}(t+b-2s)}}{-\sqrt{M}} dt$$

$$= \frac{-s}{\sqrt{M}} e^{-\sqrt{M}(b-s)} + \frac{a}{\sqrt{M}} e^{-\sqrt{M}(a+b-2s)} - \frac{1}{M} e^{-\sqrt{M}(b-s)} + \frac{1}{M} e^{-\sqrt{M}(a+b-2s)}$$

$$= \left(\frac{-s}{\sqrt{M}} - \frac{1}{M}\right) e^{-\sqrt{M}(b-s)} + \left(\frac{a}{\sqrt{M}} + \frac{1}{M}\right) e^{-\sqrt{M}(a+b-2s)},$$

and, therefore,

$$\int_a^b \int_a^s t\, e^{-\sqrt{M}(t+b-2s)} dt\, ds$$

$$= \int_a^b \left\{ \left(\frac{-s}{\sqrt{M}} - \frac{1}{M} \right) e^{-\sqrt{M}(b-s)} + \left(\frac{a}{\sqrt{M}} + \frac{1}{M} \right) e^{-\sqrt{M}(a+b-2s)} \right\} ds$$

$$= \left(-\frac{s}{\sqrt{M}} - \frac{1}{M} \right) \frac{e^{-\sqrt{M}(b-s)}}{\sqrt{M}} \Bigg]_a^b - \int_a^b \frac{e^{-\sqrt{M}(b-s)}}{\sqrt{M}} \left(-\frac{1}{\sqrt{M}} \right) ds +$$

$$+ \left(\frac{a}{\sqrt{M}} + \frac{1}{M} \right) \frac{e^{-\sqrt{M}(a+b-2b)} - e^{-\sqrt{M}(a+b-2a)}}{2\sqrt{M}}$$

$$= \left(\frac{-b}{\sqrt{M}} - \frac{1}{M} \right) \frac{1}{\sqrt{M}} - \left(\frac{-a}{\sqrt{M}} - \frac{1}{M} \right) \frac{e^{-\sqrt{M}(b-a)}}{\sqrt{M}} +$$

$$+ \frac{1}{M\sqrt{M}} (1 - e^{-\sqrt{M}(b-a)}) + \left(\frac{a}{\sqrt{M}} + \frac{1}{M} \right) \frac{1}{2\sqrt{M}} (e^{-\sqrt{M}(a-b)} - e^{-\sqrt{M}(b-a)})$$

$$= \frac{-b}{M} + \frac{a}{2M} e^{-\sqrt{M}(b-a)} + \frac{a}{2M} e^{\sqrt{M}(b-a)} +$$

$$+ \frac{1}{2M\sqrt{M}} e^{\sqrt{M}(b-a)} - \frac{1}{2M\sqrt{M}} e^{-\sqrt{M}(b-a)}$$

$$= \frac{-b}{M} + \frac{a}{M} \frac{e^{\sqrt{M}(b-a)} + e^{-\sqrt{M}(b-a)}}{2} + \frac{1}{M\sqrt{M}} \frac{e^{\sqrt{M}(b-a)} - e^{-\sqrt{M}(b-a)}}{2}$$

$$= \frac{1}{M} \left\{ a \cosh(\sqrt{M}(b-a)) + \frac{1}{\sqrt{M}} \sinh(\sqrt{M}(b-a)) - b \right\}.$$

To give an upper bound for

$$a \cosh(\sqrt{M}(b-a)) + \frac{1}{\sqrt{M}} \sinh(\sqrt{M}(b-a)) - b,$$

for $0 \le a < b \le T$, we choose $s = b - a \in (0, T]$, $(b = a + s)$, and define

$$\Phi(a,s) = -(a+s) + a\cosh(\sqrt{M}s) + \frac{1}{\sqrt{M}}\sinh(\sqrt{M}s),$$

where $s \in (0, T]$, $0 \le a = b - s \le T - s$. Fixing $s \in (0, T]$ and differentiating Φ with respect to the variable a, we get

$$-1 + \cosh(\sqrt{M}s) > 0, \text{ if } s > 0,$$

which implies that Φ is nondecreasing in a for $s \in (0, T]$ fixed, thus the maximum is attained at $a = T - s$:

$$\Phi(a,s) \le -(T - s + s) + (T - s)\cosh(\sqrt{M}s) + \frac{1}{\sqrt{M}}\sinh(\sqrt{M}s)$$

$$= -T + (T - s)\cosh(\sqrt{M}s) + \frac{1}{\sqrt{M}}\sinh(\sqrt{M}s) = \psi(s), s \in (0,T].$$

ψ is nondecreasing in $[0, T]$, then

$$\Phi(a,s) \le \psi(T)$$

$$= -T + (T - T)\cosh(T\sqrt{M}) + \frac{1}{\sqrt{M}}\sinh(T\sqrt{M})$$

$$= -T + \frac{1}{\sqrt{M}}\sinh(T\sqrt{M}),$$

but

$$-T + \frac{1}{\sqrt{M}}\sinh(T\sqrt{M}) < T\cosh(T\sqrt{M}) - \frac{1}{\sqrt{M}}\sinh(T\sqrt{M}).$$

In consequence, using (13), we get

$$\int_a^b \int_a^s [p(w)](t) e^{-\sqrt{M}(t+b-2s)} dt\, ds$$

$$\le \left(\max_{[0,T]} w\right) \frac{N}{M}\left(T\cosh(T\sqrt{M}) - \frac{1}{\sqrt{M}}\sinh(T\sqrt{M})\right)$$

$$\leq \left(\max_{[0,T]} w\right),$$

and (10) holds. The conclusion follows from Theorem 2.1.

Now, we present a comparison result in an appropriate may to develop the monotone method. Its proof can be obtained following the proof of Theorem 2.1. The symbol 0 also represents the constant function $0(t) = 0$, $t \in I$.

Theorem 2.2 *Let* $v \in W^{2,1}(I)$, $M > 0$, *and* p *be such that*

$$\begin{cases} -v''(t) + Mv(t) + [p(\max\{v,0\})](t) \geq 0, \;\; a.e. \;\; t \in I, \\ v(0) = v(T), \\ v'(0) \leq v'(T), \end{cases} \tag{14}$$

where (7) holds,

$$p(0) \leq 0 \; a.e. \; on \; I, \tag{15}$$

for all $a < b \in I$ *and* $w \in C(I)$, *we have*

$$\int_a^b \int_s^b [p(\max\{w,0\})](t)\, e^{\sqrt{M}(t+a-2s)} dt\, ds \leq \max_{s\in[0,b]} w(s), \;\; if \;\; \max_{[0,b]} w > 0, \tag{16}$$

and

$$\int_a^b \int_a^s [p(\max\{w,0\})](t)\, e^{-\sqrt{M}(t+b-2s)} dt\, ds \leq \max_{s\in[0,T]} w(s), \;\; if \;\; \max_{[0,T]} w > 0. \tag{17}$$

Then $v \geq 0$ *on* I.

