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On New Generalization of ψ (Digamma) Function

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This work is devoted to further development of important case of Wright's hypergeometric functions and its applications to investigation of new generalization of G–, y – functions.

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1. INTRODUCTION

In many areas of applied mathematics and in particular, in applied analysis and differential equations various types of special functions become essential tools for scientists and engineers.

A great number of papers have been devoted to special functions [1-6; etc.]. Nevertheless, further study of the special functions is very interesting and useful for different branches of science. Special functions appear when solving some boundary value problems of mathematical physics (that are solved by using separation of variables), in investigation so series, etc. The diversity of the problems generating special functions has led to a quick increase in a number of functions used in applications from the simplest transscendental functions to hypergeometric functions of the different nature. It should be noted that evaluation of importance of one or another class of special functions has changed substantially during the last hundred years. The elliptic integrals and related functions turned out to be the most interesting in the nineteenth century, but further development of mathematics has brought in the forefront another class - the functions of the hypergeometric type.

The special and degenarated cases of the hypergeometric functions, in particular, the Besselthe Legendre, the Whittaker functions, the classical orthogonal polynomials such as Jacobi, Laguerre, Kravchuk, Hermite polynomials, etc. are expressed by way of the ${}_2F_1(a, b; c; z)$. Let us write some out of them:

the Bessel function

$$J_{v}(z)=\frac{\left(\frac{z}{2}\right)^{v}}{\Gamma(v+1)}\,{}_0F_1(v+1;-\frac{1}{4}z^{2}),$$

the Legendre function

$$P_{v}^{u}(z)=\frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}}{}_2F_1\left(-v,v+1;1-\mu;\frac{1-z}{2}\right),$$

the Kravchuk polynomial

$$k_{n}(x)=q^{n}\binom{x}{n}{}_2F_1\left(-n,x-N;x-n;-\frac{P}{q}\right).$$

The complete elliptic integrals are also parsticular cases of hypergeometric functions. The hypergeometric functions occur in many different fields of mathematics from number theory [7] to geometric function theory [8]. An extensive list of special cases of the Gaussian hypergeometric functions F (a, b; c; z) for rational triples (a, b, c) is given in [9]. An comprehensive survey of the literature on computing the numerical values of special functions is in [10]. Some algorithms for computation are given, e.g., in [11].

Special functions are kernels of many integral transforms [12-17. etc.]. In turn, the theory of integral transforms is an effective mathematical instrument for solving many problems in analysis and the theory of partions, for researching boundary value problems of the mathematical physics, etc. The Fourier, Laplace, Mehler-Fock, Mellin transforms can be applied to many problems important for physics. Some new integral transforms have been introduced recently. Here it should be noted appearance of the remarkable book about **H**–transform [12]. This book deals with integral transforms involving the *H*–functions as kernels, or **H**– transforms, and their applications. The *H*– function is defined by the Mellin-Barnes type integral with the integrand containing products and quotients of the Euler gamma functions.

The continous development in mathematical physics, mechanics of solid medium, quantum mechanics, probability theory, aerodynamics, biomedecine, theory of modeling, partition theory, combinatories, astronomy, heat conduction and others has led to the discovery of new special function.

In our articles we consider the generalized confluent hypergeometric functions and its applications to the new generalization of the gamma-function, the ψ – function.

These functions commonly arise in such areas of applications as heat conduction, communication system, electro-optics, approximation theory, probability theory, electric circuit theory, etc. These functions are also found to be useful in a variety of fields including theory of partitions, combinatorial analysis,finite vector spaces, statistics, to the distributions of signal of or noise in output processed by a radar-receiver under various sets of conditions, in nuclear and molecular physicss, diffraction and plasma wave problems [18-20]. Lorentz-Doppler line broadening and design of particle accelerators can be expressed in terms of generalized gamma functions, for example, the cumulative density function is expressible in the terms of the generalized gamma-function [21]

$$G(x) = 1 - Ca^{-\alpha}\tilde{\Gamma}(\alpha, ax; ab).$$

Let us notice that the theory of hypergeometric functions is rich and wide, and certainly provides an inexhaustible field of the future research.

