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Dark Energy Models in Alternative Theories of Gravity

GIANLUCA ALLEMANDI & MAURO FRANCAVIGLIA

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This paper presents a review on new theories and new approaches, that have been recently investigated in deeper detail, to provide coherent alternative models for Dark Energy in the Cosmological framework. One of the most striking experimental result is nowadays the acceleration of the Universe. Physicists and Mathematicians have to face this experimental evidence and should give a coherent theoretical explanation to the striking experimental results. Dark Energy models are usually assumed to be a solution to this theoretical problem, even if they are (mainly formally) inconsistent from a physical viewpoint. The general procedure consists in assuming Einstein equations with matter and suitably introduce exotic matter to fit the experimental data.

New kinds of models have been recently introduced and studied, dealing with modifications of the gravitational contribution to the relativistic theory, rather than of the usual modifications of the matter contribution to the model, which are necessarily assumed to be some negative pressure dark matter fluids (so-called Dark Energy).

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I. INTRODUCTION

General Relativity (GR) was formulated by A. Einstein in 1916, on the bases of the theory of Mach and in the mathematical framework of Riemann geometry. It was originally intended as a geometric theory to describe the Universe. GR is based on three fundamental principles, which are by no means independent on each other:

- *Covariance principle* : The laws of nature are merely statements about spacetime coincidences; they therefore find their only natural expression in generally covariant equations.
- *Equivalence principle* : Inertia and weight are identical in nature. It follows necessarily from this and from the results of the Special Theory of Relativity that the symmetric fundamental metric tensor $g_{\mu\nu}$ determines the metrical properties of space, the inertial behaviour of bodies in it, as well as gravitational effects.
- *Mach's principle* : The g-field is completely determined by the masses of bodies.

Since mass and energy are identical in accordance with the results of the Special Theory of Relativity and the energy is described formally by means of the symmetric energy tensor $T_{\mu\nu}$, this means that the g-field is conditioned and determined by the energy tensor of matter.

These three principles represent the philosophical background for General Relativity and they imply a deep connection between the (curved) geometry of spacetime, gravitational forces and masses. They set the bases not only for General Relativity, but they are a general framework to describe the Physics of Nature, a general requirement that each field theory has to satisfy.

The Equivalence Principle gives $g_{\mu\nu}$ a physical meaning and requires the theory to be a field theory in which the metric tensor is one of the fundamental fields. It can be reformulated from a local point of view by saying that: *in each infinitesimal region of spacetime (such that the variation of the gravitational field can be neglected in it) there exists a local coordinate system in which each physical phenomenon is independent on gravity*.

Mach's Principle otherwise states that the components of the metric tensor are deeply related to the matter fields and in this context the matter and the geometry of spacetime interact with each other. Matter fields can influence and change the geometry of spacetime. This concept is the translitteration of the equivalence between inertial mass and gravitational mass. It is usually assumed that GR is the only geometric theory, able to describe the experimental results: new and strange experimental results are usually interpreted by means of modifications of the matter content of the Universe. However, in the framework of the same Mach's principle, it is otherwise possible to consider the case of modifications of the geometric content of the theory, reproducing somehow modifications of matter as we will see later. This is extremely important when matter contributions are physically non-consistent.

The fundamental Einstein's field equations of motion can be derived from those three fundamental principles, using a heuristic proof[1] or using a variational principle as we will see later. The metric is then related to the energy-momentum tensor of matter according to Mach's principle:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 4\pi T_{\mu\nu} \tag{1}$$

Here $G_{\mu\nu}$ is the *Einstein tensor*, $R_{\mu\nu}$ is the symmetric Ricci tensor, $R = R_{\mu\nu}g^{\mu\nu}$ is the scalar curvature and $T_{\mu\nu}$ is the stress energy tensor of matter. We stress that each object in the first two terms of (1) is deeply related to the geometry of spacetime. General Relativity can be otherwise constructed, setting the metric g and the matter fields ϕ as fundamental fields, as a Lagrangian field theory. The Lagrangian chosen is second order in the metric and first order in the matter fields and it is presented as the sum of a gravitational Lagrangian L_H and a matter Lagrangian

