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Stochastic Self-Similar Universe, Oscillating Solutions and Quantum Particles

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In this paper, starting from the hierarchical self-similar Universe, we analyze the cosmological Schrödinger equation. Thanks to the first quantization, it is possible to recast Einstein equations in a Schrödinger-like form. By performing a numerical analysis, we find that, when assigned the value of a mass, the probability to find a particle in fixed mass range is the same everywhere in Universe models: open-close-flat, with or without cosmological constants. Considering different masses the probability changes, but they seem to be self-similar. In addition we have the question of if dark matter really exists, and if could consist of by ultra-light particles.

1. INTRODUCTION

Understanding the experimental rotation curves of galaxies is one of the main open problems in astrophysics. We know that it required a condition of equilibrium between gravitational force and centrifugal one for a body in rotation on itself to be dynamically stable. If the body rotates faster than a certain maximum velocity, it will disintegrate by its own centrifugal force. There are galaxies which rotate faster than the theoretical maximum velocity and, beyond this, the velocity of the stars along the arms does not seem to decrease in a Keplerian way. To justify the equilibrium one can hypothesize the existence of dark matter which increases the mass. In [1] and [2] by using E-Infinity theory as developed by Mohamed El Naschie [3], [4], [5], we have proposed an approach based on the Fantappiè's transformations; in fact, the study of projected spacetime, acknowledges a new law of time dilatation that we associated with the two different time scales. That is, seen through electromagnetic time, the processes such as rotation of celestial bodies were strongly accelerated in the past, while the behaviour of light and atomic processes remained invariable. In this paper we want to follow an approach based on quantum cosmology. The enigma of quantum gravity is the most crucial problem of modern theoretical physics and at present quantum cosmology does not exist but we know that gravity induces quantum effects as, for example, the evaporation of black holes. We will analyze the cosmological Schrödinger equation as it has been proposed in different contexts by the first author of this paper in [6] and other authors in [7], [8]. These authors, following Rosen [9], reduced the Einstein's equations to a quantum mechanical system and we want to show that, the solutions of this equation, give a different oscillations of the probability function to find the quantum particles at a given scale factor $a(t)$.

The paper is organized as follows: we first discuss about the large scale structure of Universe in Sect.2; Sect.3 presents the scale factor $a(t)$ and its consequences; Sect.4 is devoted to the cosmological Schrodinger equation; in Sect.5 we analyze the numerical solutions of this equation; conclusions are drawn in Sect.6.

2. THE LARGE SCALE STRUCTURE OF THE UNIVERSE

Observation shows that the Universe has a structure with scaling rules, where the clustering properties of cosmological objects reveals a form of hierarchy. In the previous paper, the first author considers the compatibility of a Stochastic Self-Similar, Fractal Universe with the observation and the consequences of this model. In particular, it was demonstrated that the observed segregated Universe is the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, $R(N) = (h / Mc)N^{\phi}$, where R is the radius of the astrophysical structures, h is the Planck constant, M is the total mass of the self-gravitating system, c the speed of light, N the number of nucleons within the structures, and $\phi = \frac{\sqrt{5}-1}{2}$ [6]. This expression agrees with the Golden Mean and with the gross law of Fibonacci and Lucas [10],[11]. If the distribution of galaxies is random, with a mean density of n galaxies for unitary volume, the relative probability to find a near galaxy in a volume dV to a distance r from any galaxy is $dP = ndV$. Instead the distribution is certainly not random and we have [12]

$$dP = ndV\,[1 + \zeta(r)], \tag{1}$$

where the correlation function $\zeta(r)$ characterizes the excess of probability and the experimental observations show that

$$\zeta(r) = \left(\frac{5Mpc}{r}\right)^{1,8}, \tag{2}$$

for $r < 20Mpc$. For clusters of galaxies we have

$$\zeta(r) = \left(\frac{26Mpc}{r}\right)^{1,8}, \tag{3}$$

for $r < 100Mpc$. Finally Broadhurst et al. found a periodicity in the three dimensional distributions of galaxy superclusters with a characteristic scale of $128Mpc$[13]. It seems to exist a regular network of superclusters and voids with a step size of $128Mpc$ where chains of superclusters are separated by voids of almost equal size.

