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NONEXISTENCE RESULTS OF NONTRIVIAL SOLUTIONS FOR A CLASS OF NONLINEAR EQUATIONS

Brahim Khodja

ABSTRACT: Let $f : \overline{D} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$\begin{cases} f(y, o) = 0, \ y \in \overline{D}, \\ 2F(y, u) - uf(y, u) \leq 0, \end{cases}$$

where $F(y, u) = \int_0^u f(y, s)\, ds$. If we assume $u = u(x, y) \in H^2(\mathbb{R}^m \times D) \cap L^\infty(\mathbb{R}^m \times D)$ to be a solution of equation

$$-\sum_{j=1}^{m} \frac{\partial^2 u(x, y)}{\partial x_j^2} - \sum_{i=1}^{n} c_i \frac{\partial^2 u(x, y)}{\partial y_i^2} + f(y, u(x, y)) = 0, (x, y) \in \mathbb{R}^m \times D,$$

where c_i $(i = 1, ..., n)$ are given real constants and $u(x, y)$ satisfies the following boundary conditions

$$u(x, s) = 0, \ (x, s) \in \mathbb{R}^m \times \partial D,$$

or

$$\frac{\partial u}{\partial n}(x, s) = 0, (x, s) \in \mathbb{R}^m \times \partial D,$$

or

$$\left(u + \varepsilon \frac{\partial u}{\partial n}\right)(x, s) = 0, (x, s) \in \mathbb{R}^m \times \partial D,$$

Then the function $E(x) = \int_D |u(x, y)|^2 dy$ verifies $\Delta E \geq 0$ in $\mathbb{R}^m$. Here

$$\Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_m^2} \text{ and } D = \prod_{i=1}^{n}]\alpha_i, \beta_i[$$

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1. INTRODUCTION

The problem of existence and nonexistence of nontrivial solutions of problems of the form

$$\begin{vmatrix} -\Delta u + f(u) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{vmatrix}$$

has been investigated by many authors under various situations. Previous works have been reported by H. Berestycky, T. Gallouet and O. Kavian [2], G.Caristi and E. Mitidieri [3], M.J. Esteban and P.L. Lions [4] and S.I. Pohozaev [14]. To illustrate some of the typical known results, let us consider the Dirichlet problem

$$\begin{vmatrix} -\Delta u + f(u) = 0, u \in C^2\left(\overline{\Omega}\right), \\ u = 0 \text{ on } \partial\Omega, \end{vmatrix}$$

Under the hypotheses

$$\begin{cases} \nabla u \in L^2(\Omega), \\ f(0) = 0, \\ F(u) = \int_0^u f(s)\,ds \in L^1(\Omega), \end{cases}$$

where Ω is a connected unbounded domain of $\mathbb{R}^N$ such as

$$\exists \Lambda \in \mathbb{R}^N, \|\Lambda\| = 1, \langle n(x), \Lambda\rangle \geq 0 \text{ on } \partial\Omega, \langle n(x), \Lambda\rangle \neq 0,$$

($n(x)$ is the outward normal to $\partial\Omega$ at the point x) M.J. Esteban and P.L. Lions [4] established that the Dirichlet problem does not have nontrivial solutions.

The Pohozaev identity published in 1965 for solutions of the Dirichlet problem proved absence of nontrivial solutions for some elliptic equations when Ω is a star shaped bounded domain in $\mathbb{R}^n$ and f is a continuous function on $\mathbb{R}$ satisfying:

$$(n-2)F(u) - 2nuf(u) > 0,$$

where, $n = \dim\mathbb{R}^n$.

When

$$\Omega = J \times \omega,$$

where $J \subset \mathbb{R}$ is unbounded interval and $\omega \subset \mathbb{R}^n$ domain, A. Haraux and B. Khodja [6] established under the assumption

$$\begin{cases} f(0) = 0, \\ 2F(u) - uf(u) \leq 0, \end{cases}$$

if we assume that $u \in H^2(J \times \omega) \cap L^\infty(J \times \omega)$ is a solution of the problems

$$\left| \begin{array}{l} -\Delta u + f(u) = 0 \text{ in } \Omega, \\ \left(u \text{ or } \dfrac{\partial u}{\partial n} \right) = 0 \text{ on } \partial(J \times \omega). \end{array} \right.$$

Then these two problems (Dirichlet and Neumann) do have only trivial solution.