L_{mat}:

$$L = L_H(g, \partial g, \partial^2 g) + L_{\text{mat}}(g, \phi, \partial\phi) \tag{2}$$

The standard Hilbert Lagrangian for the gravitational field is set to be:

$$L_H = \frac{1}{2k}\sqrt{g}Rds \tag{3}$$

where R is the Ricci's curvature scalar, $\sqrt{g}$ denotes $|\det \| g_{\mu\nu} \||^{\frac{1}{2}}$ and ds is a local volume element of spacetime. Hamilton's principle, a variational principle based on the minimal action principle, will provide the correct Einstein's field equations.

Even if General Relativity is worldwide considered as the geometric theory for spacetime some alternative theories, based on the same principles, but on different Lagrangians, have recently assumed some importance. In this paper we try to better understand and to analyze possible cosmological applications of these alternative theories of Gravity in relation with their capability to explain the cosmological acceleration of the Universe, both in early times (inflation) and in present time universes. This is equivalent to state that these models represent alternatives for Dark Energy, as we will explain in the following. We will focus our attention on the possible theoretical explanations of the present cosmological acceleration.

Recent astronomical observations have shown in fact that the universe is accelerating at present time (see [1] and [2] for supernova observation results; see [3] for the observations about the anisotropy spectrum of the cosmic microwave background (CMBR); see [4] for the results about the power spectrum of large-scale structure). Physicists have thus to face the evidence of the acceleration of the Universe and should give a coherent theoretical explanation to those experimental results. The first attempts to explain accelerating models of the Universe where made in the context of *dark energy theories*. The real nature of dark energy, which is required by General Relativity in this cosmological context, is unknown but it is fairly well accepted that dark energy should behave like a fluid with a large negative pressure. The dark energy models with effective equation of state w_{eff} (which determines the relation between pressure p and density of matter p) smaller than $w_{eff} < -1$ are currently preferrable, owing to the experimental results of [3].

Another possibility to explain the physical evidence is to assume that we do not yet understand Gravity at large scales, which suggests to us that General Relativity should be modified or replaced by alternative gravitational theories of Gravity when the curvature of spacetime is small (see for example [8], [9], [10] and references therein), thus providing modified Friedmann equations. Hints in this direction are suggested moreover from the quantization on curved spacetimes, when interactions among the quantum fields and the background geometry or the self interaction of the gravitational field are considered. It follows that the standard Hilbert-Einstein Lagrangian has to be suitably modified by means of corrective terms, which are essential in order to remove divergences [8]. These corrective terms turn out to be higher-order terms in the curvature invariants (such as R, $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, $R\Box^{l}R$), or non-minimally coupled terms between scalar fields and the gravitational field. It is moreover interesting that such corrective terms to the standard Hilbert-Einstein Lagrangian can be predicted in higher dimensions by some time-dependent compactification in string/M-theory (see [9]) and corrective terms of this type arise surely in brane-world models with large spatial extra dimensions [10]. As a matter of facts, if these brane models are the low energy limit of string theory, it is likely that the field equations should include in particular the Gauss-Bonnet term, which in five dimensions is the only non-linear term in the curvature which yields second order field equations. In this framework Gauss-Bonnet corrections should be seriously taken into account and cosmological models deriving from the Gauss-Bonnet term have been recently studied; see [12] and references therein.

