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PERIODIC BOUNDARY VALUE PROBLEMS OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS IN BANACH ALGEBRAS

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ABSTRACT: In this paper an existence theorem for the periodic boundary value problems of first order differential equations is proved in Banach algebras under the mixed generalized Lipschitz and Caratheodory conditions. The existence of extremal positive solutions is also proved under certain monotonicity conditions.

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1. INTRODUCTION

First order ordinary differential equations (ODE) with periodic boundary value conditions are considered in many works. See Bernfeld and Lakshmikantham [1], Ladde et. al. [16], Omari and Zanolin [19] and the references therein. The study of periodic boundary value problems of nonlinear first order differential equations with discontinuous nonlinearity has been exploited in the works of Heikkilä and Lakshmikantham [15]. But the study of periodic boundary value problems of ordinary differential equations in Banach algebras involving Caratheodory as well as discontinuous nonlinearity has not been made so far in the literature. The study of initial value problems of nonlinear differential equations in Banach algebras is initiated in the works of Dhage [2], Dhage and O'Regan [10] and discussed the existence theory for first order differential equations. The study of such equations has been further exploited in the works of Dhage [3, 4, 5, 7] and Dhage et. al. [11] for various aspects of the solutions. In this paper, we deal with the periodic boundary value problems of nonlinear first order differential equations in Banach algebras and discuss the existence as well as existence results for extremal solutions under mixed Lipschitz, Caratheodory and monotonic conditions. The main tools used in the study are the hybrid fixed point theorems of Dhage [3, 4, 6, 7]. We claim that the nonlinear differential equation as well as the existence results of this paper are new to the literature on the theory of nonlinear ordinary differential equations.

Let $\mathbb{R}$ denote the real line. Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, consider the periodic boundary value problems (in short PBVP) of first order ordinary differential equations

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] &= g(t,x(t)| \ \ a.e. \ \ t \in J \\ x(0) &= x(T), \end{aligned}\right\} \tag{1.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R} - \{0\}$ and $g : I \times \mathbb{R} \to \mathbb{R}$.

By a solution of PBVP (1.1) we mean a function $x \in AC(J,\mathbb{R})$ that satisfies

(i) the function $t \to \left(\dfrac{x(t)}{f(t,x(t))}\right)$ is absolutely continuous on J, and

(ii) x satisfies the equations in (1.1),

where $AC(J,\mathbb{R})$ is the space of continuous functions whose first derivative exists and is absolutely continuous real-valued functions on J.

Our method of study is to convert the PBVP (1.1) into an equivalent integral equation and apply the fixed point theorems of Dhage [3, 4, 6, 7] under suitable conditions on the nonlinearities f and g involved it. In the following section 2, we give some preliminaries needed in the sequel.

2. AUXILIARY RESULTS

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \to X$ is called D-Lipschitz if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \le \psi(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r \, (\alpha > 0)$, A is called a Lipschitz with a Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with a contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called a nonlinear D-contraction on X. Sometimes we call the function ψ a D-function for convenience.

An operator $T : X \to X$ is called **compact** if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. Similarly $T : X \to X$ is called totally bounded if T maps a bounded subset of X into the relatively **compact** subset of X. Finally $T : X \to X$ is called **completely continuous** operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true. The nonlinear alternative of Schaefer type recently proved by Dhage [7] is embodied in the following theorem. Also see Dhage and Ntouyas [8], Dhage et. al. [9] and the references therein.

Theorem 2.1. (Dhage[7]). *Let* $B_r(0)$ and $\overline{\mathcal{B}_r(0)}$ *be respectively open and closed balls in a Banach algebra X centered at origin* 0 *and of radius r. Let* $A, B : \overline{\mathcal{B}_r(0)} \to X$ *be two operators satisfying*

(a) *A is Lipschitz with a Lipschitz constant* α,

(*b*) *B is compact and continuous, and*

(c) $\alpha M < 1$, *where* $M = \| B(\overline{\mathcal{B}_r(0)}) \| := \sup\{\| Bx \| : x \in \overline{(\mathcal{B}_r(0)}\}$.

Then either

(i) *the equation* $\lambda[Ax\, Bx] = x$ *has a solution for* $\lambda = 1$, *or*

(ii) *there exists an* $u \in X$ such that $\|u\| = r$ satisfying $\lambda[A_u B_x] = u$ for some $0 < \lambda < 1$.

A non-empty closed set *K* in a Banach algebra *X* is called a **cone** if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\in \mathbb{R}$, $\lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of *X*. A cone *K* is called to be **positive** if (iv) $K \text{ o } K \subseteq K$, where "o" is a multiplication composition in *X*. We introduce an order relation $\leq$ in *X* as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone *K* is called to be **normal** if the norm $\| \cdot \|$ is semi-monotone increasing on *K*, that is, there is a constant $N > 0$ such that $\| x \| \leq N \| y \|$ for all $x, y \in K$ with $x \leq y$. It is known that if the cone *K* is normal in *X*, then every order-bounded set in *X* is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantham [14].

