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A COMMON FIXED POINT THEOREM SATISFYING IMPLICIT RELATIONS AND ITS APPLICATION

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ABSTRACT: In this note, a general common fixed point theorem for a class of pairwise weakly compatible mappings satisfying implicit relations is obtained in a complete *s*-metric space using the notion of two mappings satisfying the (E.A) property. The result improves and extends several known results. An application of our result to product spaces is also discussed.

Keywords and Phrases: *s*-metric space, common fixed points, weakly compatible mappings, implicit relations, the (E.A) property.

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1. INTRODUCTION

Commutativity of two mappings is weakened by Sessa [11] with weakly commuting mappings. Jungck [4] enlarged the class of non-commuting mappings by *compatible mappings* which asserts that a pair of self mappings S and T of metric space (X, d) is said to be compatible if

$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0, \tag{1.1}$$

whenever $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \in X$. The concept of compatible mappings is further improved by Jungck and Rhoades [6] with *coincidentally commuting (or weakly compatible) mappings* which merely commute at their coincidence points.

In recent years, several common fixed point theorems for contractive type mappings have been established by several authors (see, for instance, Jachymski [3], Jungck *et al.* [5] and Pant [7]).

Recently, Pant [7] weakened the continuity of one mapping by the notion of reciprocal continuity which asserts that two mappings S and T are said to be *reciprocally continuous* if

$$\lim_{n\to\infty} STx_n = St \text{ and } \lim_{n\to\infty} TSx_n = Tt \tag{1.2}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Using the concept of reciprocal continuity, Popa [9, 10] proved some fixed point theorems satisfying some implicit relations. Recently, A. Aliouche *et al.* [2]

established a general common fixed point theorem for pair of reciprocally continuous mappings satisfying an implicit relation.

2. IMPLICIT RELATIONS

Let $\mathcal{F}_6$ be the set of all continuous functions $F(t_1, \ldots, t_6): \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

$$(F_a): F(u, 0, u, 0, u, 0) \le 0 \quad \text{implies } u = 0;$$

$$(F_b): F(u, 0, 0, u, 0, u) \le 0 \quad \text{implies } u = 0;$$

The function $F(t_1, \ldots, t_6): \mathbb{R}_+^6 \to \mathbb{R}$ satisfies the condition (F_1) if

$$(F_1): F(u, u, 0, 0, u, u) > 0; \text{ for all } u > 0.$$

Example 2.1: Let $F(t_1 \ldots, t_6) : t_1 - a(t_2 + t_4) - bt_5 - c(t_3 + t_6)$, where $a, b, c \ge 0$, $a + b + c < 1$.

Proof:

$(F_a) : F(u, 0, u, 0, u, 0) = u(1 - a - b - c) \le 0$ implies $u = 0$;

$(F_b) : F(u, 0, 0, u, 0, u) = u(1 - a - c) \le 0$ implies $u = 0$;

and

$(F_1) : F(u, u, 0, 0, u, u) = u(1 - a - b - c) > 0$; for all $u > 0$.

Example 2.2: Let $F(t_1, \ldots, t_6) : \{t_1 - \min\{t_5, (t_2 + t_4)\}$; where $0 < k < 1$.

Proof:

$(F_a) : F(u, 0, u, 0, u, 0) = u(1 - k) \le 0$ implies $u = 0$;

$(F_b) : F(u, 0, 0, u, 0, u) = u(1 - k) \le 0$ implies $u = 0$;

and

$(F_1) : F(u, u, 0, 0, u, u) = u(1 - k) > 0$; for all $u > 0$.

Aliouche *et al.* [2] proved the following theorem:

Theorem 2.3: Let $\{A_i\}$, $i = 1, 2, \ldots, S$ and T be self mappings of a complete metric space (X, d) such that

$$A_1(X) \subset T(X) \text{ and } A_i(X) \subset S(X), \text{ if } i > 1; \tag{2.1}$$

There exists $F \in \mathcal{F}_6$ such that

$$F(d(A_1x, A_iy), d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), d(A_1x, Ty), d(Sx, A_iy)\}) \leq 0,$$

($i = 1, 2, ...$) for all $x, y \in X$, where F satisfies the following properties:

(F_1) : F is non-increasing in t_5 and t_6.