As an alternative to extra dimensions it is also possible to explain the modification to Friedmann equations (which could provide a theoretical explanation for the acceleration of the Universe) by means of a modified theory of four-dimensional gravity. The first attempts in this direction were performed by adding to the standard Hilbert-Einstein Lagrangian some suitable analytical term in the Ricci scalar curvature invariant [11]. A simple task to modify General Relativity, when the curvature is very small, is hence to add to the Lagrangian of the theory a piece which is proportional to the inverse of the scalar curvature $\frac{1}{R}$ or to replace the standard Hilbert-Einstein action by means of polynomial-like Lagrangians, containing both positive and negative powers of the Ricci scalar R and logarithmic-like terms. Such theories have been analyzed and studied both in the metric [13] and the Palatini formalisms [14], [16]. It turns out hat both in the metric and the Palatini formalism they can provide a possible theoretical explanation to the present time acceleration of the universe. In these paper we also discuss briefly cosmological applications of $f(R)$ theories, referring to [14] and [15] for further references and discussions.

We start hereafter discussing $f(R)$ theories in the metric formalism. We then consider the case of $f(R)$ theories in the Palatini formalism and their cosmological applications as Dark Energy models. We will see as it is possible to interpret the present acceleration of the Univerrse in the framework of this class of models.

II. ALTERNATIVE THEORIES OF GRAVITY IN THE METRIC FORMALISM

We consider the case of $f(R)$ gravitational theories in the metric formalism. In general, fourthorder theories of gravity are given by an action of the general kind

$$\mathcal{A} = \int \sqrt{g} f(R), \tag{4}$$

where $f(R)$ is an arbitrary (analytic) function of the Ricci curvature scalar R. We are here considering the simplest case of fourth-order gravity but we could construct such kind of theories also using other invariants in $R_{\mu\nu}$ or $R^{\alpha}_{\gamma\mu\nu}$. The standard Hilbert–Einstein action is of course recovered for $f(R) = R$. Varying with respect to $g_{\alpha\beta}$, we get the field equations

$$f'(R)R_{\alpha\beta} - \frac{1}{2} f(R) g_{\alpha\beta} = f'(R)^{;\mu\nu}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}), \tag{5}$$

which are fourth-order equations due to the term $f'(R)^{;\mu\nu}$; the prime indicates the derivative with respect to R, while ; denotes covariant derivative with respect to the metric. By a suitable manipulation the above equation can be rewritten under the form:

$$G_{\alpha\beta} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\alpha\beta}[f(R) - Rf'(R)] + f'(R)_{;\alpha\beta} - g_{\alpha\beta} f'(R) \right\}, \tag{6}$$

where the gravitational contribution due to higher-order terms can be simply reinterpreted as a stress-energy tensor contribution. This means that additional and higher order terms in the gravitational action produce in the theory the same effects of a stress-energy tensor of matter, related to the chosen form of $f(R)$. This allows to reinterpret these theories as possible models for Dark Energy [11], [13]. The analytical form of $f(R)$, which can suitably chosen, rules the corresponding stress energy tensor of matter. In the more general case of matter theories, where a minimal interaction between the gravitational field and matter fields is present, i.e.

$$\mathcal{A} = \int \sqrt{g} f(R) + L_{mat}(\Psi), \tag{7}$$

than the stress-energy tensor due to higher order gravitational terms in the Lagrangian adds to the true stress-energy tensor $T_{\alpha\beta}$ of matter, giving:

$$G_{\alpha\beta} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\alpha\beta}[f(R) - Rf'(R)] + f'(R)_{;\alpha\beta} - g_{\alpha\beta} f'(R) \right\} + T_{\alpha\beta}, \tag{8}$$

where $T_{\alpha\beta} = -\frac{2}{\sqrt{g}}\frac{\delta L_{mat}}{\delta g_{\alpha\beta}}$. The case of standard General Relativity is of course reproduced for $f(R) = R$. In this particular case, in fact, the higher order terms in field equations vanish identically, due to the fact that the derivatives of $f(R)$ vanish. This is related with the degeneration of the Hilbert-Einstein Lagrangian.

