Mathematical Methods, Physical Methods and Simulation Science and Technology
Vol. 3 (January-December, 2019)

Einstein's Variational Approach to the Space Time Geometry

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Received: 23rd March 2018 Revised: 14th July 2018 Accepted: 10th September 2018

Einstein's variational approach to general relativity during the years 1913-1916 is analysed, focussing on the fundamental contribution of the Einstein-Levi Civita 1915 correspondence. During these years, Einstein did many efforts to clarify the differential properties of space-time geometry. Thanks to Levi-Civita he was able to derive the right mathematical formulation of the variational principle.

1. INTRODUCTION

A first variational approach to the gravitational field equations of General Relativity was unsuccessfully sketched by A. Einstein and M.Grossmann in 1913-1914 [10,11] and subsequently by Einstein himself in 1914 (the so-called *Entwurf Theory*) [12]. But Einstein's 1914 theory was confused by a misconception related to the physically unjustified requirement of restricting the covariance of the gravitational field equations [31, 34] and by some mathematical mistakes in a crucial proof in the theory [5, 8]. Between March and May 1915, the Italian mathematician T.Levi-Civita, in his private correspondence with Einstein, singled out the mathematical flaws of the *Entwurf* theory setting Einstein back on the right path of general covariance [5,8], which eventually brought him, in November 1915, to the correct formulation of the gravitational field equations [14, 16, 17]. In November 1915, D. Hilbert published an article in which he correctly showed that Einstein's gravitational field equations could be obtained easily from a variational principle, at least in presence of an electromagnetic field [21]. Five days later, independently of Hilbert, Einstein obtained the same results [19] thus obtaining the definitive variational formulation of the fields equations.

2. "ENTWURF" THEORY

In 1913 Einstein and Grossmann [10] restricted the covariance of their gravitational theory because the so-called hole argument, an uncorrected physical reasoning based on the uniqueness requirement for the gravitational field [31, 34], and mathematical arguments [5, 8, 3]. For the first time, however, they made use of a variational principle in general relativity [11] written in the form[1]

$$\int_{\Sigma} \left(\delta H - 2\chi \sqrt{-g}\, T_{\mu\nu} \delta g^{\mu\nu} \right) d\tau = 0 \qquad (1)$$

where ÷ is the universal constant, T_{mn} is the stress-energy tensor for matter, $g^{\mu n}$ are the components of the metric tensor (gravitational potentials), *d*t is the elementary "volume" of the 4-dimensional domain Σ. The lagrangian *H* is defined as ([11] p. 219)

$$H = \frac{1}{2} g^{\alpha\beta} \frac{\partial g_{\tau\rho}}{\partial x^{\alpha}} \frac{\partial g^{\tau\rho}}{\partial x^{\beta}} \qquad (2)$$

By requiring that the arbitrary variations $\delta g^{\mu n}$ of g^{μ} are mutually independent within the four dimensiona1 domain of integration S and vanish on the boundary ∂Σ, they easily obtain the limited covariant equations of the *Entwurf* gravitational theory [10]. A few months later[2] Einstein completed a new article, which he presented at the *Preussiche Akademie der Wissenshaften* of Berlin the 29th of October, 1914 [12]. In this paper he tried to extend the covariance properties of the *Entwurf theory* by redefining and generalizing his previous variational approach, and found what he (wrongly) believed[3] to be a more satisfactory derivation of his old *Entwurf* field equations. He first discussed the transformation properties of the action integral:

$$J = \int_{\Sigma} H \sqrt{-g}\, d\tau \qquad (3)$$

under an infinitesimal transformation D of coordinates $x^{\mu} \rightarrow x^{\mu} \stackrel{def}{=} x^{\mu} + \Delta x^{\mu}$ (where *H* is considered as an unspecified lagrangian density, supposed to be a function of the metric tensor $g^{ìí}$ and of its first derivatives

$$g_{\sigma}^{\mu\nu} \stackrel{def}{=} \frac{\partial g^{\mu\nu}}{\partial x^{\alpha}}$$

on the boundary of S the following conditions

$$\Delta x^{\mu} = 0, \qquad \frac{\partial}{\partial x^{\sigma}} \Delta x^{\mu} = 0$$

hold. By assuming the invariance of H under linear transformations Einstein was able to specify the form of DH