3. FRACTAL SCALE FACTOR $a(t)$ AND QUANTUM PARTICLES

Below the Plank scale, we find the quantum fluctuations of the space-time geometry. For this reason, the scale of the order 10^{-40}cm represents the quantum memory that we find againg in the present Universe. Indeed, it is well-known that at the Plank scale, or equivalently after 10^{-43} seconds after the Big Bang, Universe starts emerging from the fluctuating quantum geometry in the way that we know and then producing the "building blocks" of nature, like quarks, leptons, fermions and bosons [14], [15], [12]. Could it be detected any trace of these processes? We will see that the Einstein's equations could have a solution for the scale factor $a(t)$ which is equivalent in form to the length $R(N)$, which generalizes the Compton wavelength. In some sense we can say that a kind of quantum imprinting is present in the actual Universe.

Let us start from the well-known Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = (8\pi G / c^4) T_{\mu\nu} + \Lambda g_{\mu\nu}, \tag{4}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the stress-energy tensor, and Λ is the cosmological constant; we assume a standard perfect fluid matter so that we may write

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu - p g_{\mu\nu}, \tag{5}$$

where p is the pressure, ρ the energy density and u_μ the velocity.

In [6] it was considered the case with cosmological constant $\Lambda = 0$. By taking into account the Friedman-Robertson-Walker Metric

$$ds^2 = c^2 dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right), \tag{6}$$

the author obtained the following relation

$$a(t) = \beta t^{1/2.} \tag{7}$$

In this way it is proved that the scale factor $a(t)$ has a scale law in time, where β is the following evaluated constant

$$\beta = \left(\frac{32\pi G}{3c^4} \tilde{a}^4 \tilde{\rho} \right)^{1/4}, \tag{8}$$

where the symbol $\tilde{x}$ denotes quantities at the present time. Besides it was found

$$\rho(t) = \frac{3c^4}{32\pi G} t^{-2}. \tag{9}$$

It means that $\rho(t)$ has no dependence from the present condition of the Universe. If we assume that also the scale factor is a function of the number of the constituents, $a = a(N)$, that is

$$a(N) = \frac{h}{m_n c} N^{\phi}, \tag{10}$$

by considering that when Universe passes from the dominance of radiation to the dominance of matter $\rho_{mat} = \rho_{rad}$. Consequently, we have

$$\tilde{\rho}_{mat} \frac{\tilde{a}^3}{a^3} = \tilde{\rho}_{rad} \frac{\tilde{a}^4}{a^4} \rightarrow m_n \sim \tilde{\rho}_{rat} \frac{\tilde{a}^4}{a} \tag{11}$$

and so

$$\frac{a(N)}{a(t)} = \frac{h}{c \tilde{\rho}_{rad} \tilde{a}^4} N^{\phi}. \tag{12}$$

It is obvious that $a(N) / a(t) = 1$ for every physical time; then, it follows

$$N = \left(\frac{c}{h} \tilde{\rho}_{rad} \tilde{a}^4 \right)^{1/\phi}. \tag{13}$$

In other words, the number of components N into a structure is constant as it could be suspected.

We now consider the general case represented by curvature $k = 0, \pm 1$ and cosmological constant $\Lambda = 0$ and $\Lambda \neq 0$. Starting from the following Bianchi's identities

$$T^{\mu}_{\nu;\mu} = 0, \tag{14}$$

we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3p) + \frac{\Lambda c^2}{3}, \tag{15}$$

$$(\frac{\dot{a}}{a})^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}\rho + \frac{\Lambda c^2}{3}. \tag{16}$$

By multiplying for $\frac{1}{2}ma^2$, finally we rewrite the previous (0, 0) Einstein equation as

$$\frac{1}{2}m\dot{a}^2 - \frac{m}{6}(\frac{8\pi G}{c^2}\rho + \Lambda c^2)a^2 = -\frac{1}{2}mkc^2, \tag{17}$$

where m is the mass of a quantum test particle.

4. COSMOLOGICAL SCHRODINGER EQUATION

In the usual approach to quantum gravity Universe is assumed to be a quantum system [16], [17], [18], [19] or as a classical background with primordial quantum processes, as in the context of quantum field theory on curved spacetime [20]. This approach is based on the Wheeler-De Witt equation

$$(G_{ijkl}\frac{\partial^2}{\partial h_{ij}\partial h_{kl}} - {}^{(3)}Rh^{1/2} + 2\Lambda h^{1/2})\psi(h_{ij}) = 0, \tag{18}$$

where h_{ij} is the spatial metric, ${}^{(3)}R$ is the scalar curvature of the intrinsic geometry of the three-surface and

$$G_{ijkl}\frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jl} - h_{ij}h_{kl}). \tag{19}$$

Here we have considered Universe as a fractal spacetime background, where primordial quantum processes have given rise to the present segregated macroscopic structures.