III. FIRST ORDER NON LINEAR $f(R)$ GRAVITY

We study hereafter the so-called Palatini formalism for alternative theories of Gravity, that we deeply investigated in view of cosmological applications [14, 15]. The action for $f(R)$ Gravity is introduced to be:

$$A = A_{grav} + A_{mat} = \int (\sqrt{\det g}\, f(R) + 2\kappa L_{mat}(\Psi)) d^4x \tag{9}$$

where $R \equiv R(g, \Gamma) = g^{\alpha\beta}R_{\alpha\beta}(\Gamma)$ is the *generalized Ricci scalar and* $R_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsionless connection Γ. The gravitational part of the Lagrangian is again controlled by a given real analytic function $f(R)$ of one real variable, that now depends on the two independent variable g and Γ. The total Lagrangian contains also a matter part L_{mat} in minimal interaction with the gravitational field, depending on matter fields Ψ together with their first derivatives and equipped with a gravitational coupling constant $\kappa = 8\pi G$.

Equations of motion, ensuing from the first order á la Palatini formalism are (we assume the spacetime manifold to be a Lorentzian manifold M with dim $M = 4$; see [39]):

$$f'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu} \tag{10}$$

$$\nabla_\alpha^\Gamma(\sqrt{\det g} f'(R)g^{\mu\nu}) = 0 \tag{11}$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta L_{mat}}{\delta g_{\mu\nu}}$ denotes again the matter source stress-energy tensor and ∇^Γ means now covariant derivative with respect to Γ.

We shall use the standard notation denoting by $R_{(\mu\nu)}$ the symmetric part of $R_{\mu\nu}$, i.e. $R_{(\mu\nu)} \equiv \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu})$; notice that for an arbitrary torsionless Γ the Ricci tensor is not a priori symmetric. In order to get (11) one has to additionally assume that L_{mat} is functionally independent on Γ; however it may contain metric covariant derivatives $\nabla^g\Psi$ of fields. This means that the matter stress-energy tensor $T_{\mu\nu} = T_{\mu\nu}(g, \Psi)$ depends on the metric g and some matter fields denoted here by Ψ, together with their derivatives. From (11) one sees that $\sqrt{\det g}\, f'(R) g^{\mu\nu}$ is a symmetric twice contravariant tensor density of weight 1, so that if not degenerate one can use it to define a metric $h_{\mu\nu}$ such that the following holds true

$$\sqrt{\det g}\, f'(R) g^{\mu\nu} = \sqrt{\det h}\, h^{\mu\nu} \tag{12}$$

This means that both metrics h and g are conformally equivalent. The corresponding conformal factor can be easily found to be $f'(R)$ (in dim $M = 4$) and the conformal transformation results to be:

$$h_{\mu\nu} = f'(R) g_{\mu\nu} \tag{13}$$

Therefore, as it is well known, equation (11) implies that $\Gamma = \Gamma_{LC}(h)$, i.e. Γ coincides with the Levi-Civita connection Γ_{LC} of the metric h defined by (12) and $R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(h) \equiv R_{\mu\nu}$. Let us now introduce a (1,1)-tensorfield P by

$$P^{\mu}_{\nu} = g^{\mu\alpha} R_{\alpha\nu}(h) \tag{14}$$

so that (10) re-writes as

$$f'(R) P^{\nu}_{\mu} - \frac{1}{2} f(R) \delta^{\nu}_{\mu} = \kappa \widehat{T}^{\nu}_{\mu} \tag{15}$$

where we set $\widehat{T} = \widehat{T}^{\nu}_{\mu} = g^{\mu\alpha} T_{\alpha\nu}$ and from (14) we obtain that $R = \mathrm{tr}P$.

Equation (15) can be supplemented by the scalar-valued equation obtained by taking the trace of (15); i.e.

$$f'(R)R - 2f(R) = \kappa g^{\alpha\beta} T_{\alpha\beta} \equiv \kappa\tau \tag{16}$$

which controls solutions of (15) (we define $\tau = tr\widehat{T}$). We shall refer to this scalar-valued equation as the *structural equation* of spacetime. The structural equation (13), if explicitly solvable, provides an expression of $R = F(\tau)$ and consequently both $f(R)$ and $f'(R)$ can be expressed in terms of τ. More precisely, for any real solution $R = F(\tau)$ of (16) one has that the operator P can be obtained from the matrix equation (15):

$$P = \frac{f(F(\tau))}{2f'(F(\tau))} I + \frac{\kappa}{f'(F(\tau))} \widehat{T} \tag{17}$$