Lemma 2.1 (Dhage [4]). *Let K be a positive cone in a real Banach algebra X and let* $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. *Then* $u_1 u_2 \leq v_1 v_2$.

For any $a, b \in X$, $a \leq b$, the order interval [a, *b*] is a set in *X* given by

$$[\mathrm{a}, b] = \{x \in X : \mathrm{a} \leq x \leq b\}.$$

Definition 2.1: *A mapping* $T : [a, b] \to X$ *is said to be* **nondecreasing** *or* **monotone increasing** *if* $x \leq y$ *implies* $Tx \leq Ty$ for all $x, y \in [a, b]$.

We use the following fixed point theorems of Dhage [3, 4, 6] for proving the existence of extremal solutions for the BVP (1.1) under certain monotonicity conditions.

Theorem 2.2: (Dhage [3]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A,B : [a, b] \to K$ are two operators such that*

(a) *A is Lipschitz with a Lipschitz constant α,*

(b) *B is completely continuous,*

(c) *$Ax\,Bx \in [a, b]$ for each $x \in [a, b]$, and*

(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has a least and a greatest positive solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Theorem 2.3: (Dhage [6]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A,B : [a, b] \to K$ are two operators such that*

(a) *A is completely continuous,*

(b) *B is totally bounded,*

(c) *$Ax\,By \in [a, b]$ for each $x \in [a, b]$, and*

(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has a least and a greatest positive solution in $[a, b]$.

Theorem 2.4. (Dhage [6]). *Let K be a cone in a Banach algebra X and let $a, b \in X$.* Suppose that *$A,B : [a, b] \to K$ are two operators such that*

(a) *A is Lipschitz with a Lipschitz constant α,*

(b) *B is totally bounded,*

(c) *$Ax\,By \in [a, b]$ for each $x \in [a, b]$, and*

(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has least and a greatest positive solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Remark 2.1: *Note that hypothesis (c) of Theorems 2.2, 2.3, and 2.4 holds if the operators A and B are positive, monotone increasing and there exist elements a and b in X such that $a \le Aa\,Ba$ and $Ab\,Bb \le b$.*

In the following sections we prove the main existence results of this paper.

3. EXISTENCE THEORY

Let $B(J, \mathbb{R})$ denote the space of bounded real-valued functions on J. Let $C(J, \mathbb{R})$, denote the space of all continuous real-valued functions on J. Define a norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $C(J, \mathbb{R})$ by

$$\| x \| = \sup_{t\in J} | x(t) | \text{ and } (x, y)(t) = x(t)\, y(t) \text{ for } t \in J.$$

Clearly $C(J,\mathbb{R})$ becomes a Banach algebra with respect to above norm and multiplication. By $L^1(J,\mathbb{R})$ we denote the set of Lebesgue integrable functions on J and the norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ is defined by

$$\| x \|_{L^1} = \int_0^T | x(t) | \, ds.$$

The following useful lemma is obvious and the details may be found in Nieto [18].

Lemma 3.1. *For any* $k \in L^1(J,\mathbb{R}^+)$ *and* $\sigma \in L^1(J,\mathbb{R})$, x *is a solution to the differential equation*

$$\left.\begin{aligned} x' + k(t)x(t) &= \sigma(t) \quad a.\,e.\, t \in J \\ x(0) &= x(T), \end{aligned}\right\} \tag{3.1}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_k(t,s)\sigma(s)ds \tag{3.2}$$

where

$$G_k(t,s) = \begin{cases} \dfrac{e^{K(s)-K(t)}}{1-e^{-K(T)}}, & 0 \le s \le t \le T, \\ \dfrac{e^{K(s)-K(t)-K(T)}}{1-e^{-K(T)}}, & 0 \le t < s \le T, \end{cases} \tag{3.3}$$

where $K(t) = \int_0^t k(s)ds.$

Notice that the Green's function G_k is nonnegative on $J \times J$ and the number

$$M_k := \max \{ |G_k(t, s)| : t, s \in [0, T] \}$$

exists for all $L^1(J,\mathbb{R}^+)$. Note also that $K(t) > 0$ for all $t > 0$.

We need the following definition in the sequel.

Definition 3.1: *A mapping* $\beta : J \times \mathbb{R} \to \mathbb{R}$ *is said to be* Carathéodory if

(i) $t \mapsto \beta(t, x)$ *is measurable for each* $x \in \mathbb{R}$, *and*

(ii) $x \mapsto \beta(t, x)$ *is continuous almost everywhere for* $t \in J$.