Following Rosen, we write

$$T + V = E, \tag{20}$$

where T has the role of the kinetic energy, V of the potential energy, and E of the total energy with the following expression

$$T = \frac{1}{2}m\dot{a}^2, V(a) = -\frac{1}{6}m[8\pi G/c^2)\rho + \Lambda c^2]a^2, E = -\frac{1}{2}mkc^2 \qquad (21)$$

The classical equation of mass motion is

$$m\ddot{a} = -\frac{dV}{da}. \qquad (22)$$

By defining the momentum of mass as

$$P = m\dot{a}, \qquad (23)$$

the Hamiltonian as

$$\mathcal{H} = \frac{P^2}{2m} + V(a), \qquad (24)$$

and by using the standard procedure of first quantization, we obtain the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}\mathcal{H}\psi, \qquad (25)$$

where ψ is the wave function. In other words, a quantum particle of mass m has the probability $|\psi|^2$ to be at a given scale factor $a(t)$. Taking into account the relation between the scale factor and the red-shift

$$\frac{\dot{a}}{a} = H = -\frac{\dot{z}}{z+1}, \qquad (26)$$

and so

$$\frac{\tilde{a}}{a} = 1 + z, \qquad (27)$$

we can easily obtain the red-shift z to observe the quantum particles. To complete this analysis, by using the standard quantum mechanics, the Schrödinger stationary equation

$$\mathcal{H}\psi = E\psi, \qquad (28)$$

has the stationary state of energy E

$$\psi(a, t) = \psi(a)e^{-iEt/h}. \tag{29}$$

Therefore, we can write

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{da^2} + V(a)\psi \tag{30}$$

and the Schrödinger stationary equation can be written in the form

$$\psi'' + [\frac{8\pi m^2 G}{3\hbar^2 c^2}\rho a^2 + \frac{m\Lambda c^2 a^2}{6} - \frac{m^2 kc^2}{\hbar^2}]\psi = 0 \tag{31}$$

where the prime indicates the derivative with respect to a.

Since

$$\rho = \frac{M}{2\pi^2 a^3}, \tag{32}$$

where M is the mass of all Universe, we have

$$\rho = \frac{\tilde{\rho}\tilde{a}^3}{a^3} \tag{33}$$

and finally we obtain

$$\psi'' + [\frac{A}{a} + Ba^2 + C]\psi = 0 \tag{34}$$

where

$$A = \frac{8\pi m^2 G}{3\hbar^2 c^2}\tilde{\rho}\tilde{a}^3. \tag{35}$$

Remembering that

$$\rho_c = \frac{3\tilde{H}^2 c^2}{8\pi G}, \tag{36}$$

we obtain

$$A = \frac{m^2 c^2 \tilde{a} \Omega_M}{\hbar^2} \tag{37}$$

5. SELF-SIMILAR NUMERICAL SOLUTIONS

We have analyzed the numerical solutions of the cosmological Schrödinger equation for different models of Universe and we have considered the oscillating solutions in a red shift range from $z = 0$ to $z = 1$. Let us remember the useful relations for the study of the equation

$$1 = \frac{\tilde{\rho}}{\rho_c} + \frac{\Lambda c^2}{3\tilde{H}^2} - \frac{kc^2}{\tilde{a}\tilde{H}^2} = \Omega_M + \Omega_\Lambda + \Omega_k$$

$$\tilde{H} = 100h \frac{Km}{Mpc} \text{ with } 0.4 \le h \le 1$$

$$\tilde{a} = \frac{c}{\tilde{H}} = 3000 h^{-1} Mpc \text{ for } k = 0$$

$$\tilde{a} = \frac{c}{\tilde{H}\sqrt{\Omega_M + \Omega_\Lambda - 1}} = \frac{3000 h^{-1} Mpc}{\sqrt{\Omega_M + \Omega_\Lambda - 1}} \text{ for } k = 1$$

$$\tilde{a} = \frac{c}{\tilde{H}\sqrt{1 - \Omega_M + \Omega_\Lambda}} = \frac{3000 h^{-1} Mpc}{\sqrt{1 - \Omega_M + \Omega_\Lambda}} \text{ for } k = -1$$