When

$$f(u) = u\,(u + 1)\,(u + 2),$$

and

$$\Omega = \mathbb{R} \times]0, a[(a < \pi),$$

Neumann problem

$$\left| \begin{array}{l} -\Delta u + u(u+1)(u+2) = 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{array} \right.$$

is still open.

This has motivated us to investigate furthermore this question.

Let f be a real continuous function

$$f : \overline{D} \times \mathbb{R} \to \mathbb{R},$$

verifying

$$f(y, 0) = 0 \text{ in } \overline{D},$$

so that

$$u \equiv 0,$$

is a solution of the equation

$$-\sum_{j=1}^{m}\frac{\partial^2 u}{\partial x_j^2}-\sum_{i=1}^{n}c_i\frac{\partial^2 u}{\partial y_i^2}+f(y,u)=0 \text{ in } \Omega=\mathbb{R}^m\times D, \tag{1.1}$$

where $c_i (i = 1, ..., n)$ are real constants.

We assume that

$$u = u(x, y) \in H^2(\mathbb{R}^m \times D) \cap L^\infty(\mathbb{R}^m \times D),$$

satisfies

$$u(x, s) = 0,\ (x, s) \in \mathbb{R}^m \times \partial D \text{ (Dirichlet boundary condition)}, \tag{1.2}$$

or

$$\frac{\partial u}{\partial n}(x, s) = 0,\ (x, s) \in \mathbb{R}^m \times \partial D \text{ (Neumann boundary condition)}, \tag{1.3}$$

or

$$\left(u+\varepsilon\frac{\partial u}{\partial n}\right)(x, s) = 0,\ (x, s) \in \mathbb{R}^m \times \partial D \text{ (Robin boundary condition)}. \tag{1.4}$$

The purpose of this paper is to extend our previous result [6] to problems (1.1)–(1.2), (1.1)–(1.3) and (1.1)–(1.4).

Let us denote by:

$$\Gamma = \mathbb{R}^m \times \partial D = \Gamma_{\alpha_1} \cup \Gamma_{\beta_1} \cup \ldots \cup \Gamma_{\alpha_n} \cup \Gamma_{\beta_n},$$

where

$$\Gamma_{\alpha_i} = \{(x_1, ..., x_m, y_1, ... y_{i-1}, \alpha_i, y_{i+1}, ..., y_n)\},$$

the boundary of $\Omega = R^m \times \prod_{i=1}^{n}]\alpha_1, \beta_i[$, $(x, y) = (x_1, ..., x_m, y_1, ..., y_n)$, the generic point of Ω, $n(x, s) = (0, ..., 0, n_1(x, s), n_2(x, s), ..., n_n(x, s))$, the outward normal to $\partial\Omega$ at the (x, s) point and $\frac{\partial^2 u(x, y)}{\partial x_i^2}$ the second derivative of u with respect to $x_i (i = 1, ..., n)$ at (x, y) point.

Let us define for

$$k = 1, 2, ...n \text{ and } \tau \in \{\alpha_1, \beta_1, \alpha_2, \beta_2, ..., \alpha_n, \beta_n\},$$
$$y_k^\tau = (y_1, y_2, ..., y_{k-1}, \tau, y_{k+1}, ..., y_n),$$
$$dy_k^* = dy_1...dy_{k-1}dy_{k+1}...dy_n,$$

$$\int_{\alpha_1}^{\beta_1}...\int_{\alpha_{-1}^i}^{\beta_{i-1}}\int_{\alpha_{i+1}}^{\beta_{i+1}}...\int_{\alpha_n}^{\beta_n} f(y,s)\,dy_1...dy_{i-1}dy_{i+1}\dots dy_n = \int_{D_i^*} f(y,s)\,dy_i^*.$$