Again a Carathéodory function $\beta(t, x)$ *is called* L^1*-Carathéodory if*

(iii) *for each real number* $r > 0$ *there exists a function* $h_r \in L^1(J,\mathbb{R})$ such that

$$|\beta(t, x)| \le h_r(t), \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

Finally a Carathéodory function $\beta(t, x)$ is called L^1_X*-Carathéodory if*

(iv) *there exists a function* $h \in L^1(J,\mathrm{R})$ *such that*

$$|\beta(t, x)| \le h(t), \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$.

For convenience, the function h is referred to as a **bound function** *of* β.

We will use the following hypotheses in the sequel.

(A_0) The function $t \mapsto f(t, x)$ is periodic of period T for all $x \in \mathbb{R}$.

(A_1) The function $x \mapsto \dfrac{x}{f(0,x)}$ is injective in $\mathbb{R}$.

(A_2) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A_3) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\ell \in B(J,\mathbb{R})$ such that

$$|f(t, x) - f(t, y)| \le \ell(t)\,|x - y| \text{ a.e. } t \in J$$

for all $x, y \in \mathbb{R}$.

(A4) The function g is Carathéodory.

Note that hypotheses (A_0) through (A_4) are much common in the literature on the theory of nonlinear differential equations. Actually the function $f : J \times \mathbb{R} \to \mathbb{R}$ defined by $f(t, x) = \alpha + \beta|x|$ for some $\alpha, \beta \in \mathbb{R}$ satisfies the hypotheses (A_0)-(A_3).

Now consider the PBVP

$$\left.\begin{aligned}\left(\frac{x(t)}{f(t,x(t))}\right)' + k(t)\left(\frac{x(t)}{f(t,x(t))}\right) = g_k(t,x(t)) \;\; a.e. \;\; t \in J \\ x(0) = x(T)\end{aligned}\right\} \tag{3.4}$$

where $k \in L^1(J,\mathbb{R}^+)$ and the function $g_k : J \times \mathbb{R} \to \mathbb{R}$ is defined by

$$g_k(t,x) = g(t,x) + k(t)\left(\frac{x}{f(t,x)}\right). \tag{3.5}$$

Remark 3.1. Note that the PBVP (1.1) is equivalent to the PBVP (3.4) and a solution of the PBVP (1.1) is the solution for the PBVP (3.4) on J and vice versa.

Remark 3.2. Assume that hypotheses (A_2) and (A_4) hold. Then the function g_k defined by (3.5) is Carathéodory on $J \times \mathbb{R}$.

Lemma 3.2. *Assume that hypothesis $(A_1) - (A_1)$ holds. Then for any $k \in L^1(J,\mathbb{R}^+)$, x is a solution to the differential equation (3.4) if and only if it is a solution of the integral equation*

$$x(t) = [f(t,x(t))]\left(\int_0^T G_k(t,s) g_k(t,x(s))ds\right), \tag{3.6}$$

where the Green's function $G_k(t,$ s) is defined by (3.3).

Proof. Let $y(t) = \dfrac{x(t)}{f(t,x(t))}\cdot$ Since $f(t,x)$ is periodic in t of period T for all $x \in \mathbb{R}$, we have

$$y(0) = \frac{x(0)}{f(0,x(0))} = \frac{x(T)}{f(T,x(T))} = y(T).$$

Now an application of Lemma 3.1 yields that the solution to differential equation (3.4) is the solution to integral equation (3.6). Conversely, suppose that x is any solution to the integral equation (3.6), then

$$y(0) = \frac{x(0)}{f(0,x(0))} = y(T) = \frac{x(T)}{f(T,x(T))} = \frac{x(T)}{f(0,x(T))}.$$

Since the function $x \mapsto \dfrac{x}{f(0,x)}$ is injective, one has $x(0) = x(T)$ and so, x is a solution to PBVP (1.1). The proof of the lemma is complete.

We make use of the following hypothesis in the sequel.

(A_5) There exists a continuous and nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $\gamma L^1(J,\mathbb{R})$ such that $\gamma(t) > 0$, a.e. $t \in J$ satisfying

$$|g_k(t, x)| \le \gamma(t)\, \psi\, (|x|), \text{ a. e. } t \in J,$$

for all $x \in \mathbb{R}$.

Theorem 3.1. *Assume that the hypotheses* (A_0) – (A_1), (A_3) – (A_5) *hold. Suppose that there exists a real number* $r > 0$ *such that*

$$r > \frac{FM_k \,\|\gamma\|_{L^1}\, \psi(r)}{1 - LM_k \,\|\gamma\|_{L^1}\, \psi(r)} \tag{3.7}$$

where $LM_k \,\|\gamma\|_{L^1}\, \psi(r) < 1$, $F = \sup_{t\in[0,T]} |f(t,0)|$ and $L = \max_{t \in J} \ell(t)$. *Then the PBVP (1.1) has a solution on J.*