$$\hbar = 1{,}05459.10^{-27} erg\,.\,s$$

$$c = 3.10^{10} \frac{cm}{s}$$

$$B = \frac{m^2 c^2 \Lambda}{3\hbar^2}$$

$$C = -\frac{m^2 c^2 k}{\hbar^2}$$

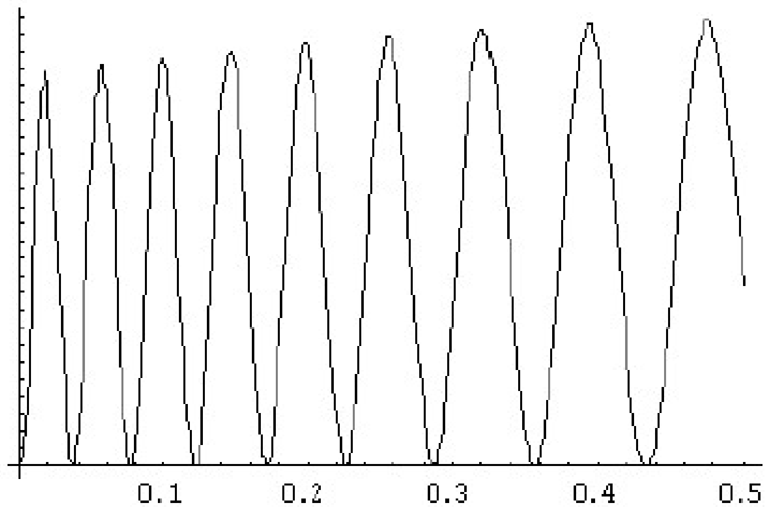

Figure 1: Probability wawes to find a particle of mass $10^{-52}g$ in a red shift range from z = 0 to z = 0.5.

If the cosmological constant $\Lambda = 0$, the equation becomes

$$\begin{cases} \psi'' + \dfrac{A}{a}\psi = 0 \\ \psi'' + [\dfrac{A}{a} + C]\psi = 0. \end{cases}$$

The first equation is valid in an Euclidean space, the second in a not Euclidean space. Vice versa, if the cosmological constant $\Lambda \neq 0$ we obtain

$$\begin{cases} \psi'' + [\dfrac{A}{a} + Ba^2]\psi = 0 \\ \psi'' + [\dfrac{A}{a} + Ba^2 + C]\psi = 0. \end{cases}$$

By observing the numerical solutions, we note that, the oscillations of probability to find a quantum particle, depend only on the value of the mass (see fig.1-5). That is, they do not depend on the Universe model and besides they are not tied to the initial conditions. The oscillations increase as the mass grows, but they appear self-similar. Inside the solar system, the peaks of $|\psi|^2$ correspond to a masses range from $m \cong 10^{-24}g$ to $m \cong 10^{-37}g$. Then, in the solar system, the particles that have a greater probability to be found, are nucleons, electrons and neutrinos. The ultra light particles, if they exist, have probability to be found on galactic and extragalactic scale and they could form halos of dark matter. Fig.3 is a zoom of fig.2; thanks to it we can observe the self-similarity of the probability. The same results is shown in fig.4-5 at different mass scale.

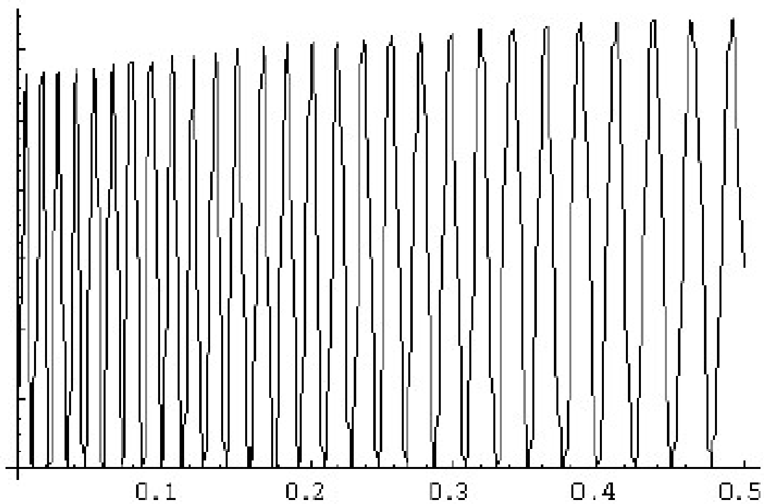

Figure 2: Probability wawes to find a particle of mass $10^{-51}g$ in a red shift range from $z = 0$ to $z = 0.5$.