Corollary 6: *Let* $M > 0$, $N \geq 0$, and $\theta : I \to I$, $\theta(t) \leq t$, *a.e.* $t \in I$, *such that the map* p *given by*

$$[p(w)](t) = Nw(\theta(t)), \; t \in I,$$

verifies (7). *If* $v \in W^{2,1}(I)$ *is such that*

$$\begin{cases} -v''(t) + Mv(t) + N\max\{v(\theta(t)),0\} \geq 0, \;\; a.e. \;\; t \in I, \\ v(0) = v(T), \\ v'(0) \leq v'(T), \end{cases}$$

and estimate (11) holds, then $v \geq 0$ *on* I.

Proof: Condition (15) is trivially valid. Let $a < b \in I$ and $w \in C(I)$ with $\max_{[0,b]} w > 0$. Taking into account that $\theta(t) \le t$, a.e. $t \in I$, we get, for $t \in [a, b]$,

$$[p(\max\{w, 0\})](t) = N \max\{w(\theta(t)), 0\}$$

$$= \begin{cases} Nw(\theta(t)), & if \quad w(\theta(t)) \ge 0, \\ 0, & if \quad w(\theta(t)) < 0 \end{cases} \le N \max_{[0,b]} w.$$

So that we can conclude that conditions (16) and (17) hold and, therefore, Theorem 2.2 implies that v ≥ 0 on I.

Corollary 7: *Suppose that $M > 0$ and p are such that (3), (4) and (11) hold. If $v \in W^{2,1}(I)$ verifies (14), then $v \ge 0$ on I.*

Proof: Condition (15) holds, since

$$[p(0)](t) \le ess \sup_{[0,t]}[p(0)] \le N \max_{[0,t]} 0 = 0, \text{ for a.e. } t \in I.$$

To show that (16) and (17) hold, we only have to use estimate (11) and the fact that, for $a < b \in I$ and $w \in C(I)$ with $\max_{[0,b]} w > 0$,

$$[p(\max\{w, 0\})](t) \le \text{ess sup}_{[0,t]} [p(\max\{w, 0\})]$$

$$\le N \max_{[0,t]}(\max\{w, 0\}) \le N \max_{[0,b]} w, \text{ a.e. } t \in [a, b].$$

Corollary 8: *Suppose that $M > 0$ and p is such that there exists $N \ge 0$ with*

$$[p(w)](t) \le N \int_0^t w(r)dr, \text{ a.e. } t \in I,$$

and that (7) and (13) hold. If $v \in W^{2,1}(I)$ verifies conditions in (14), then $v \ge 0$ on I.

Proof: Hypothesis (15) is valid, since

$$[p(0)](t) \le N \int_0^t 0 \, dr = 0, \text{ for a.e. } t \in I.$$

Using the estimate on the constants and that, for $a < b \in I$, $w \in C(I)$ with $\max_{[0,b]} w > 0$, and a.e. $t \in [a, b]$,

$$[p(\max\{w, 0\})](t) \le N \int_0^t \max\{w(r), 0\} dr \le N \max_{[0,b]} w \int_0^t dr = Nt \max_{[0,b]} w,$$

we prove the validity of (16) and (17).

According to the ideas in Theorem 2.2 [3], we can prove the following comparison result.

Theorem 2.3: *Let* $v \in C^2(I)$, $M > 0$, *and* $p : C(I) \to C(I)$ *be such that*

$$\begin{cases} -v''(t) + Mv(t) + [p(v)](t) \geq 0, \quad t \in I, \\ v(0) = v(T), \\ v'(0) \leq v'(T), \end{cases} \tag{18}$$

with

$$p(w) \leq 0 \text{ on } I, \text{ if } w \in C(I),\ w \leq 0 \text{ on } I, \tag{19}$$

and suppose that there exists $N \geq 0$ *such that,*

$$[p(w)](t) \leq N \max_{s\in[0,T]} w(s),\ t \in I, \quad \text{for all } w \in C(I) \text{ with } \max_{[0,T]} w > 0, \tag{20}$$

where

$$\frac{2N}{M}\left(\sinh \frac{T\sqrt{M}}{4}\right)^2 \leq 1. \tag{21}$$

Then $v \geq 0$ *on* I.

Proof: We distinguish two cases:

Case 1: If $\max\limits_{[0,T]} v \leq 0$, then $v \leq 0$ on I and

$$v''(t) \leq Mv(t) + [p(v)](t) \leq 0, \text{ for } t \in I,$$

thus, v' is nonincreasing. Since $v'(0) \leq v'(T)$, then v' is a constant function. This fact joint to $v(0) = v(T)$ implies that $v(t) = k \leq 0$, but then

$$0 \leq Mk + [p(k)](t) \leq Mk,$$

so that $k \geq 0$ and $v \equiv 0$ on I.

Case 2: Suppose that $\max\limits_{[0,T]} v > 0$. Let $z(t) = -v(t)$, $t \in I$, and take $\xi \in [0, T]$ with

$$z(\xi) = \min_{[0,T]} z = \min_{[0,T]}(-v) = -\max_{[0,T]} v < 0.$$

Then, we obtain

$$z''(t) - Mz(t) - Nz(\xi) = -v''(t) + Mv(t) - Nz(\xi)$$

$$= -v''(t) + Mv(t) + N \max_{[0,T]} v \geq -v''(t) + Mv(t) + [p(v)](t) \geq 0, \ t \in I.$$

$$z(0) = -v(0) = -v(T) = z(T), z'(0) = -v'(0) \geq -v'(T) = z'(T).$$

Following the procedure in Theorem 2.2 [3], we can deduce that $z \leq 0$ on I and thus $v \geq 0$ on I.

Remark 2: *This result improves Theorem 2.2 [3], which is restricted to the case*

$$[p(v)](t) = N\,v(\theta(t)),\ \theta : I \to I.$$

It is important to remark that condition (20) is different from

$$[p(w)](t) \leq N \max_{s\in[0,T]} w(s),\ \ t \in I,\ \ \textit{for all}\ \ w \in C(I),$$

and therefore it is valid even for the integral case.

Corollary 9: *Let* $v \in C^2(I)$, $M > 0$, $N \geq 0$ and $\theta : I \to I$ *continuous be such that*

$$\begin{cases} -v''(t) + Mv(t) + Nv(\theta(t)) \geq 0, \ \ t \in I, \\ v(0) = v(T), \\ v'(0) \leq v'(T), \end{cases}$$

and (21) holds. Then $v \geq 0$ on I.

Proof: If $w \in C(I)$, $w \leq 0$ on I,

$$[p(w)](t) = Nw(\theta(t)) \leq 0 \text{ on } I,$$

and for all $w \in C(I)$ with $\max_{[0,T]} w > 0$,

$$[p(w)](t) = Nw(\theta(t)) \leq N \max_{s\in[0,T]} w(s).$$

This last result extends Theorem 2.2 in [3].

Corollary 10: *Let* $v \in C^2(I)$, $M > 0$, *and* $p : C(I) \to C(I)$ *such that (18) holds and suppose that there exists* $N \geq 0$ *with*

$$[p(w)](t) \leq N \max_{[0,t]} w,\ \ t \in I,\ \ w \in C(I),$$

where (21) is valid. Then $v \geq 0$ *on* I.