2. INTEGRAL IDENTITY

We begin this section by giving an integral identity useful in the sequel

Lemma 1: Let

$$u = u(x, y) \in H^2(\mathbb{R}^m \times D) \cap L^\infty(\mathbb{R}^m \times D),$$

be a solution of problem (1.1)–(1.4). Then for each $x \in \mathbb{R}^m$ and $\varepsilon > 0$, the solution u verifies

$$\int_D \left(-\frac{1}{2}\sum_{j=1}^m \left|\frac{\partial u(x,y)}{\partial x_j}\right|^2 + \frac{1}{2}\sum_{i=1}^n c_i \left|\frac{\partial u(x,y)}{\partial y_i}\right|^2 + F(y, u(x,y)) \right) dy$$

$$+\frac{1}{2\varepsilon}\sum_{i=1}^n c_i \int_{D_i^*}\left(\left|u\left(x, y_i^{\alpha_i}\right)\right|^2 + \left|u\left(x, y_i^{\beta_i}\right)\right|^2 \right) dy_i^* = 0. \tag{2.1}$$

Proof: Let the function defined by

$$I : \mathbb{R}^m \to \mathbb{R},$$

$$I(x) = \int_D \left(-\frac{1}{2}\sum_{j=1}^m \left|\frac{\partial u(x,y)}{\partial x_j}\right|^2 + \frac{1}{2}\sum_{i=1}^n c_i \left|\frac{\partial u(x,y)}{\partial y_j}\right|^2 + F(y, u(x,y)) \right) dy.$$

The hypotheses on u and f imply that I is absolutely continuous and thus differentiable almost everywhere on $\mathbb{R}^m$, we have

$$\frac{\partial}{\partial x_j} I(x) = \int_D \left(-\sum_{j=1}^m \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j^2} + \sum_{i=1}^n c_i \frac{\partial u}{\partial y_i}\frac{\partial^2 u}{\partial y_i x_j} + f(y,u)\frac{\partial u}{\partial x_j} \right)(x,y)\,dy. \tag{2.2}$$

A simple use of Fubini's theorem and an integration by parts yields

$$\int_D \sum_{i=1}^{n} c_i \frac{\partial u}{\partial y_i} \frac{\partial^2 u}{\partial y_i x_j}(x, y)\, dy = \sum_{i=1}^{n} \int_{D_i^*} \left(\int_{\alpha_i}^{\beta_i} c_i \frac{\partial u}{\partial y_i} \frac{\partial^2 u}{\partial y_i x_j}(x, y)\, dy_i \right) dy_i^*$$

$$= \int_D -\sum_{i=1}^{n} c_i \frac{\partial^2 u}{\partial y_i^2} \frac{\partial u}{\partial x_j}(x, y)\, dy + \sum_{i=1}^{n} c_i \int_{D_i^*} \left(\frac{\partial u}{\partial y_i} \frac{\partial u}{\partial x_j}\left(x, y_i^{\beta_i}\right) - \frac{\partial u}{\partial y_i} \frac{\partial u}{\partial x_j}\left(x, y_i^{\alpha_i}\right) \right) dy_i^*.$$

Substituting these formulas in (2.2), one obtains

$$\frac{\partial}{\partial x_j} I(x) = \int_D \left(-\sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} + \sum_{i=1}^{n} c_i \frac{\partial^2 u}{\partial y_i^2} + f(y, u) \right) \left(\frac{\partial u}{\partial x_j} \right)(x, y)\, dy$$

$$+ \sum_{i=1}^{n} \int_{D_i^*} c_i \left(\frac{\partial u}{\partial x_j} \frac{\partial u}{\partial y_i}\left(x, y_i^{\beta_i}\right) - \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial y_i}\left(x, y_i^{\alpha_i}\right) \right) dy_i^*.$$

As *u* satisfies equation (1.1), the above expression reduces to

$$\frac{\partial}{\partial x_j} I(x) = + \sum_{i=1}^{n} \int_{D_i^*} c_i \left(\frac{\partial u}{\partial x_j} \frac{\partial u}{\partial y_i}\left(x, y_i^{\beta_i}\right) - \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial y_i}\left(x, y_i^{\alpha_i}\right) \right) dy_i^*. \tag{2.3}$$