Proof. Let $X = C(J,\mathbb{R})$. Define an open ball $\mathcal{B}_r(0)$ centered at origin 0 of radius r, where the real number r satisfies the inequality (3.7). Define two mappings A and B on $\overline{\mathcal{B}_r(0)}$ by

$$Ax(t) = f(t, x(t)),\ t \in J, \tag{3.8}$$

and

$$Bx(t) = \int_0^T G_k(t,s) g_k(s, x(s)) ds,\ \ t \in J. \tag{3.9}$$

Obviously A and B define the operators $A,B : \overline{\mathcal{B}_r(0)} \to X$. Then the integral equation (3.6) is equivalent to the operator equation

$$x(t) = Ax(t)Bx(t),\ t \in J. \tag{3.10}$$

We shall show that the operators A and B satisfy all the hypotheses of Theorem 2.1.

We first show that A is a Lipschitz on X. Let $x, y \in X$. Then by (A_3),

$$\begin{aligned} |Ax(t) - Ay(t)| &\le |f(t, x(t)) - f(t, y(t))| \\ &\le \ell(t)\, |x(t) - y(t)| \\ &\le L\, \|x - y\| \end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain

$$\|Ax - Ay\| \le L\|x - y\|$$

for all $x, y \in X$. So A is a Lipschitz on X with Lipschitz constant L. Next we show that B is completely continuous on X. Using the standard arguments as in Granas et. al. [13], it is shown that B is a continuous operator on X. We shall show that $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded and equicontinuous set in X. Let $x \in B(\overline{\mathcal{B}_r(0)})$ be arbitrary. Since g is Carathéodory, we have

$$
\begin{aligned}
|Bx(t)| &\le \int_0^T G_k(t,s) g_k(s, x(s)) ds| \\
&\le M_k \int_0^T [\gamma(s)\, \psi(|x(s)|)]\, ds \\
&\le M_k \int_0^T \gamma(s)\, \psi(|x(s)|)\, ds \\
&\le M_k \|\gamma\|_{L^1}\ \psi(r).
\end{aligned}
$$

Taking the supremum over t, we obtain $\|Bx\| \le M$ for all $x \in \overline{B_r(0)}$, where $M = M_k \|\gamma\|_{L^1}\ \psi(r)$. This shows that $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded set in X. Next we show that $B(\overline{\mathcal{B}_r(0)})$ is an equicontinuous set. To finish it is enough to show that $y' = (Bx)'$ is bounded on $[0, T]$. Now for any $t \in [0, T]$, one has

$$
\begin{aligned}
|y'(t)| &\le |\int_0^T \frac{\partial}{\partial t} G_k(t, s) g_k(s, x(s))\, ds| \\
&= |\int_0^T (-k(s)) G_k(t, s) g_k(s, x(s))\, ds| \\
&\le K\, M_k \|\gamma\|_{L^1}\ \psi(r) \\
&= c,
\end{aligned}
$$

where $K = \max_{t\in J} k(t)$. Hence for any $t, \tau \in [0, T]$ one has

$$|Bx(t) - Bx(\tau)| \le c\, |t - \tau| \to 0 \text{ as } t \to \tau.$$

This shows that $B(\overline{\mathcal{B}_r(0)})$ is a equi-continuous set X. Now $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded and equi-continuous set in X, so it is compact by Arzelá-Ascoli theorem. As a result B is a compact and continuous operator on $B(\overline{\mathcal{B}_r(0)})$. Thus all the conditions of Theorem 2.1 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be a solution to PBVP (1.1) such that $\|u\| = r$. Then we have, for any $\lambda \in (0, 1)$,

$$u(t) = \lambda[f(t, x(t))]\left(\int_0^T G_k(t,s) g_k(s, x(s))\, ds\right)$$

for $t \in J$. Therefore,

$$\begin{aligned}
|u(t)| &\leq \lambda\, |f(t,u(t))| \left(\left|\int_0^T G_k(t,s) g_k(s, x(s))\, ds\right|\right) \\
&\leq \lambda\big(|f(t,x(t)) - f(t,0)| + |f(t,0)|\big) \\
&\quad \times \left(\int_0^T G_k(t,s)\, |g_k(s, x(s))|\, ds\right) \\
&\leq [\ell(t)\,|u(t)| + F]\left(\int_0^T M_k\, |g_k(s, x(s))|\, ds\right) \\
&\leq LM_k\, |u(t)| \left(\int_0^T \gamma(s)\, \psi\,(|u(s)|\, ds\right) \\
&\quad + FM_k \left(\int_0^T \gamma(s)\, \psi\,(|u(s)|) ds\right) \\
&\leq LM_k \,\|\gamma\|_{L^1}\, \psi\,(\|u\|)\, |u(t) + FM_k\, \|\gamma\|_{L^1}\, \psi\,(\|u\|).
\end{aligned} \tag{3.11}$$

Taking the supremum in the above inequality (3.11) yields

$$\|u\| \leq \frac{FM_k \,\|\gamma\|_{L^1}\, \psi\,(\|u\|)}{1 - LM_k\, \|\gamma\|_{L^1}\, \psi\,(\|u\|)}.$$

Substituting $\|u\| = r$ in above inequality yields

$$r \leq \frac{FM_k \,\|\gamma\|_{L^1}\, \psi\,(r)}{1 - LM_k\, \|\gamma\|_{L^1}\, \psi\,(r)}$$

This is a contradiction to (3.7). Hence the conclusion (ii) of Corollary 2.1 does not hold. Therefore the operator equation $Ax\ Bx = x$ and consequently the PBVP (1.1) has a solution on J. This completes the proof.