Proof: For $w \in C(I)$, $w \leq 0$ on I,

$$[p(w)](t) \leq N \max_{[0,t]} w \leq 0, t \in I,$$

and, for all $w \in C(I)$ with $\max\limits_{[0,T]} w > 0$,

$$[p(w)](t) \leq N \max_{s\in[0,t]} w(s) \leq N \max_{s\in[0,T]} w(s), \ \ t \in I.$$

Corollary 11: *Let* $v \in C^2(I)$, $M > 0$, *and* $p : C(I) \to C(I)$ *be such that (18) is verified and assume that there exists* $N \geq 0$ *with*

$$[p(w)](t) \leq N \int_0^t w(s)\, ds, \ \ t \in I, \ \ w \in C(I),$$

where

$$\frac{2NT}{M}\left(\sinh \frac{T\sqrt{M}}{4}\right)^2 \leq 1 \tag{22}$$

holds. Then $v \geq 0$ *on* I.

Proof: *Let* $w \in C(I)$, $w \leq 0$ *on* I, *then*

$$[p(w)](t) \leq N \int_0^t w(s)\, ds \leq 0, \ \ t \in I,$$

and, if $w \in C(I)$ with $\max\limits_{[0,T]} w > 0$, we get

$$[p(w)](t) \leq N \int_0^t w(s) ds \leq Nt \max_{s\in[0,T]} w(s) \leq N\,T \max_{s\in[0,T]} w(s).$$

Note that NT plays the role of the constant N in Theorem 2.3.

Remark 3: *Estimate (22) is weaker than condition (13).*

Using these results, we extend the type of functional dependence (it is allowed the case of equations with advanced arguments) and the estimates on the constants improve previous conditions in the case of $v \in C^2(I)$. Extension to functions in $W^{2,1}(I)$ is obtained following the lines in [3]. In comparison with Theorem 2.2 in [3], Theorem 2.2 has the advantage of dealing with more general functional dependence including maximum and integral type dependence.

The appropriate formulation of the comparison result for application to the development of upper and lower solutions method and monotone iterative technique is the following. Here, 0 also denotes the constant function $0(t) = 0$, $t \in I$.

Theorem 2.4 *Let* $v \in C^2(I)$, $M > 0$, *and* $p : C(I) \to C(I)$ *such that*

$$\begin{cases} -v''(t) + Mv(t) + [p(\max\{v, 0\})](t) \geq 0, \quad t \in I, \\ v(0) = v(T), \\ v'(0) \leq v'(T), \end{cases} \tag{23}$$

$$p(0) \leq 0 \text{ on } I, \tag{24}$$

and suppose that there exists $N \geq 0$ *such that, for all* $w \in C(I)$ with $\max\limits_{[0,T]} w > 0$,

$$[p(\max\{w, 0\})](t) \leq N \max_{s\in[0,T]} w(s), \ \ t \in I, \tag{25}$$

where (21) is valid. Then $v \geq 0$ *on I.*

Proof: If $\max\limits_{[0,T]} v \leq 0$, then $v \leq 0$ on I, and

$$v''(t) \leq Mv(t) + [p(\max\{v, 0\})](t) = Mv(t) + [p(0)](t) \leq 0, \text{ for } t \in I,$$

thus, v' is nonincreasing, but $v'(0) \leq v'(T)$ implies that v' is a constant function. Since $v(0) = v(T)$, then $v(t) = k \leq 0$, but

$$0 \leq Mk + [p(0)](t) \leq Mk,$$

which implies that $k \geq 0$ and $v \equiv 0$ on I.

If $\max\limits_{[0,T]} v > 0$, take $z(t) = -v(t)$, $t \in I$ and $\xi \in [0, T]$ with

$$z(\xi) = \min_{[0,T]} z = \min_{[0,T]}(-v) = -\max_{[0,T]} v < 0,$$

then we obtain

$$z''(t) - Mz(t) - Nz(\xi) \geq -v''(t) + Mv(t) + [p(\max\{v, 0\})](t) \geq 0, \ t \in I.$$

The proof is concluded similarly to Theorem 2.3.

Corollary 12: *Let* $v \in C^2(I)$, $M > 0$, $N \geq 0$, *and* $\theta : I \to I$ *continuous such that*

$$\begin{cases} -v''(t) + Mv(t) + N \max\{v(\theta(t)), 0\} \geq 0, \ t \in I, \\ v(0) = v(T), \\ v'(0) \leq v'(T), \end{cases}$$

and (21) holds. Then $v \geq 0$ *on* I*.*

Proof: Defining $[p(w)](t) = Nw(\theta(t))$, $t \in I$, we check that

$$[p(0)](t) = 0, \ t \in I,$$

and if $w \in C(I)$ with $\max_{[0,T]} w > 0$,

$$\begin{aligned} [p(\max\{w, 0\})](t) &= N \max\{w(\theta(t)), 0\} \\ &= \begin{cases} N\, w(\theta(t)), & \text{if } w(\theta(t)) \geq 0, \\ 0, & \text{if } w(\theta(t)) < 0 \end{cases} \leq N \max_{s \in [0,T]} w(s). \end{aligned}$$

Corollary 13 *Let* $v \in C^2(I)$, $M > 0$, *and* $p : C(I) \to C(I)$ *such that there exists* $N \geq 0$ *with*

$$[p(w)](t) \leq N \max_{[0,t]} w, \ t \in I, \ w \in C(I),$$

and suppose that (23) and (21) hold. Then $v \geq 0$ *on* I*.*

Proof: We check conditions (24) and (25),

$$[p(0)](t) \leq N \max_{[0,t]} 0 = 0, \ t \in I,$$

and, for all $w \in C(I)$ with $\max_{[0,T]} w > 0$,

$$[p(\max\{w, 0\})](t) \leq N \max_{s \in [0,t]} (\max\{w, 0\})(s) \leq N \max_{s \in [0,T]} w(s), \ t \in I.$$

Corollary 14 *Let* $v \in C^2(I)$, $M > 0$, *and* $p : C(I) \to C(I)$ such that there exists $N \geq 0$ *with*

$$[p(w)](t) \leq N \int_0^t w(s)ds, \ t \in I, \ w \in C(I),$$

where (23) and (22) hold. Then $v \geq 0$ *on* I*.*

Proof: It is easy to show that conditions (24) and (25) are valid. Indeed,

$$[p(0)](t) \leq N\int_0^t 0(s)ds = 0,\ \ t \in I,$$

and, if $w \in C(I)$ verifies that $\max\limits_{[0,T]} w > 0,$ then

$$\begin{aligned}[p(\max\{w, 0\})](t) &\leq N\int_0^t \max\{w, 0\}(s)ds \\ &\leq N\left(\max_{s\in[0,T]} w(s)\right)\int_0^t ds \leq NT \max_{s\in[0,T]} w(s),\ \ t \in I.\end{aligned}$$

3. QUASI-LINEAR PROBLEM

Consider the following periodic boundary value problem for a second-order quasi-linear differential equation

$$\begin{cases} -v''(t) + Mv(t) + [p(v)](t) = \sigma(t),\ \ a.e.\ \ t \in I, \\ v(0) = v(T), \\ v'(0) = v'(T), \end{cases} \tag{26}$$

where $M > 0$, p satisfies (7), and $\sigma \in L^1(I)$.