$$\frac{1}{2}\Delta H = g^{\nu\alpha}\frac{\partial H}{\partial g_{\sigma}^{\mu\nu}}\frac{\partial^2 \Delta x^{\mu}}{\partial x^{\sigma}\partial x^{\alpha}} \tag{4}$$

and ΔJ [12] pp. 1069-1070:

$$\frac{1}{2}\Delta J = \int_{\Sigma}\left(B_{\mu}\Delta x^{\mu}\right)d\tau + F, \tag{5}$$

where

$$B_{\mu} = \frac{\partial^2}{\partial x^{\sigma}\partial x^{\alpha}}\left(g^{\nu\alpha}\frac{\partial H\sqrt{-g}}{\partial g_{\sigma}^{\mu\nu}}\right) \tag{6}$$

and F is an integral over $\partial\Sigma$ that vanishes of the boundary conditions. Then Einstein defined a set of coordinate systems *K, K', K''*.... , infinitesimally close to each other, so that in the transformation from one system to the next the same boundary conditions hold. This also implies that, for all those systems with coinciding boundary conditions, the relation F=0 holds, so that equation (5) can be written as:

$$\frac{1}{2}\Delta J = \int_{\Sigma}\left(B_{\mu}\Delta x^{\mu}\right)d\tau, \tag{7}$$

The coordinate system, *adapted* to a given gravitational field, were thus defined by Einstein as those systems of the series K, K',K'' , for which J is an extremal, i.e. DJ = 0, and then

$$B_{\mu} = 0 \tag{8}$$

From the arbitrariness of the variations Δx^{μ}, Einstein easily re-obtained the necessary and sufficient conditions for the definition of the *adapted* coordinate systems. Einstein's next step was to consider arbitrary infinitesimal variations of the metric tensor $g^{\mu\nu} \to g^{\nu\mu}$ defined as:

$$g^{\mu\nu'} = g^{\mu\nu} + \delta g^{\mu\nu} \tag{9}$$

in such a way that *äg*ìí and its derivatives $\delta g_{\sigma}^{\mu\nu} \stackrel{def}{=} \frac{\partial}{\partial x^{\sigma}} \delta g^{\mu\nu}$ vanish on ∂Σ.

He then calculated the corresponding increment *dJ* of the integral of action *J*

$$\delta J = \int_{\Sigma} \varepsilon_{\mu\nu} \delta g^{\mu\nu} d\tau \tag{10}$$

where

$$\varepsilon_{\mu\nu} \stackrel{def}{=} \frac{\partial H\sqrt{-g}}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^{\sigma}} \frac{\partial H\sqrt{-g}}{\partial g_{\sigma}^{\mu\nu}} \tag{11}$$

is the gravitational tensor. His aim was to show the invariance of *J* under *justified* transformations of the metric tensor (i.e. those arbitrary variations of the metric tensor from an *adapted* coordinate system), that is:

$$\Delta \delta J = 0\,; \tag{12}$$

this was considered by Einstein a *(crucial theorem)* because from the invariance of *J* follows the covariance of the gravitational tensor $\varepsilon_{\mu\nu}$ In fact, from the invariance of *J* under justified transformations and since $\delta g^{\mu n}$ is a tensor and $\sqrt{-g}d\tau$ is a scalar, it follows that $\frac{\varepsilon_{\mu\nu}}{\sqrt{-g}}$ is a covariant tensor in any system of *adapted* coordinates. The importance of theorem (12) thus lies in the demonstration of the tensorial nature of $\frac{\varepsilon_{\mu\nu}}{\sqrt{-g}}$, a crucial result which connects the gravitational tensor to the matter stress-energy tensor (and other physical phenomena) through equation

$$\varepsilon_{\mu\nu} = -\chi \mathcal{T}_{\mu n} \tag{13}$$

The starting point of Einstein's proof was the decomposition of the arbitrary variations of the metric tensor ägìí into two parts

$$\delta g^{\mu\nu} = \delta_1 g^{\mu\nu} + \delta_2 g^{\mu\nu} \tag{14}$$

The $\delta_1 g^{\mu\nu}$ represent infinitesimal variations of the metric tensor in the same system of *adapted* coordinates. According to Einstein, the ten $\delta_1 g^{\mu\nu}$ *are not independent,* since they are connected by four differential equations. Since K_1 is *adapted* to both the original gravitational field $g^{ìí}$ and to the new varied field $g^{\mu\nu} + \delta_1 g^{\mu\nu}$, Einstein drew the conclusion that in addition to the usual conditions $B = 0$ the four equations

$$d_1 B_\sigma = 0$$

hold.