(F_a) : $F(u, 0, u, 0, u, 0) \leq 0$ implies $u = 0$;

(F_b) : $F(u, 0, 0, u, 0, u) \leq 0$ implies u = 0;

(F_2) : There exists a $h \in (0, 1)$ such that for $u, v \geq 0$ with

(F_{2a}) : $F(u, v, v, u, u + v, 0) \leq 0$ or

(F_{2b}) : $F(u, v, u, v, 0, u + v) \leq 0$ implies $u \leq hv$.

(F_3) : $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

Let S be compatible with A_1 and T be compatible with A_k, for some $k > 1$. If one of the mapping of the compatible pairs $\{A_1, S\}$ and $\{A_k, T\}$ are reciprocally continuous, then $\{A_i\}$, S and T have a common fixed point.

Before presenting our main result, we have the following definition:

Definition 2.4[1]: Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy the property $(E.A)$, if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Example 2.5: Let X be the closed interval $[0, 1]$ with usual metric. Define functions S and T by

$$Sx = \begin{cases} 1-2x & \text{for } x \in \left[0, \frac{1}{2}\right) \\ 0, & \text{for } x \in \left[\frac{1}{2}, 1\right] \end{cases},$$

and

$$Tx = \begin{cases} 1-\frac{1}{2}x & \textit{for } x \in \left[0, \frac{1}{2}\right) \\ 1 & \textit{for } x = \left[\frac{1}{2}, 1\right] \end{cases}.$$

Here $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 1$ for the sequence $\{x_n\} = \frac{1}{n}$. Thus the pair of mappings S and T satisfy the (E.A) property. On the other hand $\lim_{n\to\infty} STx_n = -1$ and $\lim_{n\to\infty} TSx_n = \frac{1}{2}$; show that S and T are not reciprocal continuous mappings.

Now we introduce the notion of s–metric space.

Definition 2.6: A *symmetric function* on a set X is a real-valued function d on $X \times X$ such that for all $x, y \in X$,

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

We call the pair (X, d) an *s–metric space* if d is a symmetric function on X.

Example 2.7: Let $X = \mathbb{R}$ and d is a real-valued function on $X \times X$ defined by

$$d(x, y) = (x - y)^2 \text{ for all } x, y \text{ in } X$$

It is easy to prove that d is a symmetric function on X.

The main object of this paper is to prove a common fixed point theorem for a quadruple of mappings satisfying an implicit relation in a complete s-metric space. The reciprocally continuity of pair of mappings in the main result of Aliouche *et al.* [2] is replaced by the notion of the pair of mappings satisfying the (E.A) property.

3. MAIN RESULTS

Now we state and prove our main result as follows.

Theorem 3.1: Let (X, d) be a complete s-metric space and $\{A_i\}$, $i = 1, 2, ..., S$, and T be self mappings on X such that the following hold:

$$A_1(X) \subset T(X) \text{ and } A_i(X) \subset S(X), \text{ for } i > 1; \tag{3.1}$$

There exists $F \in F_6$ such that

$$F(d(A_1x, A_iy), d(Sx, Ty), d(A_1x, Sx), d(A_1y, Ty), d(A_1x, Ty), d(Sx, A_iy)\}) \leq 0, \tag{3.2}$$

for all $x, y \in X$, where F satisfies properties (F_a), (F_b) and (F_1).

$$\{A_1, S\} \text{ and } \{A_k, T\},\ k > 1 \text{ be weakly compatible mappings,} \tag{3.3}$$

$$\{A_1, S\} \text{ or } \{A_k, T\},\ k > 1 \text{ satisfies the property (E.A),} \tag{3.4}$$

$$S(X) \text{ or } T(X) \text{ is complete,} \tag{3.5}$$

then $\{A_i\}$, $i = 1, 2, ...$, S and T have a unique common fixed point.

Proof: Suppose $\{A_k, T\}$, $k > 1$ satisfies the property (E.A). Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} A_k x_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

Since $A_k(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in X, such that $A_k x_n = Sy_n$. Thus $\lim_{m\to\infty} Sy_n = t$.