By a solution of (26) we mean a function $u \in W^{2,1}(I)$ satisfying conditions in (26).

Problem (3.2) in [1],

$$\begin{cases} -v''(t) + Mv(t) + Nv(w(t)) = \sigma(t),\ \ t \in I, \\ v(0) = v(T), v'(0) = v'(T), \\ v(t) = v(0), t \in [-r, 0], \end{cases}$$

where $w \in C(I, [-r, T])$, $r > 0$, $t - r \leq w(t) \leq t$, $\forall t \in I$, can be written in terms of problem (26), taking $[p(v)](t) = Nv(\theta(t))$, $t \in I$, for

$$\theta(t) = \max\{w(t), 0\} = \begin{cases} w(t), & if\ \ w(t) \geq 0, \\ 0, & if\ \ w(t) < 0. \end{cases}$$

Definition 1: *A function $\alpha \in W^{2,1}(I)$ is said to be a lower solution to problem (26) if it satisfies*

$$\begin{cases} -\alpha''(t) + M\alpha(t) + [p(\alpha)](t) \le \sigma(t), \ \ a.e. \ \ t \in I, \\ \alpha(0) = \alpha(T), \\ \alpha'(0) \ge \alpha'(T). \end{cases}$$

Analogously, $\beta \in W^{2,1}(I)$ *is an upper solution for (26) if it satisfies the reversed inequalities.*

If $y_1, y_2 \in L^1(I)$, we say that $y_1 \le y_2$ a.e. on I, if

$$y_1(t) \le y_2(t), \text{ for a.e. } t \in I.$$

For $y_1, y_2 \in L^1(I)$ with $y_1 \le y_2$ a.e. on I, we denote

$$[y_1, y_2] = \{u \in L^1(I) : y_1 \le u \le y_2 \text{ a.e. on } I\}.$$

Next, we define a truncation operator that is useful to prove that problem (26) has a unique solution between a lower and an upper solutions.

Given $\alpha, \beta \in L^1(I)$, with $\alpha \le \beta$ a.e. on I, we define $q : L^1(I) \to L^1(I)$ by

$$[q(v)](t) = \max\left\{\alpha(t), \min\{v(t), \beta(t)\}\right\} = \begin{cases} \alpha(t), & \text{if } v(t) < \alpha(t), \\ v(t), & \text{if } \alpha(t) \le v(t) \le \beta(t), \\ \beta(t), & \text{if } v(t) > \beta(t), \end{cases}$$

for $v \in L^1(I)$ and $t \in I$.

Theorem 3.1: *If there exist* $\alpha, \beta \in W^{2,1}(I)$, *respectively, lower and upper solutions to (26) such that* $\alpha \le \beta$ *a.e. on I, and p satisfies (7), conditions (15)-(17) in Theorem 2.2, and*

$$p : \{w \in C(I) : \alpha \le w \le \beta\} \subseteq C(I) \to L^1(I) \text{ is continuous,} \tag{27}$$

$$p(\{w \in C(I) : \alpha \le w \le \beta\}) \text{ is bounded in } L^1(I), \text{ that is, there exists}$$

$$\tau \in L^1(I) \text{ such that } |[pw](t)| \le \tau(t), \ \forall w \in C(I), \ w \in [\alpha, \beta], \tag{28}$$

$$p(\max\{w - \alpha, 0\}) \ge p(q(w)) - p(\alpha), \text{ a.e. on } I, \text{ for } w \in C(I), \tag{29}$$

$$p(\max\{\beta - w, 0\}) \ge p(\beta) - p(q(w)), \text{ a.e. on } I, \text{ for } w \in C(I), \tag{30}$$

$$p(\max\{f - g, 0\}) \ge p(f) - p(g), \text{ a.e. on } I, \text{ for } f, g \in C(I), f, g \in [\alpha, \beta], \tag{31}$$

then problem (26) has a unique solution in $[\alpha, \beta]$.

Proof: Consider the problem

$$\begin{cases} -v''(t)+Mv(t)=\sigma(t)-[p(q(v))](t), \ \ a.e. \ \ t\in I, \\ v(0)=v(T), \\ v'(0)=v'(T), \end{cases} \tag{32}$$

where q has been defined above and is continuous. Note that, since α, β are continuous, $q(w) \in C(I)$, $\forall w \in [\alpha, \beta]$, w continuous. Now, define the operator

$$\mathrm{T} : C(I) \to C(I)$$

$$w \to T(w),$$

$$[T(w)](t)=\int_0^T G(t,s)\{\sigma(s)-[p(q(w))](s)\}\, ds, \quad \text{for } t \in I,$$

where

$$G(t,s)=\frac{1}{2\sqrt{M}(e^{\sqrt{M}T}-1)}\begin{cases} e^{\sqrt{M}(t-s)}+e^{\sqrt{M}(T-t+s)}, 0\le s\le t\le T, \\ e^{\sqrt{M}(s-t)}+e^{\sqrt{M}(T-s+t)}, 0\le t\le s\le T. \end{cases}$$

Note that $\sigma - [p(q(w))] \in L^1(I)$, for all $w \in C(I)$. T is a continuous and completely continuous operator. Hence, by Schauder's Fixed Point Theorem, we obtain the existence of a fixed point for T, that is, problem (32) has a solution $y \in W^{2,1}(I)$. Indeed, let $u \in C(I)$, then

$$\begin{aligned} &\|Tw - Tu\| \\ &= \sup_{t\in I}\left|\int_0^T G(t,s)\{\sigma(s)-[p(q(w))](s)\}\, ds - \int_0^T G(t,s)\{\sigma(s)-[p(q(u))](s)\}\, ds\right| \\ &= \sup_{t\in I}\left|\int_0^T G(t,s)\{[p(q(u))](s)-[p(q(w))](s)\}\, ds\right| \\ &\le \sup_{t\in I}\int_0^T |G(t,s)|\ |[p(q(u))](s)-[p(q(w))](s)|\, ds \\ &\quad \le \gamma \sup_{t\in I}\int_0^T |[p(q(u))](s)-[p(q(w))](s)|\, ds, \end{aligned}$$

where $|G(t, s)| \le \gamma$, $\forall (t, s)$. Let $\varepsilon > 0$. Using hypothesis (27), and the fact that $q(u)$, $q(w) \in [\alpha, \beta]$ are continuous, then there exists $\delta > 0$ such that if $\|q(u) - q(w)\| < \delta$ implies $\int_0^T |p(q(u))-p(q(w))|\, dt < \varepsilon$. But $\|u - w\| < \delta$ implies $\|q(u) - q(w)\| \le \|u - w\| < \delta$, and $\int_0^T |p(q(u))-p(q(w))|\, dt < \varepsilon$. This shows that T is continuous.