The $\delta_2 g^{\mu\nu}$ represent the variations of the *same gravitational field* due to the change of coordinate system. Since, according to Einstein, the $\delta_2 g^{\mu\nu}$ *are determined by four independent functions* (the variations of the coordinates) ([12] p. 1072), the sum of the two variations $\delta_1 g^{\mu\nu} + \delta_2 g^{\mu\nu}$ is determined by (10-4) + 4 =10 independent functions, which are equivalent to the ten arbitrary variations $\delta g^{\mu\nu}$. As a consequence Einstein obtained the proof of the crucial theorem by splitting equation (12) into two parts:

$$\Delta\delta_1 J = 0 \tag{16}$$

and

$$\Delta\delta_2 J = 0 \tag{17}$$

The proof of equation (16) is straightfoward:by applying the variation $\ddot{a}_1$ to equation (5), Einstein directly obtained the expression

$$\frac{1}{2}\Delta(\delta_1 J) = \int_\Sigma \left(\partial_1 B_\mu \Delta x^\mu\right) d\tau + \delta_1 F, \tag{18}$$

From the conditions $\delta_1 g^{\mu\nu} = 0$, $\delta_1 g^{\mu\nu}{}_\sigma = 0$ which hold on $\partial\Sigma$, the surface integral $\delta_1 F$ vanishes. Equation (15), then gives equation (16). Einstein's proof of equation (17) is rather cumber some. He first considered the variation $\delta_2 J$, corresponding to an infinitesimal transformation of coordinates $x^\mu \to x^\mu + \delta x^\mu$, which reduces to the identity transformation on the boundary of Σ. In this case, as a consequence of the definition of $\delta_2 g_{\mu\nu}$, the same unvaried field will be expressed as $g_{\mu\nu}$ and $g_{\mu\nu} + \delta_2 g_{\mu\nu}$ respectively, in the old and in the new infinitesimally varied coordinate system. Since the coordinates of the old system are adapted to the unvaried field, it follows from the definition of adapted systems (where $B_\sigma = 0$), and from the fact that an infinitesimal transformation of coordinates is considered (this is the consequence of equation (5)), that

$$\delta_2(J) = \Delta J = 0\,; \tag{19}$$

where $\delta_2(J)$ means: $\delta_2(J) = J(g^{\mu\nu} + dg^{\mu\nu}) - J(g^{\mu\nu})$; whereas $J(g^{\mu\nu})$ and $J(g^{\mu\nu} + dg^{\mu\nu})$ represent the action integrals relative to the same field, calculated in the old adapted system and in the new infinitesimally varied system, respectively. Then Einstein considered that particular transformation of the field $g^{*\mu\nu} = g^{\mu\nu} + dg^{\mu\nu}$ in the same adapted coordinate system K_1, obtained by choosing those variations $\delta g^{\mu\nu}$ of the field $g^{\mu\nu}$ in K_1 that coincide with the $d_2 g^{\mu\nu}$ previously obtained as a result of an infinitesimal coordinate transformation, the variation of the field in K_1 is in Einstein's words a δ_2-*variation* ([12], p. 1072). The varied field in K_1 will be

$$g^{*\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu}$$

and from (19) it follows that

$$\delta_2(J_1) = 0\,; \tag{20}$$

(where $\delta_2(J_1) = J_1(g^{*\mu\nu}) - J(g^{\mu\nu}) = J_1(g^{\mu\nu} + d_2 g^{\mu\nu}) - J_1(g^{\mu\nu})$)**;** the two integrals are both calculated in K_1(for the unvaried and varied fields, respectively). Einstein's next step was to show that this variation of the field, considered in another adapted coordinate system K_2 is still a δ_2-variation. If this is the case, it follows that in K_2 the relation analogous to (20) holds:

$$\delta_2(J_2) = 0\,; \tag{21}$$

By subtracting (20) from (21), one obtains the desired result ([12], pp. 1072-1073):

$$\delta_2(J_2 - J_1) = \delta_2(\Delta J) = \delta_2 J_2 - \delta_2 J_1 = \Delta(\delta_2 J) = 0\,; \tag{22}$$

According to Einstein, the validity of theorem (12) implied, the tensorial character of $\frac{\varepsilon_{\mu\nu}}{\sqrt{-g}}$ under justified transformations and consequently the restricted covariance of the field equations (13).