H. Buchholz [22] constructed the theory of hypergeometric functions on Whittaker functions, F.Tricomi [23] took Kumer function for construction of confluent by pergeometric functions. L.Slater [24] applied both above methods.

The confluent hpergeometric functions are the solutions of the following differential equation [3]:

$$x\frac{d^2y}{dx^2} + (c - x)\frac{dy}{dx} - ay = 0.$$

The confluent hypergeometric function ${}_1\Phi_1(a; c; x)$ has the next form

$${}_1\Phi_1(a;c;x) = \frac{\Gamma(c)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(c+n)}.\frac{x^n}{n!},$$

integral representation of ${}_1\Phi_1(a; c; x)$:

$${}_1\Phi_1(a;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tx}\,dt,$$

where $Rec > Rea > 0$, $\Gamma(a)$ is gamma-function.

Let us indicate some generalizations of gamma-functions:

$$\Gamma_m(u,v) = \int_0^\infty \frac{t^{u-1}e^{-t}}{(t+v)^m}dt, \quad \text{(see [25])},$$

$$D\begin{pmatrix} a,b;c;p \\ u,v \end{pmatrix} = v^{-a}\int_0^\infty t^{u-1}e^{-pt}\,{}_2F_1(a,b;c;-\frac{t}{v})dt, \quad \text{(see [26])},$$

$$\Gamma\begin{pmatrix} a,b;c;p; \quad p,\omega,\mu \\ u,v \end{pmatrix} = v^{-a}\int_0^\infty t^{u-1}e^{-pt}\,{}_2R_1(a,b;c;\omega,\mu;-\frac{t}{v})dt,$$

where ${}_2R_1$ is the case of Wright's function [29, 30],

$$\Gamma_a^c(\alpha;\tau,\beta,\omega;b) = \int_o^\infty t^{\alpha-1}e^{-t\omega}\,{}_1\Phi_1^\tau(a;c;-\frac{b}{t^\beta})dt$$

where $Rec > Rea > 0$, $Re\alpha > 0$, $\tau \in R$, $\tau > 0$, $b > 0$, $\beta \geq 1$, $\omega > 0$, ${}_1\Phi_1^\tau(a;c;z)$ is τ–generalized confluent hypergeometric function [27].

As it known [3] the digamma-function (or psi-function) is the logarithmic derivative of the gamma-function:

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz} = \frac{\Gamma(z)}{\Gamma(z)} \quad \text{ro} \quad \ln\Gamma(z) = \int_1^z \psi(z)dz.$$

For this classical special function some properties, integral and series representations are established (see, for example, in [32-36, etc.]).

The following generalization of psi-function

$$\psi_b(\alpha) = \frac{1}{\Gamma_b(\alpha)}\int_0^\infty t^{\alpha-1}(\ln t)e^{-t-bt^{-1}}dt$$

was introduced in [37]. Here $\mathrm{Re}\,b > 0, \mathrm{Re}\,\alpha > 0, \Gamma_b(\alpha) = \int_0^\infty t^{\alpha-1}(\ln t)e^{-t-bt^{-1}}dt.$ The factor $e^{-bt^{-1}}$ in the integral plays the role of a regularizer; for $b = 0$ the function $\Gamma_b(\alpha)$ coincides with the classical gamma-function [3].

2. DISPLAYED MATHEMATICS

Let us introduced the following generalization of the gamma-function:

$$\tau,\beta\Gamma_a^c(\alpha;\gamma,\omega;b) \equiv \tilde{\Gamma}(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t^\omega} \, {}_1\Phi_1^{\tau,\beta}\left(a;c;-\frac{b}{t^\gamma}\right) dt, \tag{1}$$

where $Rec > Rea > 0$, $Re\alpha > 0$, $\tau \in R$, $\tau > 0$, $\beta \in R$, $\beta > 0, b > 0$, $\gamma \geq 0$ $\omega > 0$, ${}_1\Phi_1^{\tau,\beta}(a;c;z)$ is the (τ, β)– generalized confluent hypergeometric function:

$$ {}_1\Phi_1^{\tau,\beta}(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+\tau n) z^n}{\Gamma(c+\beta n) z!}, \tag{2}$$

$$ {}_1\Phi_1^{\tau,\beta}(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} {}_1\Psi_1 \left[\begin{matrix} (c;\tau) & zt^\tau \\ (c;\beta); & \end{matrix}\right] dt,$$

where ${}_1\Psi_1$ is the Fox-Wright function with parameter c and the positive real parameters τ,β; $\Gamma(a)$ is the classical gamma-function [3]. As $\tau = \beta = 1$ in ${}_1\Psi_1$ we get the confluent hypergeometric function Φ (a; c; z) [3]; when $\tau = \beta$ we have ${}_1\Phi_1^\tau(z)$ [27,28].