Now, let $B \subseteq C(I)$ be a bounded set ($\|u\| \le k$, for all $u \in B$), $T(B) \in C(I)$. Then

$$\| Tu \| = \sup_{t\in I} \left| \int_0^T G(t,s)\{\sigma(s) - [p(q(w))](s)\}\, ds \right|$$

$$\le \sup_{t\in I} \int_0^T |G(t,s)|\, |\sigma(s) - [p(q(w))](s)|\, ds$$

$$\le \gamma \sup_{t\in I} \int_0^T |\sigma(s) - [p(q(w))](s)|\, ds$$

$$\le \gamma \left(\sup_{t\in I} \int_0^T |\sigma(s)|\, ds + \int_0^T |[p(q(w))](s)|\, ds \right) \le \gamma(\| \sigma \|_1 + \| p(q(w)) \|_1) \le \upsilon.$$

Moreover, for $u \in B$, $Tu \in C(I)$ and $(Tu)'$ exists and is continuous on I. It is not dificult to prove that $|(Tu)'(t)| \le \kappa$, for every $u \in B$, with κ independent of u. Then $T(B)$ is bounded and equicontinuous, therefore it is relatively compact and T is compact.

Theorem 2.2 allows to show that $\alpha \le y \le \beta$ a.e. on I and, in consequence, y is a solution to (26). To check this fact, set $m = y - \alpha \in W^{2,1}(I)$. By (29), we obtain, for a.e. $t \in I$,

$$\begin{aligned}
&-m''(t) + Mm(t) + [p(\max\{m, 0\})](t) \\
&\quad = -y''(t) + \alpha''(t) + My(t) - M\alpha(t) + [p(\max\{y - \alpha, 0\})](t) \\
&\quad \ge \sigma(t) - [p(q(y))](t) - \sigma(t) + [p(\alpha)](t) + [p(\max\{y - \alpha, 0\})](t) \\
&\quad = [p(\max\{y - \alpha, 0\})](t) - [p(q(y))](t) + [p(\alpha)](t) \ge 0,
\end{aligned}$$

and

$$m(0) = y(0) - \alpha(0) = y(T) - \alpha(T) = m(T),$$
$$m'(0) = y'(0) - \alpha'(0) \le y'(T) - \alpha'(T) = m'(T).$$

Since conditions in Theorem 2.2 are valid, we obtain m $= y - \alpha \ge 0$ on I, that is, $y \ge \alpha$ on I.

A similar reasoning and condition (30) provide that $y \le \beta$ a.e. on I, since $u = \beta - y \in W^{2,1}(I)$ verifies that

$$-u''(t) + Mu(t) + [p(\max\{u, 0\})](t) \geq 0, \text{ for a.e. } t \in I,$$

$$u(0) = u(T),\ u'(0) \leq u'(T).$$

The uniqueness of solution to (26) in $[\alpha, \beta]$ follows from the application of Theorem 2.2 to the functions $v_1 = y_1 - y_2$, $v_2 = y_2 - y_1 \in W^{2,1}(I)$, where $y_1, y_2 \in [\alpha, \beta]$ are solutions to (26). Condition (31) provides that v_1 and v_2 are under the hypotheses of the comparison result Theorem 2.2:

$$-v_1''(t) + Mv_1(t) + [p(\max\{v_1, 0\})](t)$$

$$= -y_1''(t) + My_1(t) + y_2''(t) - My_2(t) + [p(\max\{y_1 - y_2, 0\})](t)$$

$$= \sigma(t) - [p(y_1)](t) - \sigma(t) + [p(y_2)](t) + [p(\max\{y_1 - y_2, 0\})](t) \geq 0, \text{ for a.e. } t \in I,$$

$$v_1(0) = y_1(0) - y_2(0) = y_1(T) - y_2(T) = v_1(T),$$

$$v_1'(0) = y_1'(0) - y_2'(0) = y_1'(T) - y_2'(T) = v_1'(T)$$

and

$$-v_2''(t) + Mv_2(t) + [p(\max\{v_2, 0\})](t)$$

$$= -y_2''(t) + My_2(t) + y_1''(t) - My_1(t) + [p(\max\{y_2 - y_1, 0\})](t)$$

$$= \sigma(t) - [p(y_2)](t) - \sigma(t) + [p(y_1)](t) + [p(\max\{y_2 - y_1, 0\})](\mathrm{t}) \geq 0, \text{ for a.e. } t \in I,$$

$$v_2(0) = y_2(0) - y_1(0) = y_2(T) - y_1(T) = v_2(T),$$

$$v_2'(0) = y_2'(0) - y_1'(0) = y_2'(T) - y_1'(T) = v_2'(T).$$

By Theorem 2.2, $y_1 = y_2$ on I.

Remark 4: *Theorem 3.1 extends Theorem 3.2 in [1], because of the sharper estimate on the constants and the type of problems considered. Indeed, they are not restricted to functional dependence given by a delay function. The importance of this result lies in the fact that we obtain existence and uniqueness of solution for nonlinear (quasi-linear) problems given by functional operators not necessarily Lipschitzian, so that we can consider functionals p different from maximum, integral or delay type.*

Theorem 3.2: *The conclusion of last theorem, Theorem 3.1, is valid if we replace hypotheses (15) - (17) of Theorem 2.2, by conditions (24), (25) and (21) in Theorem 2.4.*

Lemma 1: *The following assertions are true:*

$$\max\{w - \alpha, 0\} \geq q(w) - \alpha \textit{ on } I, \textit{ for } w \in C(I),$$

and

$$\max\{\beta - w, 0\} \geq \beta - q(w) \textit{ on } I, \textit{ for } w \in C(I).$$

Proof: If $w \in C(I)$ and $t \in I$,

$$\max\{w-\alpha,0\}(t) = \begin{cases} 0, & \textit{if } w(t) \leq \alpha(t), \\ w(t)-\alpha(t), & \textit{if } w(t) > \alpha(t) \end{cases}$$

$$\geq \begin{cases} \alpha(t)-\alpha(t), & \textit{if } w(t) \leq \alpha(t), \\ q(w)(t)-\alpha(t), & \textit{if } w(t) > \alpha(t) \end{cases} = (q(w)-\alpha)(t).$$

The second assertion follows similarly.

Theorem 3.3: *Conditions (29)–(31) hold if*

$$p \textit{ is nondecreasing,}$$

and

$$p(f - g) \geq p(f) - p(g), \textit{ a.e. on } I, \textit{ for } f, g \in C(I), f, g \in [\alpha, \beta].$$

Proof: Let $w \in C(I)$, then we can prove that

$$p(\max\{w - \alpha, 0\}) \geq p(q(w) - \alpha) \geq p(q(w)) - p(\alpha) \text{ a.e. on } I,$$

$$p(\max\{\beta - w, 0\}) \geq p(\beta - q(w)) \geq p(\beta) - p(q(w)) \text{ a.e. on I},$$

and, for $f, g \in C(I), f, g \in [\alpha, \beta]$,

$$p(\max\{f - g, 0\}) \geq p(f - g) \geq p(f) - p(g) \text{ a.e. on } I.$$

Corollary 15: *Suppose that* $\alpha \leq \beta \in W^{2,1}(I)$ are, respectively, lower and upper solutions to

$$\begin{cases} -v''(t) + Mv(t) + Nv(\theta(t)) = \sigma(t), \quad a.e.\, t \in I, \\ v(0) = v(T), \\ v'(0) = v'(T), \end{cases} \tag{33}$$

where $M > 0$, $N \geq 0$, $\sigma \in L^1(I)$, $\theta : I \to I$, $\theta(t) \leq t$, a.e. $t \in I$, the functional p defined by $[p(w)](t) = N\,w(\theta(t))$, $t \in I$, *verifies (7), and estimate (11) holds.Then, there exists exactly one solution to (33) in* $[\alpha, \beta]$. *We can avoid hypothesis* $\theta(t) \leq t$ *a.e. on I, even replacing (11) by (21).*