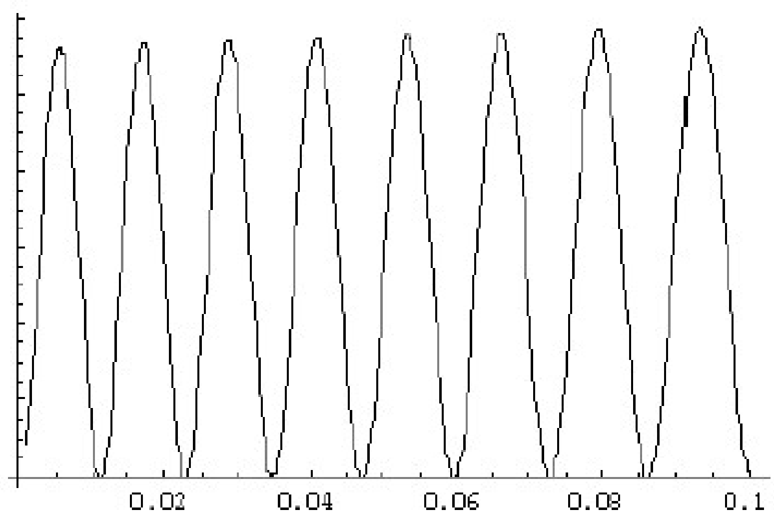

Figure 3: The preceding graph in the range five times smaller and enlarged to the same scale

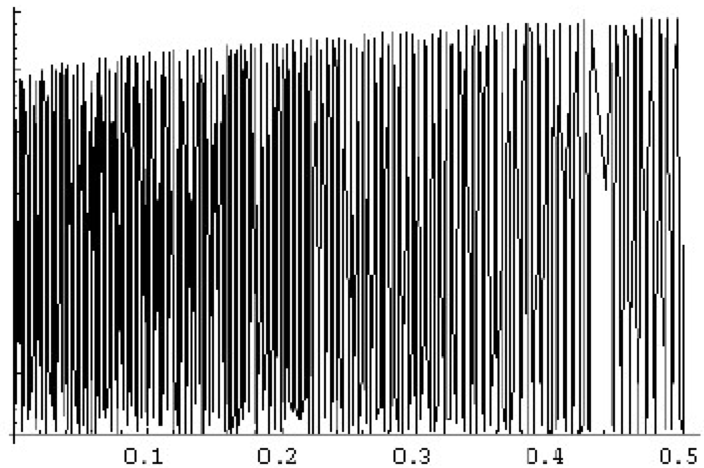

Figure 4: Probability wawes to find a particle of mass $10^{-50}g$ in a red shift range from $z = 0$ to $z = 0.5$

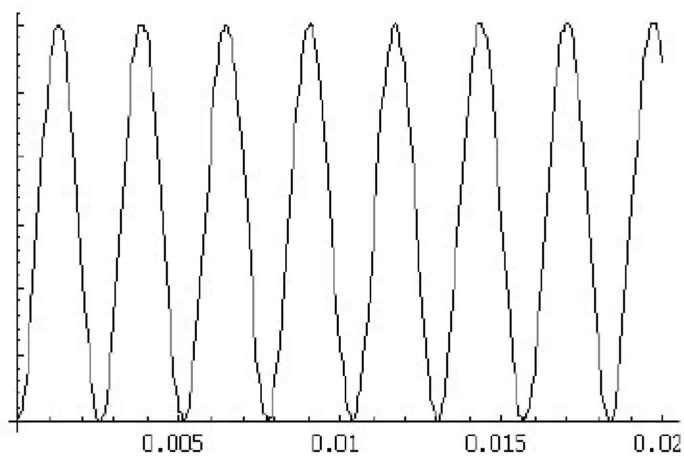

Figure 5: Probability wawes to find a particle of mass $10^{-50}g$ in a red shift range twenty five times smaller and enlarged to the same scale.

6. CONCLUSIONS

In this paper we have reduced the cosmological dynamical system to Schrödingerlike equation. Analyzing this equation numericaly, we have noted that they are Mass dependent and appear to be self-similar. It seems that particles are selfsimilar in the sense of quantum physics, that is as probability waves. Therefore, if this method is valid, the large scale structure of Universe is self-similar not only in a geometrical sense, but the particles obey in it a self-similar probabilistic distribution.

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