Now observe that $\left(u - \varepsilon \frac{\partial u}{\partial y_i} \right)(x, s) = 0$ on $\partial\Omega$, is equivalent to

$$\begin{cases} \left(u - \varepsilon \frac{\partial u}{\partial y_i} \right)\left(x, y_i^{\alpha_i}\right) = 0 \\ \left(u + \varepsilon \frac{\partial u}{\partial y_i} \right)\left(x, y_i^{\beta_i}\right) = 0 \end{cases}, \; x \in \mathbb{R}^m, \; \alpha_i < y_i < \beta_i, \tag{2.4}$$

$\varepsilon > 0$, we can write

$$\frac{\partial}{\partial x_j} I(x) \;=\; -\frac{1}{\varepsilon} \sum_{i=1}^{n} c_i \int_{D_i^*} \left(\left(\frac{\partial u}{\partial x_j} u \right)\left(x, y_i^{\beta_i}\right) + \left(\frac{\partial u}{\partial x_j} u \right)\left(x, y_i^{\alpha_i}\right) \right) dy_i^*$$

$$= -\frac{1}{\varepsilon}\sum_{i=1}^{n}\int_{D_i^*} c_i \frac{1}{2}\frac{\partial}{\partial x_j}\left(\left|u\left(x, y_i^{a_i}\right)\right|^2 + \left|u\left(x, y_i^{\beta_i}\right)\right|^2\right)dy_i^*$$

$$= -\frac{1}{2\varepsilon}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} c_i \int_{D_i^*}\left(\left|u\left(x, y_i^{a_i}\right)\right|^2 + \left|u\left(x, y_i^{\beta_i}\right)\right|^2\right)dy_i^*\right).$$

We obtain instead of (2.3)

$$\frac{\partial}{\partial x_j}\left(I\left(x\right) + \frac{1}{2\varepsilon}\sum_{i=1}^{n} c_i \int_{D_i^*}\left(\left|u\left(x, y_i^{a_i}\right)\right|^2 + \left|u\left(x, y_i^{\beta_i}\right)\right|^2\right)dy_i^*\right) = 0.$$

Integrating this expression, with respect to x_j, ($j = 1, ..., m$), one obtains

$$I\left(x\right) + \frac{1}{2\varepsilon}\sum_{i=1}^{n} c_i \int_{D_i^*}\left(\left|u\left(x, y_i^{a_i}\right)\right|^2 + \left|u\left(x, y_i^{\beta_i}\right)\right|^2\right)dy_i^* = const.$$

Since

$$u(x, y) \in H^2(\mathbb{R}^m \times D),$$

we conclude that the constant is null which is the desired result.

Lemma 2: Under the hypotheses of Lemma 1, if u satisfies either the Dirichlet condition (1.2) or the Neumann condition (1.3) on $\mathbb{R}^m \times \partial D$. Then for each $x \in \mathbb{R}^m$, u verifies the integral identity

$$\int_D\left(-\frac{1}{2}\sum_{j=1}^{m}\left|\frac{\partial u\left(x, y\right)}{\partial x_j}\right|^2 + \frac{1}{2}\sum_{i=1}^{n} c_i\left|\frac{\partial u\left(x, y\right)}{\partial y_i}\right|^2 + F\left(y, u\left(x, y\right)\right)\right)dx = 0. \qquad (2.5)$$

Proof: To prove (2.5), it suffices to check that

$$\frac{\partial}{\partial x_j} I\left(x\right) = 0,$$

for

$$u(x, s) = 0,\ (x, s) \in \mathbb{R}^m \times \partial D,$$

or

$$\frac{\partial u}{\partial n}(x, s) = 0, (x, s) \in \mathbb{R}^m \times \partial D.$$

Indeed, let us examine for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ the expressions

$$\frac{\partial u}{\partial y_i}\frac{\partial u}{\partial x_j}\left(x, y_i^{\alpha_i}\right) \text{ and } \frac{\partial u}{\partial y_i}\frac{\partial u}{\partial x_j}\left(x, y_i^{\beta_i}\right).$$