Proof: The hypotheses of Theorem 3.1 (resp., 3.2) hold, since (27), (28) are true:

$$\int_0^T |Nw(\theta(t)) - Nw_0(\theta(t))|\, dt \le N\int_0^T \|w - w_0\|\, dt = NT\|w - w_0\|,$$

$$\int_0^T |Nw(\theta(t))|\, dt \le NkT, \text{ for } w \text{ continuous, } w \in [\alpha, \beta],$$

p is nondecreasing and, for $f, g \in C(I), f, g \in [\alpha, \beta]$, and a.e. $t \in I$,

$$[p(f - g)](t) = Nf(\theta(t)) - Ng(\theta(t)) = [p(f)](t) - [p(g)](t).$$

Corollary 16: If $\alpha \le \beta \in W^{2,1}(I)$ *are, respectively, lower and upper solutions to*

$$\begin{cases} -v''(t) + Mv(t) + N\max\limits_{[0,t]} v = \sigma(t), \quad a.e.\ t \in I, \\ v(0) = v(T), \\ v'(0) = v'(T), \end{cases} \tag{34}$$

where $M > 0, N \ge 0, \sigma \in L^1(I)$, *and estimate (21) holds, then there exists exactly one solution to (34) in* $[\alpha, \beta]$.

Proof: The operator p given by $[p(w)](t) = N\max\limits_{[0,t]} w$ is nondecreasing and, for f, $g \in C(I)$, a.e. $t \in I$,

$$[p(f-g)](t) = N\max_{[0,t]}(f-g) \ge N\left(\max_{[0,t]} f - \max_{[0,t]} g\right) = [p(f)](t) - [p(g)](t).$$

Moreover, if $w \in C(I)$, $[pw](t) = N\max_{[0,t]} w$ is continuous on I, then $pw \in L^1(I)$,

$$\int_0^T |[pw](t) - [pw_0](t)|\, dt = N\int_0^T \left|\max_{[0,t]} w - \max_{[0,t]} w_0\right| dt$$

$$\le N\int_0^T \max_{[0,t]} |w(s) - w_0(s)|\, ds \le NT\|w - w_0\|,$$

and p is continuous in $\{w \in C(I) : \alpha \le w \le \beta\}$,

$$\int_0^T |[pw](t)|\, dt = N\int_0^T \left|\max_{[0,t]} w(s)\right| dt \le Nk\int_0^T dt = NkT,$$

for $w \in C(I)$, $w \in [\alpha, \beta]$, and $p(\{w \in C(I) : \alpha \le w \le \beta\})$ is bounded in $L^1(I)$.

Corollary 17 *If* $\alpha \le \beta \in W^{2,1}(I)$ *are, respectively, lower and upper solutions to problem*

$$\begin{cases} -v''(t) + Mv(t) + N\int_0^t v(s)ds = \sigma(t), \quad a.e.\ t \in I, \\ v(0) = v(T), \\ v'(0) = v'(T), \end{cases} \tag{35}$$

where $M > 0$, $N \ge 0$, $\sigma \in L^1(I)$, *and estimate (22) holds, then there exists exactly one solution to (35) in* $[\alpha, \beta]$.

Proof: We check that p given by $[p(w)](t) = N\int_0^t w(s)ds$ is nondecreasing and for every $f, g \in C(I)$, a.e. $t \in I$,

$$\begin{aligned}[p(f-g)](t) &= N\int_0^t (f-g)(s)ds \\ &= N\int_0^t f(s)ds - N\int_0^t g(s)ds = [p(f)](t) - [p(g)](t).\end{aligned}$$

Note that for $w \in C(I)$, $pw \in C(I)$, then $pw \in L^1(I)$ and (7) holds.

$$\begin{aligned}\int_0^T |[pw](t) - [pub](t)|\, dt &= N\int_0^T \left| \int_0^t w(s)ds - \int_0^t w_0(s)ds \right| dt \\ &\le N\int_0^T \int_0^t |w(s) - w_0(s)|\, ds\, dt \le N\int_0^T t\, dt\, \| w - w_0 \| = N\frac{T^2}{2} \| w - w_0 \|,\end{aligned}$$

and, for $w \in C(I)$, $w \in [\alpha, \beta]$,

$$\int_0^T |[pw](t)|\, dt = N\int_0^T \left| \int_0^t w(s)ds \right| dt \le N\int_0^T \int_0^t |w(s)|\, ds\, dt \le Nk\frac{T^2}{2},$$

and (27), (28) hold.

4. MONOTONE METHOD

Now we give a result on the existence of extremal solutions for a general second-order functional differential equation with periodic boundary value conditions in presence of a couple of well-ordered lower and upper solutions. Consider

$$\begin{cases} -v''(t) = f(t, v(t), [p(v)](t)), \ \ a.e.\, t \in I, \\ v(0) = v(T), \\ v'(0) = v'(T), \end{cases} \tag{36}$$

where $I = [0, T]$, $T > 0$, p satisfies (7), and $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$.

Definition 2: *A function $\alpha \in W^{2,1}(I)$ is said to be a lower solution to (36) if*

$$\begin{cases} -\alpha''(t) \le f(t, \alpha(t), [p(\alpha)](t)), \ \ a.e.\, t \in I, \\ \alpha(0) = \alpha(T), \\ \alpha'(0) \ge \alpha'(T), \end{cases} \tag{37}$$

An upper solution $\beta \in W^{2,1}(I)$ is defined similarly, reversing the inequalities above.

Definition 3: *We say that $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function if $f(\cdot, x, y)$ is Lebesgue-measurable on I, for all $(x, y) \in \mathbb{R}^2$, $f(t, u, v)$* continuous *in $(u, v) \in \mathbb{R}^2$, for a.e. $t \in I$ and for all $R > 0$*, there exists $\phi \in L^1(I)$ with

$$|f(t, u(t), v(t))| \le \phi(t), \text{ for a.e. } t \in I,\ u, v \in L^1(I) \text{ with } \|u\|_1, \|v\|_1 \le R.$$

Theorem 4.1: *Suppose that $\alpha, \beta \in W^{2,1}(I)$ are, respectively, lower and upper solutions for (36) with $\alpha \le \beta$ a.e. on I, f is a L^1-Carathéodory function and there exists $M > 0$ such that*

$$\begin{aligned} &f(t, x(t), [p(x)](t)) - f(t, y(t), [p(y)](t)) \\ &\qquad \ge -M(x(t) - y(t)) - ([p(x)](t) - [p(y)](t)), \\ &\text{for a.e. } t \in I,\ x, y \in L^1(I), \text{ such that } \alpha \le y \le x \le \beta \text{ a.e. on } I, \end{aligned} \tag{38}$$

where p satisfies (7), (15)–(17) (or (24), (25) and (21) instead) and (27)-(31). Moreover, assume that if $\{f_n\}$ is a sequence of continuous functions which is monotone, $\alpha \le f_n \le \beta$, $\forall n$ and $\{f_n\} \to f$ at every point of I, then $[pf_n](t) \to [pf](t)$, for almost every point $t \in [0, T]$.

Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, which converge monotonically to the extremal solutions of (36) in $[\alpha, \beta]$.

Remark 5: *In the continuous case, the hypothesis about sequences is:*

$$f_n \to f \text{ uniformly} \Rightarrow p(f_n) \to p(f) \text{ uniformly}.$$

Proof: For each $\eta \in [\alpha, \beta]$ continuous, we consider problem

$$\begin{cases} -v''(t)+Mv(t)+[p(v)](t)=\sigma_\eta(t), & a.e.\, t\in I,\\ v(0)=v(T),\\ v'(0)=v'(T), \end{cases} \tag{39}$$

where

$$\sigma_\eta(t) = f(t, \eta(t), [p(\eta)](t)) + M\eta(t) + [p(\eta)](t),\ t \in I$$

is a function in $L^1(I)$, using that f is L^1-Carathéodory and (7),

$$|\sigma_\eta(t)| \le |f(t, \eta(t), [p(\eta)](t))| + M\,|\eta(t)| + |[p(\eta)](t)| \le \phi(t) + M\,|\eta(t)| + |[p(\eta)](t)|,\ t \in I.$$

Since $\eta \in [\alpha, \beta]$, using the hypothesis on f and the definition of lower and upper solutions, we obtain that

$$\begin{aligned} &-\alpha''(t) + M\alpha(t) + [p(\alpha)](t)\\ &\quad\le f(t, \alpha(t), [p(\alpha)](t)) + M\alpha(t) + [p(\alpha)](t)\\ &\quad\le f(t, \eta(t), [p(\eta)](t)) + M\eta(t) + [p(\eta)](t) = \sigma_\eta(t), \text{ a.e. } t \in I, \end{aligned}$$

and

$$\begin{aligned} &-\beta''(t) + M\beta(t) + [p(\beta)](t)\\ &\quad\ge f(t, \beta(t), [p(\beta)](t)) + M\beta(t) + [p(\beta)](t)\\ &\quad\ge f(t, \eta(t), [p(\eta)](t)) + M\eta(t) + [p(\eta)](t) = \sigma_\eta(t), \text{ a.e. } t \in I, \end{aligned}$$

so that α and β are, respectively, lower and upper solutions to problem (39). By Theorem 3.1 (resp., Theorem 3.2), we can affirm the existence of a unique solution to (39) in $[\alpha, \beta]$. This allows to define an operator $A : [\alpha, \beta] \to [\alpha, \beta]$ which assigns to each $\eta \in [\alpha, \beta]$ the unique solution to (39) in $[\alpha, \beta]$. This operator A is nondecreasing: if $\alpha \le \eta_1 \le \eta_2 \le \beta$ a.e. on I, using (31) and (38), we achieve that $m = A\eta_2 - A\eta_1$ verifies the hypotheses of Theorem 2.2 (resp., Theorem 2.4):

$$\begin{aligned} &-m''(t) + Mm(t) + [p(\max\{m, 0\})](t)\\ &\quad= -(A\eta_2)''(t) + MA\eta_2(t) + (A\eta_1)''(t) - MA\eta_1(t) + p[\max\{A\eta_2 - A\eta_1, 0\}](t)\\ &\quad= f(t, \eta_2(t), [p(\eta_2)](t)) + M\eta_2(t) + [p(\eta_2)](t) - [p(A\eta_2)](t)\\ &\quad- f(t, \eta_1(t), [p(\eta_1)](t)) - M\eta_1(t) - [p(\eta_1)](t) + [p(A\eta_1)](t)\\ &\quad+ p[\max\{A\eta_2 - A\eta_1, 0\}](t)\\ &\quad\ge p[\max\{A\eta_2 - A\eta_1, 0\}](t) - [p(A\eta_2)](t) + [p(A\eta_1)](t) \ge 0, \text{ a.e. } t \in I, \end{aligned}$$

$m(0) = m(T)$ and $m'(0) = m'(T)$. Applying the corresponding comparison result (Theorem 2.2 or 2.4), $A\eta_1 \le A\eta_2$ on I. Then we can define sequences $\{\alpha_n\}$, $\{\beta_n\}$ by $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_{n+1} = A\alpha_n$ and $\beta_{n+1} = A\beta_n$, for $n \ge 0$, so that $\{\alpha_n\}$ is nondecreasing, $\{\beta_n\}$ is nonincreasing, and

$$\alpha = \alpha_0 \le \alpha_1 \le \ldots \le \alpha_n \le \beta_n \le \ldots \le \beta_1 \le \beta_0 = \beta \text{ a.e. on } I.$$

$\{\alpha_n\}$ is pointwise convergent to ρ. Besides,

$$\alpha_n(t) = \int_0^T G(t,s)\{f(s,\alpha_{n-1}(s),[p\alpha_{n-1}](s)) + M\alpha_{n-1}(s) + [p\alpha_{n-1}](s) - [p\alpha_n](s)\}ds, \; t \in I.$$

For each s fixed, the function

$$\{f(s, \alpha_{n-1}(s), [p\alpha_{n-1}](s)) + M\alpha_{n-1}(s) + [p\alpha_{n-1}](s) - [p\alpha_n](s)\}$$

tends to

$$\{f(s, \rho(s), [p\rho](s)) + M\rho(s) + [p\rho](s) - [p\rho](s)\} = \{f(s, \rho(s), [p\rho](s)) + M\rho(s)\}$$

as $n \to +\infty$. Moreover,

$$\begin{aligned}|G(t, s)\{f(s, \alpha_{n-1}(s), [p\alpha_{n-1}](s)) + M\alpha_{n-1}(s) + [p\alpha_{n-1}](s) - [p\alpha_n](s)\}| \\ \le \gamma\,(\phi(s) + M\Lambda + 2\tau\,(t)).\end{aligned}$$

By the dominated convergence theorem,

$$\rho(t) = \int_0^T G(t,s)\{f(s,\rho(s),[p\rho](s)) + M\rho(s)\}ds, \; t \in I,$$

and ρ is a solution to (36). We can demonstrate that $\{\alpha_n\} \uparrow \rho$ on I and that $\{\beta_n\} \downarrow \tilde{r}$ on I, where ρ, $\tilde{r} \in [\alpha, \beta]$ are the extremal solutions of (36) in $[\alpha, \beta]$, that is, if u is a solution to (36) between α and β, by induction, we show that

$$\alpha_n(t) \le u(t) \le \beta_n(t) \text{ a.e. on } I, \text{ for } n = 0, 1, \ldots$$

and the continuity of these functions imply $\rho \le u \le \tilde{r}$ a.e. on I.