From (3.2), we have for $x = y_n$ and $y = x_n$,

$$F(d(A_1 y_n, A_k x_n), d(Sy_n, Tx_n), d(A_1 y_n, Sy_n), d(A_k x_n, Tx_n),$$
$$d(A_1 y_n, Tx_n), d(Sy_n, A_k x_n)) \le 0.$$

Taking $n \to \infty$ we get

$$\lim_{n\to\infty} F(d(A_1 y_n, t), d(t, t), d(A_1 y_n, t), d(t, t), d(A_1 y_n, t), d(t, t)) \le 0,$$

which, by using the condition (F_a) implies

$$\lim_{n\to\infty} F(d(A_1 y_n, t), 0, d(A_1 y_n, t), 0, d(A_1 y_n, t), 0) \le 0$$

i.e.,

$$\lim_{n\to\infty} A_1 y_n = t.$$

Now suppose that $S(X)$ is a complete subspace of X. Then $t = Su$, for some $u \in X$. Subsequently, we have

$$\lim_{n\to\infty} A_1 y_n = \lim_{n\to\infty} A_k x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Su.$$

Again from (3.2), we have

$$F(d(A_1 u, A_k X_n), d(Su, Tx_n), d(A_1 u, Su), d(A_k x_n, Tx_n),$$
$$d(A_1 u, Tx_n), d(Su, A_k x_n)) \le 0.$$

Taking $n \to \infty$ in the above implicit relation, we get

$$F(d(A_1 u, Su), d(Su, Su), d(A_1 u, Su), d(Su, Su), d(A_1 u, Su), d(Su, Su)) \le 0.$$

which, by using the condition (F_a) implies

$$F(d(A_1 u, Su), 0, d(A_1 u, Su), 0, d(A_1 u, Su), 0) \le 0$$

i.e.,

$$A_1 u = Su.$$

Since A_1 and S are weakly compatible mappings, we have $A_1Su = SA_1u$ whenever $A_1u = Su$ and hence $A_1A_1u = A_1Su = SA_1u = SSu$. Also since $A_1(X) \subset T(X)$, there exists $v \in X$ such that $A_1u = Tv$.

Now we show that $Tv = A_kv$. From (3.2), we have

$$F(d(A_1u, A_kv), d(Su, Tv), d(A_1u, Su), d(A_kv, Tv), d(A_1u, Tv), d(Su, A_kv)) \leq 0$$

i.e.,

$$F(d(Tv, A_kv), 0, 0, d(A_kv, Tv), 0, d(Tv, A_kv)) \leq 0,$$

which, by using the condition (F_b) implies

$$A_kv = Tv.$$

Thus we have

$$A_1u = Su = A_kv = Tv.$$

The weak compatibility of A_k and T gives $A_KTv = TA_kv$ whenever $A_kv = Tv$ and hence $TTv = TA_kv = A_kTv = A_kA_kv$. Now we show that A_kv is a common fixed point of the mappings A_k, $k \leq 1$, S and T.

In view of (3.2), it follows

$$F(d(A_1u, A_kA_kv), d(Su, TA_kv), d(A_1u, Su), d(A_kA_kv, TA_kv),$$
$$d(A_1u, TA_kv), d(Su, A_kA_kv)) \leq 0,$$

i.e.,

$$F(d(A_kv, A_kA_kv), d(A_kv, A_kA_kv), 0, 0, d(A_kv, A_kA_kv), d(A_kv, A_kA_kv)) \leq 0,$$

which contradicts (F_1). Therefore we have $A_kv = A_kA_kv = TA_kv$. Thus A_kv is a common fixed point of A_k and T. Similarly A_1u is a common fixed point of A_1 and S. Since $A_kv = A_1u$, we conclude that A_kv is a common fixed point of $\{A_i\}$, $i = 1, 2, ..., S$ and T. The proof is similar when $T(X)$ is assumed to be a complete subspace of X.