Remark 3.3. We note that in theorem 3.1, we only require the hypothesis (A_1) to hold in $[-r, r]$.

4. EXISTENCE OF EXTREMAL SOLUTIONS

We equip the space $C(J,\mathbb{R})$ with the order relation $\leq$ with the help of the cone defined by

$$K = \{x \in C(J,\mathbb{R}) : x(t) \geq, \forall t \in J\}. \tag{4.1}$$

It is well known that the cone K is positive and normal in $C(J,\mathbb{R})$. We need the following definitions in the sequel.

Definition 4.1. *A function $a \in AC^1(J,\mathbb{R})$ is called a lower solution of the PBVP* (1.1) *on J if*

$$\left.\begin{array}{c} \dfrac{d}{dt}\left[\dfrac{a(t)}{f(t,a(t))}\right] \leq g(t,a(t)) \;\; a.e. \;\; t \in J \\ a(0) \leq a(T). \end{array}\right\}$$

Again a function $b \in AC^1(J,\mathbb{R})$ is called an upper solution of the PBVP (1.1) on J if

$$\left.\begin{array}{c} \dfrac{d}{dt}\left[\dfrac{b(t)}{f(t,b(t))}\right] \geq g(t,b(t)) \;\; a.e. \;\; t \in J \\ b(0) \geq b(T). \end{array}\right\}$$

Definition 4.2. *A solution x_M of the PBVP (1.1) is said to be maximal if for any other solution x to PBVP (1.1) one has $x(t) \leq x_M(t)$,* for all *$t \in J$. Again a solution x_m of the PBVP (1.1) is said to be minimal if $x_m(t) \leq x(t)$, for all $t \in J$, where x is any solution of the PBVP (1.1) on J.*

Remark 4.1. The upper and lower solutions of the PBVP (1.1) are respectively the upper and lower solutions of the PBVP (3.4) and vice-versa. Similarly the maximal and minimal solutions of the PBVP (1.1) are respectively the upper and lower solutions of the PBVP (3.4) and vice-versa.

4.1. **Carathéodory case.** We consider the following set of assumptions:

(B_0) $f : J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$, $g : J \times \mathbb{R} \to \mathbb{R}^+$.

(B_1) The function $x \mapsto \dfrac{x}{f(0,x)}$ is increasing in the interval $\left[\min\limits_{t\in J} a(t),\ \max\limits_{t\in J} b(t)\right]$

(B_2) The functions $f(t, x)$ and $g(t, x)$ are nondecreasing in x and y almost everywhere for $t \in J$.

(B_3) The PBVP (1.1) has a lower solution a and an upper solution b on J with $a \leq b$.

(B_4) The function $h : J \to \mathbb{R}$ defined by

$$h(t) = |g_k(t, b(t))|,$$

is Lebesgue integrable.

We remark that hypothesis (B_4) holds in particular if f is continuous and g is L^1-Carathéodory on $J \times \mathbb{R}$.

Remark 4.2. *Assume that* (B_0)-(B_4) *hold. Then the function* $t \mapsto g_k(t, x(t))$ *is Lebesgue integrable on J and*

$$|g_k(t, x(t))| \leq h(t), \quad \text{a.e. } t \in J,$$

for all $x \in [a, b]$.

Theorem 4.1. *Suppose that the assumptions* (A_1), (A_3) *through* (A_4) and (B_0) *through* (B_4) *hold. Further if* $LT \| h \|_{L^1} < 1$, *where h is given in Remark 4.1 and L =* $\max_{t \in J} \ell(t)$, *then PBVP (1.1) has a minimal and a maximal positive solution on* J.