If we suppose that $u(x, s) = 0$, for $(x, s) \in \mathbb{R}^m \times \partial D$, it is known that

$$\nabla u = \frac{\partial u}{\partial n} \cdot n \text{ on } \partial\Omega,$$

i.e.

$$\begin{bmatrix} \frac{\partial u}{\partial x_1}(x, s) \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial u}{\partial x_m}(x, s) \\ \frac{\partial u}{\partial y_1}(x, s) \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial u}{\partial y_n}(x, s) \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ n_1 \frac{\partial u}{\partial n}(x, s) \\ \cdot \\ \cdot \\ \cdot \\ n_n \frac{\partial u}{\partial n}(x, s) \end{bmatrix}, (x, s) \in \mathbb{R}^m \times \partial D.$$

Consequently,

$$\frac{\partial u}{\partial x_j}\left(x, y_i^{\alpha_i}\right) = \frac{\partial u}{\partial x_j}\left(x, y_i^{\beta_i}\right) = 0, 1 \le i \le n, 1 \le j \le m.$$

Now if the boundary condition is $\frac{\partial u}{\partial n}(x, s) = 0$, for $(x, s) \in \mathbb{R}^m \times \partial D$, then

$$\frac{\partial u}{\partial n} = \langle \nabla u, n \rangle = 0 \text{ on } \Gamma_{\alpha_1} \cup \Gamma_{\beta_1} \cup ... \cup \Gamma_{\alpha_n} \cup \Gamma_{\beta_n},$$

i.e

$$\frac{\partial u}{\partial y_i}\left(x, y_i^{\alpha_i}\right) = \frac{\partial u}{\partial y_i}\left(x, y_i^{\beta_i}\right) = 0,\, x \in \mathbb{R}^m,\, i = 1, ..., n.$$

In both cases,

$$\frac{\partial}{\partial x_j} I(x) = 0,\, x \in \mathbb{R}^m,$$

and we conclude as was done in the previous lemma.

3. MAIN RESULTS

We examine now our main result which is stated as follows.

Theorem 1: Let

$$u : \Omega \to \mathbb{R},$$

be a solution of problem (1.1)–(1.4). Assume that

$$u \in H^2(\Omega) \cap L^\infty(\Omega);$$

and f verifying the following condition

$$2F(y, u) - uf(y, u) \leq 0. \tag{3.1}$$

Then

$$E(x) = \int_D |u(x, y)|^2 \, d_y \text{ is subharmonic in } \mathbb{R}^m.$$

Proof: It is easy to see that almost everywhere in $\mathbb{R}^m \times D$, we have

$$\left(u \frac{\partial^2 u}{\partial x_j^2}\right)(x, y) = \left(\frac{1}{2} \frac{\partial^2 (u^2)}{\partial x_j^2} - \left|\frac{\partial u}{\partial x_j}\right|^2\right)(x, y). \tag{3.2}$$

In fact by multiplying equation (1.1) by $\frac{u}{2}$ and integrating the new equation over D, we obtain

$$0=\int_D\left(-\frac{1}{4}\sum_{j=1}^{m}\frac{\partial^2\left(u^2\right)}{\partial x_j^2}-\frac{1}{2}\sum_{j=1}^{m}\left|\frac{\partial u}{\partial x_j}\right|^2+\frac{1}{2}\sum_{i=1}^{n}c_i u\frac{\partial^2 u}{\partial y_i^2}+\frac{1}{2}uf\left(y,u\right)\right)\left(x,y\right)dy.$$