Remark 6: *This result extends Theorem 3.4 in [1]. Note that in the case where* $\alpha, \beta \in C^2(I)$ *and f continuous satisfying the rest of the hypotheses, the convergence of the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *is uniform on I. Our formulation is not restricted to the case where the functional dependence is of the type* $[p(w)](t) = N\,w(\theta(t))$, *for* $t \in I$ *and* $N \ge 0$, *as it was considered in [1] and [3]. Compare this result with Theorem 3.2 [3].*

Corollary 18: *Suppose that there exist* $\alpha, \beta \in W^{2,1}(I)$, $\alpha \le \beta$ a.e. on I, respectively, lower and upper solutions of

$$\begin{cases} -v''(t) = f(t, v(t), Nv(\theta(t))), \ \ a.e.\, t \in I, \\ v(0) = v(T), \\ v'(0) = v'(T), \end{cases} \tag{40}$$

where $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *is a* L^1*-Carathéodory function,* $\theta : I \to I$, $N \geq 0$ *and the functional defined by* $[p(w)](t) = N\, w(\theta(t))$, $t \in I$, is such that (7) holds. *Moreover, assume that there exists* $M > 0$ *such that*

$$f(t, x(t), N\, x(\theta(t))) - f(t, y(t), N\, y(\theta(t)))$$
$$\geq -M(x(t) - y(t)) - N(x(\theta(t)) - y(\theta(t))),$$
$$\text{for a.e. } t \in I,\ x, y \in L^1(I),\ \alpha \leq y \leq x \leq \beta \text{ a.e. on } I, \tag{41}$$

and that estimate (21) holds. Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that $\alpha_0 = \alpha$, $\beta_0 = \beta$, which converge monotonically to the extremal solutions of (40) in $[\alpha, \beta]$.

Remark 7: *The expression of problem (40) in terms of problem (1.2) in [1], [3] leads to a new equation*

$$-v''(t) = f(t, v(t), N\, v(\theta(t))) = g(t, v(t), v(\theta(t))),\ a.e.\ t \in I,$$

where $g(t, x, y) = f(t, x, N\, y)$, *then condition (41) can be written as*

$$g(t, x(t), x(\theta(t))) - g(t, y(t), y(\theta(t)))$$
$$\geq - M(x(t) - y(t)) - N(x(\theta(t)) - y(\theta(t))),$$
$$\text{for a.e. } t \in I,\ x, y \in L^1(I),\ \alpha \leq y \leq x \leq \beta \text{ a.e. on } I,$$

and, therefore, it coincides with condition (B_1) *in Theorem 4.2 [1] and Theorem 3.2 [3].*

Corollary 19: *Suppose that* α, $\beta \in W^{2,1}(I)$, $\alpha \leq \beta$ *a.e. on I are, respectively, lower and upper solutions for*

$$\begin{cases} -v''(t) = f\left(t, v(t), N \max\limits_{s \in [0,t]} v(s)\right), \ \ a.e.\, t \in I, \\ v(0) = v(T), \\ v'(0) = v'(T), \end{cases} \tag{42}$$

where $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, $N \geq 0$, there exists $M > 0$ *such that*

$$f\left(t, x(t), N \max_{[0,t]} x\right) - f\left(t, y(t), N \max_{[0,t]} y\right)$$

$$\geq -M(x(t)-y(t)) - N\left(\max_{[0,t]} x - \max_{[0,t]} y\right),$$

$$\text{for a.e. } t \in I,\ x, y \in L^1(I),\ \alpha \leq y \leq x \leq \beta \text{ a.e. on } I, \tag{43}$$

and estimate (21) is valid. Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$, *such that* $\alpha_0 = \alpha$, $\beta_0 = \beta$, *which converge monotonically to the extremal solutions of (42) in* $[\alpha, \beta]$.

Corollary 20: *Suppose that* $\alpha, \beta \in W^{2,1}(I)$, $\alpha \leq \beta$ *a.e. on I are, respectively, lower and upper solutions for*

$$\begin{cases} -v''(t) = f\left(t, v(t), N\int_0^t v(s)\,ds\right), \text{a.e. } t \in I, \\ v(0) = v(T), \\ v'(0) = v'(T), \end{cases} \tag{44}$$

where $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-*Carathéodory function,* $N \geq 0$, *there exists* $M > 0$ *with*

$$f\left(t, x(t), N\int_0^t x(s)ds\right) - f\left(t, y(t), N\int_0^t y(s)ds\right)$$

$$\geq -M(x(t)-y(t)) - N\left(\int_0^t x(s)ds - \int_0^t y(s)ds\right),$$

$$\text{for a.e. } t \in I,\ x, y \in L^1(I),\ \alpha \leq y \leq x \leq \beta \text{ a.e. on } I, \tag{45}$$

and estimate (22) holds. Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$, *such that* $\alpha_0 = \alpha$, $\beta_0 = \beta$, *converging monotonically to the extremal solutions of (44) in* $[\alpha, \beta]$.

An analogous study can be made for problem

$$\begin{cases} -v''(t) = f(t, v(t), v_t), \text{a.e. } t \in I, \\ v(t) = v(0) = v(T), t \in [-r, 0], \\ v'(0) = v'(T), \end{cases}$$

where $I = [0, T]$, $T > 0$, $r > 0$, $f : [0, T] \times \mathbb{R} \times C([-r, 0]) \to \mathbb{R}$ and

$$v_t(s) = v(t + s),\ s \in [-r, 0],\ t \in I.$$

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REFERENCES

[1] D. Jiang, J. Wei, Monotone method for first-and second-order periodic boundary value problems and periodic solutions of functional differential equations, Nonlinear Analysis **50** (2002) 885–898.

[2] X. Xu, J.J. Nieto, D. Jiang, Differential inequalities for functional perturbations of second order differential equations, Arch. Inequal. Appl. 1 (2003), no. 3-4, 351–359.

[3] D. Jiang, J.J. Nieto, W. Zuo, On monotone method for first and second order periodic boundary value problems and periodic solutions of functional differential equations, J. Math. Anal. Appl. **289** (2004) 691–699.

[4] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman, Boston, 1985.

[5] V. Lakshmikantham, S.G. Leela, Differential and Integral Inequalities: Theory and Applications, Volumes I and II, Academic Press, New York, 1969.

[6] E. Liz, J.J. Nieto, Periodic boundary value problems for a class of functional differential equations, *Journal of Mathematical Analysis and Applications* **200** (1996) 680–686.

[7] E. Liz, J.J. Nieto, An abstract monotone iterative method and applications, Dynamic Systems and Applications **7** (1998) 365–375.

[8] J.J. Nieto, R. Rodríguez-López, Existence and approximation of solutions for nonlinear functional differential equations with periodic boundary value conditions, Computers and Mathematics with Applications **40** (2000) 433–442.

[9] J.J. Nieto, Differential inequalities for functional perturbations of first-order ordinary differential equations, Appl. Math. Letters **15** (2002) 173–179.

[10] J.J. Nieto, R. Rodríguez-López, Remarks on periodic boundary value problems for functional differential equations, *Journal of Computational and Applied Mathematics* **158** (2003) 339–353.

[11] H.K. Xu, E. Liz, Boundary value problems for functional differential equations, Nonlinear Analysis; Theory, Methods and Applications **41** (2000) 971–988.

Juan J. Nieto & Rosana Rodríguez-López
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Santiago de Compostela, 15782, Spain
E-mail: amnieto@usc.es; amrosana@usc.es