Proof. Now PBVP (1.1) is equivalent to integral equation (3.6) on J. Let $X = C(J,\mathbb{R})$. Define two operators A and B on X by (3.8) and (3.9) respectively. Then integral equation (3.6) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_1) implies $A,B : [a, b] \to K$. Since the cone K in X is normal, $[a, b]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem 3.1, that A is a Lipschitz with a Lipschitz constant L and B is completely continuous operator on $[a, b]$. Again the hypothesis (B_2) implies that A and B are nondecreasing on $[a, b]$. To see this, let $x, y \in [a, b]$ be such that $x \leq y$. Then by (B_2),

$$Ax(t) = f(t, x(t)) \leq f(t, y(t)) = Ay(t)$$

for all $t \in J$. Similarly, we have

$$\begin{aligned} Bx(t) &= \int_0^T G_k(t,s) g_k(s, x(s)) ds \\ &\leq \int_0^T G_k(t,s) g_k(s,s)\, ds \\ &= By(t) \end{aligned}$$

for all $t \in J$. So A and B are nondecreasing operators on $[a, b]$. Again Lemma 4.1 and hypothesis (B_3) together imply that

$$\begin{aligned} a(t) &\le [f(t, a(t))]\left(\int_0^T G_k(t,s) g_k(s, a(s)))\, ds\right) \\ &\le [f(t, x(t))]\left(\int_0^T G_k(t,s) g_k(s, x(s))\, ds\right) \\ &\le [f(t, b(t))]\left(\int_0^T G_k(t,s) g_k(s, b(s)))\, ds\right) \\ &\le b(t), \end{aligned}$$

for all $t \in J$ and $x \in [a, b]$. As a result $a(t) \le Ax(t)\, Bx(t) \le b(t)$, for all $t \in J$ and $x \in [a, b]$. Hence $Ax\, Bx \in [a, b]$ for all $x \in [a, b]$. Again,

$$\begin{aligned} M &= \|B([a, b])\| \\ &= \sup\{\|Bx\| : x \in [a, b]\} \\ &\le \sup\left\{\sup_{t \in J} \int_0^T G_k(t,s)\, |\, g_k(s, x(s))\, |\, ds \,|\, x \in [a,b]\right\} \\ &\le M_k \int_0^T h(s) ds \\ &= M_k \, \| h \|_{L^1} . \end{aligned}$$

Since $\alpha M \le L M_k \, \| h \|_{L^1} < 1,$ we apply Theorem 2.2 to the operator equation $Ax\, Bx = x$ to yield that the PBVP (1.1) has a minimal and a maximal positive solution on J. This completes the proof.

4.2. **Discontinuous case.** We need the following definition in the sequel.

Definition 4.3. *A mapping* $\beta : J \times \mathbb{R} \to \mathbb{R}$ *is said to be Chandrabhan if*

(i) $t \mapsto \beta(t, x)$ *is measurable for each* $x \in \mathbb{R}$, and

(ii) $x \mapsto \beta(t, x)$ *is nondecreasing almost everywhere for* $t \in J$.

Again a Chandrabhan function $\beta(t, x)$ *is called* L^1*-Chandrabhan if*

(iii) *for each real number* $r > 0$ *there exists a function* $h_r \in L^1(J,\mathbb{R})$ such that

$$|\beta(t, x)| \le h_r(t), \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$ *with* $|x| \le r$.

Finally a Chandrabhan function $\beta(t, x)$ *is called* L^1_X*-Chandrabhan if*

(iv) *there exists a function* $h \in L^1(J,\mathbb{R})$ *such that*

$$|\beta(t, x)| \le h(t), \text{ a.e. } t \in I$$

for all $x \in \mathbb{R}$.

For convenience, the function h is referred to as a bound function *of* β.

We consider the following hypotheses in the sequel.

(C_1) The function $f(t, x)$ is nondecreasing in x almost everywhere for $t \in J$.

(C_2) The function g_k defined by (3.5) is Chandrabhan.

Theorem 4.2: *Suppose that the assumptions* (A_1)-(A_2), (B_0)-(B_3) and (C_1)-(C_2) *hold. Then PBVP (1.1) has a minimal and a maximal positive solution on* J.

Proof. Now PBVP (1.1) is equivalent to integral equation (3.6) on J. Let $X = C(J,\mathbb{R})$. Define two operators A and B on X by (3.8) and (3.9) respectively. Then integral equation (3.6) is transformed into an operator equation $Ax(t)\,Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A,B : [a, b] \to K$. Since the cone K in X is normal, $[a, b]$ is a norm bounded set in X.

Step I : First we show that A is completely continuous on $[a, b]$. Now the cone K in X is normal, so the order interval $[a, b]$ is norm-bounded in X. Hence there exists a constant $r > 0$ such that $\|x\| \le r$ for all $x \in [a, b]$. As f is continuous on compact $J \times [-r, r]$, it attains its maximum, say M. Therefore for any subset S of $[a, b]$ we have

$$\begin{aligned}\|A(S)\|p &= \sup\{\|Ax\| : x \in S\}\\ &= \sup\left\{\sup_{t\in J} |f(t, x(t))| : x \in S\right\}\\ &\le \sup\left\{\sup_{t\in J} |f(t, x)| : x \in [-r, r]\right\}\\ &\le M.\end{aligned}$$

This shows that $A(S)$ is a uniformly bounded subset of X.