However,

$$\begin{aligned}\sum_{i=1}^{n}\int_D c_i u\frac{\partial^2 u}{\partial y_i^2}\left(x,y\right)dy &= -\sum_{i=1}^{n}c_i\int_D\left|\frac{\partial u}{\partial y_i}\right|^2\left(x,y\right)dy\\ &\quad+\sum_{i=1}^{n}c_i\int_{D_i^*}\left(\left(u\frac{\partial u}{\partial y_i}\right)\left(x,y_i^{\beta_i}\right)-\left(u\frac{\partial u}{\partial y_i}\right)\left(x,y_i^{\alpha_i}\right)\right)dy_i^*\\ &= -\sum_{i=1}^{n}c_i\int_{D_i^*}\left|\frac{\partial u}{\partial y_i}\right|\left(x,y\right)dy\\ &\quad-\frac{1}{\varepsilon}\sum_{i=1}^{n}c_i\int_{D_i^*}\left(\left|u\left(x,y_i^{\alpha_i}\right)\right|^2+\left|u\left(x,y_i^{\beta_i}\right)\right|^2\right)dy_i^*\end{aligned}$$

Thus we get

$$\int_D\left(-\frac{1}{4}\sum_{j=1}^{m}\frac{\partial^2\left(u^2\right)}{\partial x_j^2}+\frac{1}{2}\sum_{j=1}^{m}\left|\frac{\partial u}{\partial x_j}\right|^2+\frac{1}{2}\sum_{i=1}^{n}c_i\left|\frac{\partial u}{\partial y_i}\right|^2+\frac{1}{2}uf\left(y,u\right)\right)\left(x,y\right)dy$$

$$=\frac{1}{2\varepsilon}\sum_{i=1}^{n}c_i\int_{D_i^*}\left(\left|u\left(x,y_i^{\alpha_i}\right)\right|^2+\left|u\left(x,y_i^{\beta_i}\right)\right|^2\right)dy_i^*$$

Lemma 1 implies that

$$\int_D\left(-\frac{1}{4}\sum_{j=1}^{m}\frac{\partial^2\left(u^2\right)}{\partial x_j^2}+\frac{1}{2}\sum_{j=1}^{m}\left|\frac{\partial u}{\partial x_j}\right|^2+\frac{1}{2}\sum_{i=1}^{n}c_i\left|\frac{\partial u}{\partial y_i}\right|^2+\frac{1}{2}uf\left(y,u\right)\right)\left(x,y\right)dy$$

$$= \int_D \left(-\frac{1}{2}\sum_{j=1}^{m}\left|\frac{\partial u}{\partial x_j}\right|^2 + \frac{1}{2}\sum_{i=1}^{n} c_i \left|\frac{\partial u}{\partial y_i}\right|^2 + F(y,u) \right)(x,y)\,dy$$

Hence we have

$$\int_D \left(-\frac{1}{4}\sum_{j=1}^{m}\frac{\partial^2 (u^2)}{\partial x_j^2} + \sum_{j=1}^{m}\left|\frac{\partial u}{\partial x_j}\right|^2 - F(y,u) + \frac{1}{2}uf(y,u) \right)(x,y)\,dy = 0,$$

i.e

$$-\frac{1}{4}\sum_{j=1}^{m}\frac{\partial^2}{\partial x_j^2} + \left(\int_D |u(x,y)|^2\,dy \right) = \int_D \left(-\sum_{j=1}^{m}\left|\frac{\partial u}{\partial x_j}\right|^2 + F(y,u) - \frac{1}{2}uf(y,u) \right)(x,y)\,dy.$$

The hypothesis (3.1) implies that

$$\sum_{j=1}^{m}\frac{\partial^2 E(x)}{\partial x_j^2} \geq 4\sum_{j=1}^{m}\int_D \left|\frac{\partial u(x,y)}{\partial x_j}\right|^2 dy \geq 0 \text{ in } \mathbb{R}^m,$$

This completes the proof.

Remark 1: The Maximum Principle implies the triviality of the solution $u(x, y)$ of the problem (1.1)–(1.4).

Theorem 2: The result of Theorem 1 is still true if Robin condition (1.4) is replaced by Dirichlet condition (1.2) or Neumann condition (1.3).