In the remainder of his article Einstein tackled the problem of deriving the explicit form of H**,** in order to obtain the gravitational field equations. According to equation (13), since the divergence of the energy-momentum for matter is zero the same holds for the gravitational tensor $\frac{\varepsilon_{\mu\nu}}{\sqrt{-g}}$:

$$\frac{\partial}{\partial x^{\nu}}\left(g^{\tau\nu}\varepsilon_{\sigma\tau}\right) + \frac{1}{2} g_{\sigma}^{\mu\nu}\varepsilon_{\mu\nu} = 0$$

which reduces to

$$\frac{\partial}{\partial x^{\nu}}\left(S_{\sigma}^{\nu}\right) - B_{\sigma} = 0$$

being

$$S_{\sigma}^{\nu} \overset{def}{=} g^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g^{\sigma\tau}} + g_{\mu}^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g_{\mu}^{\sigma\tau}} + \frac{1}{2}\delta_{\sigma}^{\nu} H\sqrt{-g} - \frac{1}{2} g_{\sigma}^{\mu\tau} \frac{\partial H\sqrt{-g}}{\partial g_{\nu}^{\mu\tau}}$$

Taking into account of condition (8), there follows

$$S_{\sigma}^{\nu} = 0$$

It can be easily shown that function (2) fulfils the above. Indeed Einstein selected five functions similar to (2), and it has been proved [31] that all of them satisfy the above, but Einstein choose (2) without giving any explanation. Once we have the expression of *H,* from the definition (11) and the equation (13) we get

$$-\frac{\partial}{\partial x^{\alpha}}\left(g^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g^{\sigma\tau}}\right) = \chi T_{\sigma}^{\nu} + \left(-g^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g^{\sigma\tau}} + g_{\sigma}^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g_{\alpha}^{\sigma\tau}}\right)$$

or explicitly

$$\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{-g}\, g^{\alpha\beta} \Gamma_{\sigma\beta}^{\nu}\right) = -\chi\left(T_{\sigma}^{\nu} + t_{\sigma}^{\nu}\right) \tag{23}$$

with

$$\Gamma_{\sigma\beta}^{\nu} \overset{def}{=} \frac{1}{2} g^{\nu\tau} \frac{\partial g_{\sigma\tau}}{\partial x^{\beta}}$$

and

$$t_{\sigma}^{\nu} \overset{def}{=} \frac{1}{\chi}\left(-g^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g^{\sigma\tau}}\right) + g_{\sigma}^{\nu\tau} \frac{\partial H\sqrt{-g}}{\partial g_{\alpha}^{\sigma\tau}}$$

is the gravitational energy pseudo-tensor. In Einstein's opinion, $\Gamma^{\nu}_{\alpha\beta}$ are the components of the gravitational field and the equations (23) must be considered as the gravitational field equations, in fact, deriving the left hand side of (23) with respect to x^{i} and thanks to equations (8)-(6), we obtain the conservation law of the total energy

$$\frac{\partial}{\partial x^{\nu}}\left(T^{\nu}_{\sigma}+t^{\nu}_{\sigma}\right)=0$$

3 1915 Correspondence Einstein-Levi Civita

M.Abraham considered Einstein' choice of the function *H,* as an arbitrary choice and his demonstration unclear thus asking Levi-Civita to enter the field. On February 23, 1915 Max Abraham, a tenacious opponent of Einstein's theories [7], wrote to his friend Levi-Civita:

"*Really I did not understand on which hypothesis his* [Einstein's] *new demonstrations is based" [1]* pp. 3-4 attracting the interest of Levi-Civita to Einstein's article [12]. During the spring of 1915, in private correspondence with Einstein, Levi-Civita focused his criticism mainly on the tensorial character of $\varepsilon_{\mu\nu}$, finding fault with Einstein's proof of it. Levi-Civita denied the tensorial character of the variations $\delta g^{\mu\nu}$ and, from the first, Einstein's reply appeared to be weak:

"*In variational calculus people always work with the same method that I have used*." $[13]_1$

Thus Einstein focused his defence on a limiting process that should warrant the tensorial character of the $\delta g^{\mu\nu}$. However the weakness of Einstein's defence of his theorem lay not so much in the limiting approximation as in the assumption of the independence of the $\delta g^{\mu\nu}$ in addition to their covariance. With various examples and counter-examples ($[13]_{2,3}$), Levi-Civita tried to focus on the main objection regarding the independence of the $\delta g^{\mu\nu}$, until March 23rd [25], he showed

"with a concrete example that by means of a justified transformation starting from a Euclidean ds^2 one finds some no vanishing $\varepsilon_{\mu\nu}$ in contrast with the requirement of the covariance [of it]" [25], p.1.