Next we note that, the function $f(t, x)$ is uniformly continuous on $[0, T] \times [-r, r]$. Therefore for any $t, \tau \in [0, T]$ we have

$$|f(t, x) - f(\tau, x)| \to 0 \text{ as } t \to \tau$$

for all $x \in [-r, r]$. Similarly for any $x, y \in [-r, r]$

$$|f(t, x) - f(t, y)| \to 0 \text{ as } x \to y$$

for all $t \in [0, T]$. Hence any $t, \tau \in [0, T]$ and for any $x \in S$ one has

$$\begin{aligned} |Ax(t) - Ax(\tau)| &= |f(t, x(t)) - f(\tau, x(\tau))| \\ &\le |f(t, x(t)) - f(\tau, x(t))| + |f(\tau, x(t)) - f(\tau, x(\tau))| \\ &\to 0 \text{ as } t \to \tau. \end{aligned}$$

This shows that $A(S)$ is an equi-continuous set in X. Now an application of Arzelá-Ascoli theorem yields that A is a completely continuous operator on $[a, b]$.

Step II : Next we show that B is totally bounded operator on [a, b]. To finish, we shall show that $B(S)$ is uniformly bounded ad equi-continuous set in X for any subset S of $[a, b]$. Since the cone K in X is normal, the order interval $[a, b]$ is norm-bounded. Let $y \in B(S)$ be arbitrary. Then,

$$y(t) = \int_0^T G_k(t,s) g_k(s, x(s))\, ds$$

for some $x \in S$. By hypothesis (B_2) one has

$$\begin{aligned} |\, y(t) \,| &= \int_0^T G_k(t,s)\, |\, g_k(s, x(s)) \,|\, ds \\ &\le M_k \int_0^T h(s)\, ds \\ &\le M_k \,\|\, h \,\|_{L^1} . \end{aligned}$$

Taking the supremum over t,

$$\|\, y \,\| \le M_k \,\|\, h \,\|_{L^1},$$

which shows that $B(S)$ is a uniformly bounded set in X. Similarly let $t, \tau \in J$. To finish it is enough to show that y' is bounded on $[0, T]$. Now for any $t \in [0, T]$,

$$\begin{aligned} |\, y(t) \,|\, y(t) \,| &\le \left| \int_0^T \frac{\partial}{\partial t} G_k(t,s)\, |\, g_k(s, x(s)) \,|\, ds \right| \\ &= \left| \int_0^T (-k(s)) G_k(t,s)\, |\, g_k(s, x(s)) \,|\, ds \right| \end{aligned}$$

$$\leq K\, M_k \,\| h \|_{L^1}$$

where $K = \max_{t \in J} |k(t)|$. Hence for any t, $\tau \in [0, T]$ one has

$$|y(t) - y(\tau)| \leq c\, |t - \tau| \to 0 \text{ as } t \to \tau.$$

This shows that $B(S)$ is a equi-continuous set of functions in $[a, b]$. for all $S \subset [a, b]$. Now $B(S)$ is a uniformly bounded and equi-continuous, so it is totally bounded by Arzelà-Ascoli theorem. Thus all the conditions of Theorem 2.3 are satisfied and hence an application of it yields that the PBVP (1.1) has a maximal and a minimal positive solution on J.

Theorem 4.3: *Suppose that the assumptions* (A_3), (A_2), (B_0)-(B_3) *and* (C_1)-(C_2) *hold. Further if*

$$LM_k \,\| h \|_{L^1} < 1,$$

where h is given in Remark 4.1 and $L = \max_{t \in J} \ell(t)$, *then the PBVP (1.1) has a minimal and a maximal positive solution on* J.

Proof. Now PBVP (1.1) is equivalent to integral equation (3.6) on J. Let $X = C(J,\mathbb{R})$. Define two operators A and B on X by (3.8) and (3.9) respectively. Then integral equation (3.6) is transformed into an operator equation $Ax(t)\, Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A,B : [a, b] \to K$. Since the cone K in X is normal, $[a, b]$ is a norm bounded set in X. Now it can be shown as in the proofs of Theorem 3.1 and Theorem 2.4 that the operator A is a Lipschitz with a Lipschitz constant $\alpha = L$ and B is totally bounded with $M = \| B([a,b]) \| = M_k \,\| h \|_{L^1}$, respectively. Since $\alpha M = L\, M_k \,\| h \|_{L^1} < 1$, the desired conclusion follows by an application of Theorem 2.4.