Proof: By similar arguments as in the proof of Theorem 1, we obtain

$$\int_D \left(-\frac{1}{4}\sum_{j=1}^{m}\frac{\partial^2 (u^2)}{\partial x_j^2} + \frac{1}{2}\sum_{j=1}^{m}\left|\frac{\partial u}{\partial x_j}\right|^2 + \frac{1}{2}\sum_{i=1}^{n} c_i \left|\frac{\partial u}{\partial y_i}\right|^2 + \frac{1}{2}uf(y,u) \right)(x,y)\,dy$$

$$= -\frac{1}{2}\sum_{i=1}^{n} c_i \int_{D_i^*} \left(\left(u\frac{\partial u}{\partial y_i} \right)(x, y_i^{\beta_i}) - \left(u\frac{\partial u}{\partial y_i} \right)(x, y_i^{\alpha_i}) \right) dy_i^*$$

If $u(x, s) = 0$ or $\frac{\partial u}{\partial n}(x, s) = 0$, for $(x, s) \in \mathbb{R}^m \times \partial D$ this formula reduces to

$$\int_D \left(-\frac{1}{4}\sum_{j=1}^{m} \frac{\partial^2 (u^2)}{\partial x_j^2} + \frac{1}{2}\sum_{j=1}^{m} \left| \frac{\partial u}{\partial x_j} \right|^2 - \frac{1}{2}\sum_{i=1}^{n} c_i \left| \frac{\partial u}{\partial y_i} \right|^2 + \frac{1}{2} uf(y, u) \right)(x, y)\, dy = 0.$$

We can now employ (2.4) to transform this identity into the following form

$$-\frac{1}{4}\sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} \int_D u^2(x, y)\, dy = \int_D \left(-\sum_{j=1}^{m} \left| \frac{\partial u}{\partial x_j}^2 \right| + F(y, u) - \frac{1}{2} uf(y, u) \right)(x, y))\, dy$$

Our assumption on f implies the desired result.

4. EXAMPLES

In this section we illustrate our theoretical results by some examples:

Example 1: The problem

$$\begin{cases} -\sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} - \sum_{i=1}^{n} c_i \frac{\partial^2 u}{\partial y_i^2} + \theta(y)|u|^{p-1} u = 0 \ in\ \Omega = \mathbb{R}^m \times D, \\ \left(u + \varepsilon \frac{\partial u}{\partial n} \right)(x, s) = 0, (x, s) \in \mathbb{R}^m \times \partial D, \end{cases} \tag{4.1}$$

where

$$\theta : \overline{D} \to \mathbb{R},$$

is a nonnegative continuous real function and $p \geq 1$ does not have nontrivial solutions:

Indeed,

$$2F(y, u) - uf(y, u) = \theta(y)\left(\frac{2}{p+1} - 1 \right)|u|^{p+1} \leq 0.$$

and then applying Theorem 1.

Example 2: Let $\rho : D \to \mathbb{R}$, be a continuous function and $\lambda \in \mathbb{R}$. The problem

$$\left| \begin{array}{l} -\sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} - \sum_{i=1}^{n} c_i \frac{\partial^2 u}{\partial y_i^2} + \rho(y) u = 0 \text{ in } \mathbb{R}^m \times D, \\ \left(u + \varepsilon \frac{\partial u}{\partial n} \right)(x, s) = 0, (x, s) \in \mathbb{R}^m \times \partial D, \end{array} \right. \tag{4.2}$$

considered in $H^2(\mathbb{R}^m \times D) \cap L^\infty(\mathbb{R}^m \times D)$ does not have nontrivial solutions:

In this case we observe that

$$2F(y, u) - uf(y, u) \equiv 0,$$

and

$$-\frac{1}{4} \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} \left(\int_D |u(x, y)|^2 \, dy \right) = \int_D \left(-\sum_{j=1}^{m} \left| \frac{\partial u}{\partial x_j} \right|^2 + F(y, u) - \frac{1}{2} uf(y, u) \right)(x, y) \, dy.$$

$$= -\int_D \sum_{j=1}^{m} \left| \frac{\partial u(x, y)}{\partial x_j} \right|^2 dy$$

i.e.

$$\sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} \left(|u(x, y)|^2 \, dy \right) = -\int_D \sum_{j=1}^{m} \left| \frac{\partial u(x, y)}{\partial x_j} \right|^2 dy \geq 0.$$