Levi-Civita succeeded in demonstrating the existence of a particular justified transformation from a system of adapted coordinates where all the identically vanish to another adapted coordinate system where some components do not vanish, in contradiction with the supposed tensorial character of $\varepsilon_{\mu\nu}$. Einstein was extremely interested in Levi-Civita's letter. He replied:

"For one and a half days I had to reflect unceasingly until I understood how your example could be reconciled with my proof ..... Your deduction is completely correct...... But strangely, it does not contradict my proof for the following reason: my demonstration does not work only in the special case that you have examined." $[13]_4$, p.2

Einstein's reply is particularly important since, in order to rebut Levi-Civita's example, he is compelled to admit, for the first time, that his old proof of the crucial theorem is incorrect, at least in the special case of infinitesimal transformations. Although Einstein still believed that his old demonstration maintained its validity for all finite transformations, he was nevertheless obliged to admit that, since $\int_{\Sigma} \delta_2 g^{\mu\nu} d\tau = 0$ for every infinitesimal transformation, the proof of his fundamental theorem had to be revised, with regard to infinitesimal transformations, by using only δ_1-variations. In his own words:

"These considerations suggest a modification of the proof of covariance by using only the δ_1-variations since the δ_2-variations do not give any contribution to the fundamental quantities $\int_{\Sigma} \sqrt{-g}\ \delta g^{\mu\nu} d\tau$." $[13]_4$, p.4

It is important to stress that Einstein's admission of the uselessness of the $\delta_2 g^{\mu\nu}$ variations and his willingness to use only the $\delta_1 g^{\mu\nu}$ variations, meant that the proof of his theorem $\Delta\delta J = 0$ could not hold within the Entwurf theory. In fact, if one uses only the $\delta_1 g^{\mu\nu}$, because of the four conditions $\delta_1 B_\sigma = 0$, then the $\delta g^{\mu\nu} = \delta_1 g^{\mu\nu}$ no longer represent ten arbitrary mutually independent quantities. Einstein was obliged to accept the validity of Levi-Civita's objections and in particular he had to admit the incompleteness of his proof, as he himself wrote to Levi-Civita:

"I willingly acknowledge that you have touched the sorest spot of my proof, namely the independence of the $A^{\mu\nu}$[= $\int_{\Sigma} \sqrt{-g}\ \delta g^{\mu\nu} d\tau$.]. Here. my demonstration is lacking in acumen." $[13]_8$

In the last letter of the correspondence, Einstein stressed again the same conclusion:

"My demonstration is incomplete, in the sense that the possibility of an arbitrary choice of the $A_{\mu\nu}$ remains unproved." $[13]_9$

In the 1915 correspondence with Levi-Civita [13], Einstein was obliged to admit that his variational derivation of the gravitational fields equation was uncorrected because he could not proof both the independence and the covariance of the field equations; in a later correspondence Levi-Civita attacked the untenable definition of the gravitational energy pseudo-tensor (for a deeper analysis of Einstein- Levi Civita correspondence see [3, 4, 5, 6, 8, 9]).

4. EINSTEIN'S (1916) VARIATIONAL APPROACH

In 1916, Einstein proposed a new variational method giving credit to Lorenz and Hilbert[4] but he considered his approach more general, because he reduced some hypotheses about matter. The starting point is the universal function $\mathcal{H} \overset{def}{=} H\sqrt{-g}$ which Einstein for the first time assumes not only as a function of $g^{\mu\nu}$, $g^{\mu\nu}{}_{\sigma}$ but also as linear function of the derivatives $\frac{\partial^2 g^{\mu\nu}}{\partial x^\rho \partial x^\sigma}$. Furthermore, Einstein generalizes the variables variational principle to any physical phenomenon by assuming $\mathcal{H}$ to be dependent on the (whatsoever and not only electromagnetic) matter variables q_ρ and $\frac{\partial q_\rho}{\partial x^\alpha}$. The lagrangian chosen to replace (2), is:

$$\mathcal{H} = \mathcal{H}\left(g^{\mu\nu}, g^{\mu\nu}_{\sigma}, \frac{\partial^2 g^{\mu\nu}}{\partial x^\rho \partial x^\sigma}; q_\rho, \frac{\partial q_\rho}{\partial x^\alpha}\right) \tag{44}$$

and the principle is

$$\delta\int \mathcal{H} d\tau = 0$$

Assuming H to be a linear function of the second order derivatives of the metric tensor it is possible to transform the action integral as

$$\int \mathcal{H} d\tau = \delta\int \mathcal{H}^* d\tau + F$$

where F is an integral vanishing according to the boundary conditions

$$\delta q_\rho = 0,\ \delta g^{\mu\nu} = 0,\ \delta g^{\mu\nu}{}_{\sigma} = 0.$$

So that the variational principle reduces to:

$$\delta\int \mathcal{H}^* d\tau = 0 \tag{45}$$

where $\mathcal{H}^*$ does not depend on $\frac{\partial^2 g^{\mu\nu}}{\partial x^\rho \partial x^\sigma}$. It is important to remark that after having used the boundary conditions and partially integrating, $\mathcal{H}$ reduces to $\mathcal{H}^*$ which is deprived of the second order partial derivatives of the metric tensor. Two considerations arise:

1. it is necessary to start from (44) because, according to the general relativity principle, the lagrangian $\mathcal{H}$ must be invariant, and there are no invariant lagrangian made only by first order derivatives of the metric tensor;
2. the reduction of the original $\mathcal{H}$ to $\mathcal{H}^*$, i.e. to a lagrangian quadratic function of the first order derivatives of the metric tensor, enabled Einstein to make use of the *Entwurf* mathematical machinery.

The difference between the 1916 approach and the *Entwurf* and [12] ones stand both in the general covariance of the new theory and in the arbitrariness choice of the gravitational lagrangian, which was the cause of the Einstein's previous difficulties [31, 5, 8]. Einstein's next step was to split the lagrangian, into the gravitational and matter parts, assuming, at last, for the lagrangian of gravity the Riemann curvature scalar:

$$\mathcal{H} = \mathcal{G} + \mathcal{M}$$

In contrast to the Entwurf theory, Einstein was eventually obliged to admit that, in order to accomplish the general relativity principle,

"$\mathcal{G}$ (up to a constant factor) must be the linear invariant of the Riemann curvature tensor, since there is no other invariant which fulfils the required properties [19]" p.1113

The form of $\mathcal{G}^*$ is now, in term of Christoffel symbols,

$$\mathcal{G}^* = \sqrt{-g}\;\; g^{\mu\nu}\, [\begin{Bmatrix}\beta\\ \mu\alpha\end{Bmatrix}\begin{Bmatrix}\alpha\\ \nu\beta\end{Bmatrix} - \begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\beta\\ \alpha\beta\end{Bmatrix}] \tag{47}$$

It follows with a straightforward calculation that:

$$= \frac{1}{\sqrt{-g}}\;\mathcal{G}^*\; \frac{1}{4} g^{\alpha\rho}\partial_\alpha g^{\mu\nu}\partial_\rho g_{\mu\nu} - \frac{1}{2} g^{\alpha\rho}\partial_\alpha g^{\mu\nu}\partial_\nu g_{\mu\rho} + \frac{1}{\sqrt{-g}} g^{\mu\nu}\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\frac{\partial\sqrt{-g}}{\partial x^\alpha} \tag{48}$$

so that, by comparing this form of the (reduced) lagrangian with the previous (2) we have

$$\frac{1}{\sqrt{-g}}\,\mathcal{G}^*_{(1916)} = \frac{1}{2}\mathcal{H}_{(1913)} - \frac{1}{2}\; g^{\alpha\rho}\,\partial_\alpha g^{\mu\nu}\partial_\nu g_{\mu\rho} + \frac{1}{\sqrt{-g}}\; g^{\mu\nu}\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\frac{\partial\sqrt{-g}}{\partial x^\alpha} \tag{49}$$

with $\frac{1}{\sqrt{-g}}\; g^{\mu\nu}\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\frac{\partial\sqrt{-g}}{\partial x^\alpha}$ vanishing when $\sqrt{-g} = 1$.