5. AN EXAMPLE

Given the closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the nonlinear PBVP

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t, x(t))}\right] &= \frac{|\, x(t)\,|}{16} - \frac{x(t)}{1 + |\, x(t)\,|}, \quad a.e.\ t \in J \\ x(0) &= x(1), \end{aligned}\right\} \tag{5.1}$$

where the function $f : J \times \mathbb{R} - \{0\} \to \mathbb{R}$ is defined by

$$f(t, x) = 1 + |x|.$$

Obviously $f: J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$ is continuous and the function $t \mapsto f(t, x)$ is periodic of period $T = 1$ for all $x \in \mathbb{R}$. Define a function g : $J \times \mathbb{R} \to \mathbb{R}$ by

$$g(t,x) = \frac{|x(t)|}{16} - \frac{x(t)}{1+|x(t)|}.$$

Now consider the PBVP

$$\left.\begin{aligned} \left(\frac{x(t)}{f(t,x(t))}\right)' + \frac{x(t)}{1+|x(t)|} &= \frac{|x(t)|}{16} \quad a.e.\ t \in J \\ x(0) &= x(1). \end{aligned}\right\} \tag{5.2}$$

It is easy to verify that f is continuous and Lipschitz on $J \times \mathbb{R}$ with a Lipschitz function $\ell(t) = 1$ for all $t \in J$. Here $k(t) = 1$, and so $M_k = \frac{e}{e-1}$. Also, here we have

$$F = \sup_{t \in J} |f(t,0)| = 1.$$

Now the real number $r = 4$ satisfies condition (3.6) of Theorem 3.1 with $\gamma(t) = \frac{1}{16}$ for all $t \in J$ and $\psi(r) = r$ for all $r \in \mathbb{R}^+$. Besides this, the map

$$x \mapsto \frac{x}{f(0,x)} = \frac{x}{1+|x|}$$

is strictly increasing in the order interval [–4, 4]. Therefore, an application of Theorem 3.1 yields that the PBVP (5.1) has a solution u on J with $\|u\| \le 4$.

REFERENCES

[1] S. R. Bernfeld and V. Lakshmikantham, *An introduction to nonlinear boundary value problems*, Academic Press, New York-London, 1974.

[2] B. C. Dhage, *On α-condensing mappings in Banach algebras*, The Math. Student **63** (1994), 146-152.

[3] B. C. Dhage, *Fixed point theorems in ordered Banach algebras and applications*, PanAmer. Math. J. **9**(4) (1999), 93–102.

[4] B. C. Dhage, *Some nonlinear alternatives in Banach algebras with applications II*, Kyungpook Math. J. **45** (2005), 281-292.

[5] B. C. Dhage, *On a fixed point theorem in Banach algebras with applications*, Appl. Math. Lett. **18** (2005), 273-280.

[6] B. C. Dhage, *Some algebraic fixed point theorems for multi-valued mappings with applications*, Diss. Math. Differential inclusion, control & Optimization 26 (2006), 00-00.

[7] B. C. Dhage, *Nonlinear functional boundary value problems in Banach algebras involving Carathéodories*, Kyungpook Math, J. **46** (2006), 00-00.

[8] B. C. Dhage and S. K. Ntouyas, *Existence theorems for nonlinear functional boundary value problems*, Math. Sci. Res. J. **6** (8) (2002), 380-390.

[9] B. C. Dhage, U. P. Dolhare and S. *K.* Ntouyas, *Existence theorems for nonlinear first order functional differential equations in Banach algebras*, Comm. Appl. Nonlinear Anal. **10** (4) (2003), 59-69.

[10] B. C. Dhage and D. O'Regan, *A fixed point theorem in Banach algebras with applications to nonlinear integral equation*, Functional Diff. Equations 7(3-4)(2000), 259-267.

[11] B. C. Dhage, S. N. Salunkhe, Ravi Agarwal and W. Zhang, *A functional differential equation in Banach algebras*, Math. Inequ. Appl. **8** (1) (2005), 89-99.

[12] A. Granas and J. Dugundji , *Fixed Point Theory*, Springer Verlag 2003.

[13] A. Granas, R. B. Guenther and J. W. Lee, *Some general existence principles for Carathéodory theory of nonlinear differential equations*, J. Math. Pures et Appl. **70** (1991), 153-196.

[14] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Spaces*, Academic Press, New York, 1988.

[15] S. Heikkilö and V. Lakshmikantham, *Monotone Iterative Technique for Nonlinear Discontinues Differential Equations*, Marcel Dekker Inc., New York, 1994.

[16] G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, *Monotone iterative techniques for nonlinear differential equations* Pitman, New York, 1985.

[17] J. J. Nieto, *Basic theory for nonresonance impulsive periodic problems of first order*, J. Math. Anal. Appl. **205** (1997), 423–433.

[18] J. J. Nieto, *Periodic boundary value problems for fisrt order impulsive ODEs*, Nonlinear Anal. **51** (2002), 1223-1232.

[19] P. Omari and F. Zanolin, *Remarks on periodic solutions for first order nonlinear diffrential systems*, Boll. Un. Mat. Ital. *B* (6) 2 (1983), 207-218.

[20] E. Zeidler, *Nonlinear Functional Analysis: Part I*, Springer Verlag, New York, 1985.

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