Now we are in position to introduce generalized Einstein equations under the form

$$g_{\mu\alpha} P^{\alpha}_{\nu} = R_{\mu\nu}(h) \tag{18}$$

where $h_{\mu\nu}$ is given by (13) and P^{α}_{ν} is obtained from the algebraic equations (16) and (17) (for a given $g_{\mu\nu}$ and $T_{\mu\nu}$); see also [14] and [39]. For the matter-free case we find that $R = F(0)$ becomes a constant, thus implying that the two metrics are proportional and the operator P is proportional to the identity (i.e. to Kronecker delta). Equation (18) is hence nothing but Einstein equation for the metric g, almost independently on the choice of the function $f(R)$, as already obtained in [39]. Also the standard Einstein equation with a cosmological constant Λ can be recasted into the form (18). It corresponds to the choice $f(R) = R - \Lambda$. These properties justify the name of generalized Einstein equation given to (18). In the presence of matter, equation (18) expresses a deviation for the metric g to be an Einstein metric as it was discussed in [14]. It can be otherwise interpreted as an Einstein equation with additional stress-energy contributions deriving from the modified gravitational Lagrangian [16] as much as in the previous metric formalism, or possibly as a modified theory of gravity with a time dependent cosmological constant.

A. Cosmological Applications of First-order Non-linear Gravity

We give here a brief summary of the results obtained in [14] where cosmological applications of $f(R)$ Gravity were deeply discussed. Owing to the cosmological principle we assume g to be a Friedmann-Robertson-Walker (FRW) metric which (in spherical coordinates) takes the standard form:

$$g = -dt^2 + a^2(t)\left[\frac{1}{1-\mathrm{K}r^2}dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right)\right] \tag{19}$$

where $a(t)$ is the so-called *scale factor* and K is the space curvature ($K = 0, 1, -1$). We further choose a perfect fluid stress-energy tensor for matter:

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu} \tag{20}$$

where p is the pressure, ρ is the density of matter and u^{μ} is a co-moving fluid vector, which in a co-moving frame ($u^{\mu} = (1, 0, 0, 0)$) becomes simply:

$$T_{\mu\nu}\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{pa^2}{1-\mathrm{K}r^2} & 0 & 0 \\ 0 & 0 & pa^2(t)r^2 & 0 \\ 0 & 0 & 0 & pa^2(t)r^2\sin^2(\theta) \end{pmatrix} \tag{21}$$

The metric h turns out to be conformal to the FRW metric g by means of the conformal factor $f'(R)$, which can be moreover expressed in terms of τ by means of (16) and finally as a function of time

$$b(t) = f'(R(\tau)) \tag{22}$$

by an abuse of notation. From (18) we can obtain an analogue of the Friedmann equation under the form

$$\hat{H} = \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right)^2 = \frac{\kappa}{3b}\left[\rho + \frac{f(\tau)+\kappa\tau}{2\kappa}\right] - \frac{\mathrm{K}}{a^2} \tag{23}$$

which can be seen as a generalized definition of a *modified Hubble constant*. $\hat{H} = \left(\frac{\dot{a}}{a} + \frac{\dot{b}}{2b}\right)$, taking into account the presence of the conformal factor $b(t)$ which enters into the definition of the conformal metric h (see [14] for details). This equation reproduces, as expected, the standard Einstein equations in the case $f(R) = R$.