Now we introduce the next new generalization of psi-function;

$$ {}_{\tau,\beta}\psi_a^c(\alpha;\gamma,\omega;b) \equiv \tilde{\psi}(\alpha) = \frac{d \ln \Gamma(\alpha)}{d\alpha} = \frac{1}{\tilde{\Gamma}(\alpha)} \frac{d\tilde{\Gamma}(\alpha)}{d\alpha} = \tag{3}$$

$$= \frac{1}{\tilde{\Gamma}(\alpha)} \int_0^\infty t^{\alpha-1}(\ln t) e^{-t^\omega} \, {}_1\Phi_1^{\tau,\beta}\left(a;c;-\frac{b}{t^\gamma}\right) dt.$$

Let us notice that for practical applications it is the following form of (3) is more convenient:

$$\tilde{\psi}(\alpha) = \frac{b^\alpha}{\tilde{\Gamma}(\alpha)} \int_0^\infty x^{-\alpha-1}(\ln b - \ln x) \exp(-b^\omega x^{-\omega}) \, {}_1\Phi_1^\tau\left(a;c;-b^{1-\gamma} x^\gamma\right) dx \tag{4}$$

Let us give the new generalization of Dirichlet, Gauss, Malmsten Binet formulas for the $\psi(\alpha)$- function defined by (3).

Theorem 1: (Generalized Dirichlet formula). As $Re\alpha > 0$, $Re\beta \geq 0$; $\omega = 1$ $\beta = \tau$ the following formula is valid:

$$
{}_{\tau,\beta}\psi_a^c(\alpha;\gamma;b) = \int_0^\infty \left[e^{-x} - \left(1+x\right)^{-\alpha} \frac{\tilde{\Gamma}\left(\alpha, b(1+x)^{\gamma}\right)}{\tilde{\Gamma}(\alpha;b)} \right] x^{-1} dx \tag{5}
$$

Proof: The expression for $\tilde{\Gamma}(\alpha)$ (1) we multiply by $\dfrac{e^x - e^{-tx}}{x}$ and them to integrate with respect to x from 0 go ∞:

$$
\int_0^\infty \frac{e^{-x} - e^{-tx}}{x} \int_0^\infty t^{\alpha-1} e^{-t}\, {}_1\Phi_1^{\tau,\beta}\left(a;c;-\frac{b}{t^{\gamma}}\right) dt =
$$

$$
= \int_0^\infty \left[e^{-x} \int_0^\infty t^{\alpha-1} e^{-t}\, {}_1\Phi_1^{\tau,\beta}\left(a;c;-bt^{-\gamma}\right) dt - \right.
$$

$$
\left. - \int_0^\infty t^{\alpha-1} e^{-tx-t} e^{-x}\, {}_1\Phi_1^{\tau,\beta}\left(a;c;-bt^{-\gamma}\right) dt \right] x^{-1} dx =
$$

$$
= \int_0^\infty \left[t^{-x} \tilde{\Gamma}(\alpha) - (1+x)^{-\alpha} \tilde{\Gamma}(b(1+x)^{\gamma}) \right] x^{-1} dx, \tag{6}
$$

Using

$$
\int_0^\infty \frac{e^{-x} - e^{-tx}}{x} dx \int_0^\infty t^{\alpha-1} e^{-t}\, {}_1\Phi_1^{\tau,\beta}\left(a;c;-bt^{\gamma}\right) dt =
$$

$$
= \int_0^\infty t^{\alpha-1} (\ln t) e^{-t}\, {}_1\Phi_1^{\tau,\beta}\left(a;c;-bt^{-\gamma}\right) dt = \tilde{\Gamma}(\alpha)\psi(\alpha). \tag{7}
$$

Comparing the right parts of (6) and (7) we received (5).