Example 3: Let

$$\theta_1, \theta_2 : \overline{D} \to \mathbb{R},$$

be are two continuous nonnegative functions, p, q ≥ 1 and

$$f(y, u) = mu + \theta_1(y) \, |u|^{p-1} u + \theta_2(y) \, |u|^{q-1} u), \; m \in \mathbb{R}.$$

The problem

$$\left| \begin{array}{l} -\sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} - \sum_{i=1}^{n} c_i \frac{\partial^2 u}{\partial y_i^2} + f(y, u) = 0 \text{ in } \mathbb{R}^m \times D, \\ \left(u + \varepsilon \frac{\partial u}{\partial n} \right)(x, s) = 0, (x, s) \in \mathbb{R}^m \times \partial D, \end{array} \right. \tag{4.3}$$

does not have nontrivial solutions.

It suffices to see that

$$2F(y,u)-uf(y,u)=\theta_1(y)\left(\frac{2}{p+1}-1\right)|u|^{p+1}+\theta_2(y)\left(\frac{2}{q+1}-1\right)|u|^{q+1}\leq 0,$$

and then applying Theorem 1.

REFERENCES

[1] Adams (R.A.), *Sobolev Spaces*, Academic Press, (1975).

[2] H. Berestycky, T. Gallouet and O. Kavian, Equations de champs scalaires non-linéaires dans le plan. C.R. Acad. Sc. Paris, Série 1, **297** (1983), 307-310.

[3] G. Caristi and E. Mitidieri, Nonexistence of Positive Solutions of Quasilinear Equations. *Adv. in Diff. Equations*, **2** (1997), 319-359.

[4] M.J. Esteban and P.L. Lions, Existence and Nonexistence Results for Semi Linear Elliptic Problems in Unbounded Domains. Proc. Roy. Soc. Edimburgh, **93-A** (1982), 1-14.

[5] D.G. De Figueiredo and Y. Jianfu, Decay Symetry and Existence of Solutions of Semilinear Elliptic Systems, *Nonlinear Anal.*, **33**, (1998), 211-234.

[6] A. Haraux and B. Khodja, Caractère Trivial de la solution de certaines équations aux dérivées partielles non linéaires dans des ouverts cylindriques de $\mathbb{R}^N$. *Portugaliae Mathematica.*, **42** Fasc. 2, (1982), 127-136.

[7] N. Kawarno, W. NI and Syotsutani, Generalised Pohozaev Identity and Its Applications. *J. Math. Soc. Japan.*, **42(3)** (1990), 541-563.

[8] B. Khodja, Equations Elliptiques Semilinéaires dans des ouverts non bornés de RN. *Portugaliae Mathématica.*, **53** Fasc. **4** (1996), 457-470.

[9] B. Khodja, Nonexistence of Solutions for Semilinear Equations and Systems in Cylindrical Domains. *Comm. Appl. Nonlinear Anal.*, (2000), 19-30.

[10] E. Mitidieri, Nonexistence of Positive Solutions of Semilinear Elliptic Systems in RN. *Diff. and Int. Equations*, **9** (1996), 465-479.

[11] W. NI. and J. Serrin, Nonexistence Theorems for Quasilinear Partial Differential Equations. Red. Circ. Mat. Palermo, *Suppl. Math.*, **8** (1985), 171-185.

[12] M. Otani, Existence and Nonexistence of Nontrivial Solutions of Some Nonlinear Degenerate Elliptic Equations, *J. Functional Analysis*, **76(1)** (1988), 140-159.

[13] M.H. Protter and H.F. Weinberger, *Maximum Principle in Differential Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, (1967).

[14] S.I. Pohozaev, Eigenfunctions of the Equation $\Delta u+\lambda f(u)=0$. *Soviet. Math. Dokl.*, **6** (1965), 1408-1411.

[15] C. Yarur, Nonexistence of Positive Singular Solutions for a Class of Semilinear Elliptic Systems. *Electronic Journal of Diff. Equations*, **8** (1996), 1-22.

Brahim Khodja
Department of Mathematics,
Badji Mokhtar University,
Annaba, B.P.12, Algeria.
E-mail: bmkhodja@yahoo.fr