Considering the particular example $f(R) = \beta R^n$ of a pure-power Lagrangian we have obtained that the Hubble constant for the metric g can be (locally) calculated to be:

$$H^2 = \varepsilon r(n,w) a^{\frac{-3(w+1)}{n}} - s(n,w)\frac{K}{a^2} \tag{24}$$

where:

$$\begin{cases} r(n,w) = \frac{2n}{3(3w-1)(3w(n-1)+(n-3)}\left[\frac{-\kappa(3w-1)}{\beta(2-n)}\right]^{\frac{1}{n}} \\ s(n,w) = \left[\frac{2n}{3w(n-1)+(n-3)}\right]^2 \end{cases}$$

are functions of the exponent n and of the equation of state for matter, through w. We remark that the values of can be $\varepsilon = \pm 1$, in particular $\varepsilon = \mathrm{sign}R = 1$ for odd values of n and $\varepsilon = \pm 1$ for even values of n; see [14] for details. The *deceleration parameter* can be obtained from the Hubble contant by means of the following relation:

$$q(t) := -\left(1 + \frac{\dot{H}(t)}{H^2(t)}\right) = -\left(\frac{\ddot{\mathrm{a}}(\mathrm{t})}{\mathrm{a}(\mathrm{t})H^2(t)}\right) \tag{25}$$

and from (24) it turns out to be formally equal to:

$$q(t,w,n) = -1 + \frac{\frac{3(1+w)}{2n}\varepsilon r(n,w)a^{-\frac{3(1+w)}{n}} - s(n,w)Ka^{-2}}{\varepsilon r(n,w)a^{-\frac{3(1+w)}{n}} - s(n,w)Ka^{-2}} \tag{26}$$

It follows that when the term a^{-2} dominates over $a^{-\frac{3(1+w)}{2n}}$ the deceleration parameter turns out to be positive, i.e. $q(t,w,n) \to 0$. On the contrary, when the term $a^{-\frac{3(1+w)}{2n}}$ dominates over a^{-2} (or in the case $K = 0$ corresponding to spatially flat spacetimes) the deceleration parameter turns out to be:

$$q(w,n) = -1 + \frac{3(1+w)}{2n} \tag{27}$$

which is negative for $n < 0$ or $n > \frac{3(1+w)}{2} > 0$ owing to the positivity of $(1+ w)$ for standard matter; see [14]. This implies that the accelerated behavior of the Universe is predicted in a suitable limit. In particular it follows that super-acceleration ($q < -1$) can be achieved only for $n < 0$. The effective w_{eff} can be obtained (as in [11]) by means of simple calculations from (24) and (27). It turns out to be, for this theory:

$$w_{eff} = \frac{2}{3}q(n,w) - \frac{1}{3} = -1 + \frac{(w+1)}{n} \tag{28}$$

We remark that the range of $-1.45 < w_{eff} < -0.74$ for dark energy, stated in [3], can be easily recovered in this theory by choosing suitable and admissible values[2] of n. We refer to [14] for physical considerations and for more detailed discussions and examples concerning polynomial-like Lagrangians in the generalized Ricci scalar.

IV. CONCLUSIONS

We have here proven the importance of alternative theories of Gravity as alternative models for Dark Energy. These theories have been interpreted in the framework on General Relativity principles and studied both in the metric and Palatini formalism.

A striking result is the possibility, in this framework, to interpret the present acceleration of the Universe and to obtain the expected values both for the deceleration parameter and for the effective barotropic parameter w_{eff}. More General theories, containing higher order invariants, have been also studied to provide even a larger class of models and to enlarge the number of models at our disposal to theoretically explain the recent and striking experimental results.

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NOTES

1. This proof is based on a generalization of the Polsson's equation from Newton's theory to General Relativity and on the requirements that the theory should be of second order, it should be linear in the second derivatives of the metric and the divergence equations of the motion should be zero.
2. As already explained in [14] the parameter n should not be an integer, it can be any real number satisfying some reliability conditions, see [14] for further discussions and details.

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Gianluca ALLEMANDI[1,2,] & Mauro FRANCAVIGLIA[1,2,3]
[1]*Dipartimento di Matematica, Università di Torino*
Via C. Alberto 10, 10123 TORINO (Italy)
[2]*INFN, Section of Torino, Iniziativa Specifica NA12*
[3]*E.S.G. (Evolutionary Systems Group) of the University of Calabria (Italy)*
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