Corollary: As $\beta = \tau = 1$, $\omega = 1$, $b = 0$ in (5) we get the direichlet formula [3];

$$\psi(\alpha) = \int_0^\infty (e^{-x} - \frac{1}{(1+x)^\alpha}) \frac{dx}{x} \tag{8}$$

It is easy to evalutate some integrals with the help of (5). For example:

$$\int_{-\infty}^{\infty} \left[\frac{{}_\tau\Gamma_a^c(v;b(1+e^{-x}))}{(1+e^{-x})^v {}_\tau\Gamma_a^c(v;b)} - \frac{{}_\tau\Gamma_a^c(\alpha;b(1+e^{-x}))}{(1+e^{-x})^\alpha {}_\tau\Gamma_a^c(\alpha;b)} \right] dx = \tag{9}$$

$$= {}_\tau\psi_a^c(\alpha;b) - {}_\tau\psi_a^c(v;b) \quad (\operatorname{Re}\alpha > 0, \operatorname{Re} v > 0, \gamma = 1);$$

$$\int_0^1 \frac{\exp\left(1 - \frac{1}{x}\right) {}_\tau\Gamma_a^c(v;b) - x^\alpha \, {}_\tau\Gamma_a^c\left(\alpha; \frac{b}{x}\right)}{x(1-x)} dx = \tag{10}$$

$$= {}_\tau\Gamma_a^c(\alpha;b), {}_\tau\psi_a^c(\alpha;b), \quad (\operatorname{Re}\alpha > 0, \gamma = 1).$$

The famous Gauss formula [3]

$$\psi(\alpha) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-\alpha t}}{1-e^t} \right) dt \tag{11}$$

can be generalized with help of (5) in the following form:

$${}_{\tau,\beta}\psi_a^c(\alpha;\gamma;b) = \int_0^\infty \left[e^{-t}t^{-1} - \frac{\tilde{\Gamma}\left(\alpha; be^{t\gamma}\right)}{\tilde{\Gamma}(\alpha;b)} . \frac{e^{-t\alpha}}{1-e^{-t}} \right] dt \tag{12}$$

Theorem 2: As *Re* $b > 0$, $\operatorname{Re}\alpha > 0$, $\beta = \tau$, $\omega = 1$ we have the generalized Binet formula:

$${}_{\tau,\beta}\psi_a^c(\alpha;\gamma;b) = \ln\alpha + \int_0^\infty \left[\frac{1}{t} - \frac{\tilde{\Gamma}\left(\alpha; be^{t\gamma}\right)}{\tilde{\Gamma}(\alpha;b)(1-e^{-t})} \right] e^{-\alpha t} dt \tag{13}$$

Corollary. Another form of (13):

$$ {}_{\tau,\beta}\psi_a^c(\alpha;\gamma;b) = \ln\alpha - \frac{1}{2\alpha} - \int_0^\infty \left[-\frac{1}{2} - \frac{1}{t} + \frac{\tilde{\Gamma}\left(\alpha; be^{t\gamma}\right)}{\tilde{\Gamma}(\alpha;b)(1-e^{-t})} \right] e^{-\alpha t} dt \tag{14} $$

From (12) it is easy to deduce the generalized Malmsten formula:

$$ \ln\tilde{\Gamma}(\alpha) = \int_1^\alpha \psi(\alpha) d = \int_0^\infty \left[(\alpha-1) - \frac{\tilde{\Gamma}\left(\alpha; be^{t\gamma}\right)\left(1-e^{-(\alpha-1)t}\right)}{\tilde{\Gamma}(\alpha;b)(1-e^{-t})} \right] e^{-t} t^{-1} dt \tag{15} $$

CONCLUSION

The variety of problems generating specials functions has to substantial increase of their number.

Our article is devoted to the generalization (according to Wright) of confluent hypergeometric function and to useful new generalization of the gamma-function and ψ-function.

These results can be applied to the solution of new boundary value problems of mathematical physics, to evalution of new improper intergrals, etc.

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