Corollary 3.2: Let (X, d) be a complete s-metric space and A, B, S and T be self mappings on (X, d) such that the following hold:

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X); \tag{3.6}$$

There exists $F \in \mathcal{F}_6$ such that

$$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty),$$
$$d(Ax, Ty), d(Sx, By)\}) \leq 0, \tag{3.7}$$

for all $x, y \in X$, where F satisfies properties (F_a), (F_b) and (F_1). Suppose that the pair $\{A, S\}$ and $\{B, T\}$ are weakly compatible mappings and satisfying the property (E.A). Suppose $S(X)$ or $T(X)$ is complete, then A, B, S and T have a unique common fixed point.

4. AN APPLICATION TO PRODUCT SPACES

In this section we discuss the application of our result.

Theorem 4.1: Let (X, d) be a complete s-metric space and A, B, S and T be mappings of $X \times X$ into X such that the following hold:

$$A(X \times X) \subset T(X \times X) \text{ and } B(X \times X) \subset S(X \times X); \tag{4.1}$$

There exists $F \in \mathcal{F}_6$ such that

$$F(d(A(x, y), B(p, q)), d(S(x, y), T(p, q), d(A(x, y), S(x, y)), d(B(p, q), T(p, q)), d(A(x, y), T(p, q)), d(S(x, y), B(p, q))\}) \leq 0, \tag{4.2}$$

for all $x, y \in X$, where F satisfies properties (F_a), (F_b) and (F_1). Suppose that the pair $\{A, S\}$ and $\{B, T\}$ are weakly compatible mappings and satisfying the property (E.A). Suppose $S(X)$ or $T(X)$ is complete. Also, let G satisfies the following inequality

$$d(G(x, y), G(x', y')) \leq \alpha(d(y, y'))\, d(y, y') \tag{4.3}$$

for all $(x, y), (x', y'), (y, y') \in X \times X$, where G denotes one of the mappings A, B, S or T, $\alpha : [0, \infty) \to [0, 1]$ is a monotonic non-decreasing function with

$$\lim_{t \to \infty} \sup_{t \geq t_0} \alpha(t) = \alpha < 1,$$

where $t_0 = \sup_{(y, y') \in X \times X} d(y, y')$. Then A, B, S and T have a unique common fixed point.

Proof: It follows from the relation (4.2) that

$$F(d(A(x, y), B(p, y)), d(S(x, y), T(p, y)), d(A(x, y), S(x, y)), d(B(p, y), T(p, y)), d(A(x, y), T(p, y)), d(S(x, y), B(p, y))\}) \leq 0,$$

for all $x, y, p \in X$. Therefore, by Corollary 3.2, for each $y \in X$, there exist $z(y) \in X$ such that

$$A(z(y), y) = S(z(y), y) = B(z(y), y) = T(z(y), y) = z(y). \tag{4.4}$$

Now taking $x = z(y)$ and $x' = z(y')$ in (4.3), we obtain

$$d(z(y), z(y')) \leq \alpha(d(y, y'))\, d(y, y'),$$

$$\leq \alpha\left(\sup_{(y,y')\in X\times X} d(y,y')\right)d\left(y,y'\right)=\alpha\left(t_0\right)d\left(y,y'\right)$$

$$\leq \lim_{t\to\infty}\sup_{t\geq t_0}\alpha\left(t\right)d\left(y,y'\right)$$

$$\leq \alpha d(y,y').$$

Then it follows from the well known Banach contraction principle that the mapping $z(.)$ has a common fixed point $w \in X$, i.e. $w = z(w)$, which implies by (4.4) that

$$A(w, w) = S(w, w) = B(w, w) = T(w, w) = w.$$

For uniqueness of fixed point, let w and w' be two distinct common fixed points of A, B, S and T, then taking $x = y = w$ and $p = q = w'$ in (4.2), we have

$$F(d(A(w, w), B(w', w')), d(S(w, w), T(w', w')), d(A(w, w), S(w, w)),$$
$$d(B(w', w'), T(w', w')), d(A(w, w), T(w', w')), d(S(w, w), B(w', w'))) \leq 0,$$

or

$$F(d(w, w'),\ d(w, w'), d(w, w), d(w', w'), d(w, w'), d(w, w')) \leq 0,$$

or

$$F(d(w, w'), d(w, w'), 0, 0, d(w, w'), d(w, w')) \leq 0,$$

which contradicts (F_1). Hence the result.

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