The second term of the right hand side was, under the assumption $\sqrt{-g}=1$ the missed term of the limited covariance *Entwurf* theory. However Einstein followed the previous (1913) variational approach, in order to show, by using an infinitesimal transformation of coordinates, $x^{\mu'}=x^{\mu}+\Delta x^{\mu}$ that the condition $B_{\mu}=0$ is still valid, but not covariant. In fact, we have

$$\Delta g^{\mu\nu}=g^{\mu\alpha}\frac{\partial\left(\Delta x^{\nu}\right)}{\partial x^{\alpha}}+g^{\nu\alpha}\frac{\partial\left(\Delta x^{\mu}\right)}{\partial x^{\alpha}}\quad \Delta g^{\mu\nu}{}_{\sigma}=g^{\mu\alpha}\frac{\partial\left(\Delta g^{\mu\nu}\right)}{\partial x^{\sigma}}g^{\mu\nu}{}_{\alpha}\frac{\partial\left(\Delta x^{\alpha}\right)}{\partial x^{\sigma}} \tag{50}$$

so that the infinitesimal variation of the function $\mathcal{G}^*$ is

$$\sqrt{-g}\,\Delta\frac{\mathcal{G}}{\sqrt{-g}}=S_{\nu}^{\sigma}\frac{\partial\left(\Delta x^{\sigma}\right)}{\partial x^{\nu}}+2\frac{\partial\mathcal{G}^*}{\partial g_{\alpha}^{\mu\sigma}}g^{\mu\nu}\frac{\partial^{2}\left(\Delta x^{\sigma}\right)}{\partial x^{\nu}\partial x^{\alpha}}$$

where

$$S_{\nu}^{\sigma}\overset{def}{=}2\frac{\partial\mathcal{G}^*}{\partial g^{\mu\sigma}}g^{\mu\nu}+2\frac{\partial\mathcal{G}^*}{\partial g_{\alpha}^{\mu\sigma}}g\alpha^{\mu\nu}+\mathcal{G}^*\delta^{\nu}{}_{\sigma}-\frac{\partial\mathcal{G}^*}{\partial g_{\sigma}^{\mu\alpha}}g_{\sigma}{}^{\mu\alpha} \tag{51}$$

Assuming the invariance of $\mathcal{G}^*/\sqrt{-g}$ under linear transformation of the coordinates $\frac{\partial^{2}\Delta x^{\sigma}}{\partial x^{\nu}\partial x^{\alpha}}=0$ there follows

$$S_{\sigma}^{\nu}=0 \tag{52}$$

The invariance under linear transformation of the coordinates implies

$$\Delta\int\mathcal{G}d\tau=\Delta\int\mathcal{G}^*d\tau=0 \tag{53}$$

and taking into account the condition (52) the condition

$$\Delta\int\mathcal{G}^*d\tau=\int\frac{\partial\mathcal{G}^*}{\partial g_{\alpha}^{\mu\sigma}}g^{\mu\nu}\frac{\partial^{2}\left(\Delta x^{\sigma}\right)}{\partial x^{\nu}\partial x^{\alpha}}=0$$

from where the integration by part gives

$$\frac{\partial^2}{\partial x^\nu \partial x^\alpha}\left(\frac{\partial \mathcal{G}^*}{\partial g_\sigma^{\mu\sigma}} g^{\mu\nu}\right) = 0$$

which is equivalent to the condition (8). In Einstein's opinion, although B_μ is not a vector, the condition $B_\mu = 0$ is a consequence of the invariance of $\mathcal{G}$ together with the general relativity principle. The vector B_μ plays a fundamental role in deriving the gravitational equations. In fact, according to Einstein, equations (13), which explicitly are:

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{G}^*}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial \mathcal{G}^*}{\partial g^{\mu\sigma}} = \frac{\partial \mathcal{M}^*}{\partial g^{\mu\nu}} \tag{54}$$

lead to the conservation laws in general relativity, that in Einstein 's opinion have to be considered as the gravitational equations. By multiplying the above by $g^{\mu\nu}$ one obtains

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{G}^*}{\partial g_\alpha^{\mu\nu}} g^{\mu\nu}\right) = -(\mathcal{L}_\sigma^\nu + t_\sigma^\nu) \tag{55}$$

where

$$\mathcal{L}_\sigma^\nu = g^{\mu\nu} \frac{\partial \mathcal{M}}{\partial g_\alpha^{\mu\sigma}} \tag{56}$$

and

$$t_\sigma^\nu \overset{def}{=} -\left(\frac{\partial \mathcal{G}^*}{\partial g_\alpha^{\mu\sigma}} g_\alpha^{\mu\nu} + \frac{\partial \mathcal{G}^*}{\partial g^{\mu\sigma}} g^{\mu\nu}\right) = \frac{1}{2}\left(\mathcal{G}^* \delta_\sigma^\nu - \frac{\partial \mathcal{G}^*}{\partial g_\alpha^{\mu\nu}} g_\sigma^{\mu\nu}\right) \tag{57}$$

according to the condition on B_μ

Finally, due to the condition (8), the conservation law of the total energy-momentum follows:

$$\frac{\partial}{\partial x^\nu}\left(\mathcal{L}_\alpha^\nu + t_\sigma^\nu\right) = 0 \tag{58}$$

$\mathcal{L}_s^{\,n}$ are the components of the stress-energy tensor density of the matter, while $t_\sigma^{\,\nu}$ are the components of the stress-energy *tensor density* of the gravitational field ([19]p.1116); the conservation equations for the total energy (besides the corresponding equations for the matter only), are, in Einstein's opinion, *generally covariant* and are obtained only by using the general relativity principle ([19] p. 1116).

Perhaps it was this statement, plus the controversial definition of the components of the gravitational stress-energy tensor that pushed Lorenz and Schrödinger to intervene in the debate on conservation laws in general relativity (see [6]). This controversy also induced:

(a) Levi-Civita [24] to dispute not only the tensoriality of t_s^n but also the conservation equations for gravitation-plus-matter [6]; and later (b) Palatini to derive the Einstein-Levi Civita covariant equations by a variational principle which was more in accordance with the general covariance of the theory arid with the Bianchi identities.

7. CONCLUSIONS

This paper has tried to shed some light on some misunderstandings regarding the variational principle in General Relativity, which can briefly be summarized (see also comparative tables) as follows:

1. In 1913, together with Grossmann, Einstein proposed a new theory of gravitation and obtained gravitational field equations with limited covariance [10];
2. In November 1914, Einstein attempted to derive an unsound variational principle [12] in order to *characterize this theory from a formal mathematical point of view* [12] p. 1030 and to extend the limitated covariance of the field equations, but he failed due to this unfortunate choice of the lagrangian density for gravity and a fatal mistake discovered by Levi-Civita [5, 8]:
3. On November 20, 1915, Hilbert presented an article [21] in which he succeeded in unifying the gravitational and e1ectromagnetic fields via a variational principle, based mainly on Mie's formalism and Einstein's revised theory. Five days after Hilbert, Einstein published his variational principle, which he felt to be the more general one. In fact, he relieved the hypothesis on matter (and refused the complete electromagnetic origin of the matter), which were instead defended by Hilbert and (to a lesser extent) by Weyl;

The following years saw more intensive discussion of (a) the origin of matter (b) some unnecessary hypotheses in the Hilbert approach [22, 23, 30], and (c) conservation laws [6], until Levi-Civita [24] gave a plain covariant form to the gravitational equations and explained the various identities of the other authors by means of the Bianchi identities. In 1919, Palatini presented a method in which the gravitational variables are not only the components of the metric tensor but also the coefficients of the affine connection, thereby obtaining covariant equations via an invariant method.

Palatini succeeded in proposing an original variational approach which, dependent on a wider set of parameters, was covariant and led to covariant equations. The Bianchi identities and the conservation laws directly follow in a systematic covariant way in his method.

NOTES

1. For reader's sake, the original notations have been converted into modern ones, furthermore the Einstein convention on the sum over repeated indices, is assumed everywhere.
2. Einstein's scientific collaboration with Grossmann ended in April 1914, when he moved from Zurich to the Kaiser Wilhelm Institute in Berlin.
3. Einstein thought he had succeeded, ...,.to obtain the gravitational field equations in a purely covariant theoretical form [12].
4. Einstein remarks that Recently H. A. Lorentz and D. Hilbert have succeded to give to general relativity an especially transparent form by deriving their equations from a single variational principle. This will be done also in the following treatment. There is my aim to present the basic relations as transparently as possible and in a way as general as relativity allows. [19] p. 1111.

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