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Rainbow Option Pricing of Fractional version with Hurst exponet *H* being in $(\frac{1}{2},1)$

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The Label rainbow was coined by Rubinstein (1991), who emphasizes that this option was based on a combination of various assets like a rainbow is a combination of various colours. More generally, rainbow options are multi-asset options, also referred to as correlation options. Rainbow can take various other forms but the combining idea is to have a payoff that is depending on the assets sorted by their performance at maturity. In this article, we introduce Hurst exponents into the dynamics of stock log-price, and deduce the corresponding rainbow option pricing formulas with Hurst exponents being in $(\frac{1}{2},1)$.

1. INTRODUCTION

Since it's appearance in the 1970s, the Black-Scholes formula [1] has become one of the most popular method for option pricing and its generalized version has provided mathematicallly beautiful and powerful results on option pricing. However, they are stil theoretical adoptions and not necessary consistent with empirical features of financial return series, such as nonnormality, nonindependence, nonlinearity, etc. For example: Berg and Lyhagen [2], Lo [3], Hsieth [4] and Huang and Yang [5] showed that returns are of short-term (or long-term) dependency.

Lo and Mackinlay [6], Elton and Gruber [7], Frennberg and Hansson [8], Fama and French [9] and Poterba and Summer [10] reported the positive auto-correlation for stock in the short run and negative auto-correlation in the long run.

Let $S(t)$ denote the stock log-price. $S(t)$ is defined to be statistical self-similar with Hurst exponent H ($H \in (0,1)$) if $(S(\alpha t), t \geq 0)$ has the same probability law as $(\alpha^H S(t), t \geq 0)$.

As we have known, mentioned from [19], many facts show that Hurst exponent H varies in (0, 1) for different stocks returns or commodities prices or various exchange rates at al.. These inspire us to consider the option pricing with Hurst exponents ($H \in (0,1)$ (H may not equal to $\frac{1}{2}$). In fact, in this field, some models have been made. For instance, Takahashi [16], Cutland et al. [17] and Ren *et al.* [18] have considered the option pricing problems for $H \in (\frac{1}{2},1)$. In the case of $H \in (0,\frac{1}{2})$, as we have known, it has been studied by Wang et al. [19]. But they all considered the option involving one asset.

In this paper, we will study the rainbow option involving two assets with $H \in (\frac{1}{2},1)$ by using the similar way of Wang [19] , and deduce option pricing formula while the dynamics of stock log-price S of stock I and II satisfy

$$dS_i = \zeta_i(t)dt + \eta_i(t)dB_{H_i}, \quad H_i \in (\tfrac{1}{2},1)(i=1,2), H_1 \le H_2 \tag{1.1}$$

respectively, where $\zeta_i(t)$ and $\eta_i(t)$ (i = 1, 2) are deterministic functions of time t and $B_{H_i} = B_{H_i}(t,\omega)\ (i=1,2)$ is a normalized fractional Brownian motion (fBm) on a probability space (Ω, $\mathcal{F}$, P). Furthmore, we obtain generalized rainbow option pricing formula involving k ($k \ge 1$) assets.

2. SOME MATHEMATICAL PRELIMINARIES

In this section, we will consider the stochastic integral with respect to the fractional Brownian motion (fBm) that have the Hurst exponent in (0, 1) and the corresponding general Itô's formula with respect to a function $f(t,x_1(t),\ldots,x_k(t))$, where $\{x_i(t), t\in[a, b]\}$ (i = 1, . . ., k) are stochastic processes.

we work in probability space (Ω, $\mathcal{F}$, P). Fractional Brownian motion $B_H = B_H(t)$ with Hurst exponent $H \in (0, 1)$ is a continuous Gaussian process with stationary increments and with the following properties:

(1) $B_H(0) = 0$;

(2) E $B_H(t) = 0$ for all $t \ge 0$;

(3) $E[B_H(s)B_H(t)] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]$ for all $s,t \ge 0$.

The standard Brownian motion B is a fBm with Hurst exponent $H = \frac{1}{2}$. $r(n) \overset{\Delta}{=} E[B_H(1)(B_H(n+1) - B_H(n))]$, then we have the following properties from [20]:

(a) if $H \in (0, \frac{1}{2}), \sum_{n=0}^{\infty} |r(n)| < \infty$;

(b) if $H = \frac{1}{2}, \{B_H(n+1) - B_H(n)\}$ is uncorrelated;

(c) if $H \in (\frac{1}{2}, 1), \sum_{n=0}^{\infty} |r(n)| = \infty$.

Actually, if $H \in (0, \frac{1}{2}), r(n) < 0$ for $n \geq 1$ (negative correlation); if $H \in (\frac{1}{2}, 1), r(n) > 0$ $n \geq 1$ (positive correlation). The property $\sum_{n=0}^{\infty} |r(n)| = \infty$ is often referred to as long-range dependence. A fBm is also self-similar, that is, $(B_H(\alpha t), t \geq 0)$ has the same probability law as $(\alpha^H B_H(t), t \geq 0)$. A stochastic process $x = \{x(t)\}_{t \in R}$ is self-similar with index $H > 0$ (H-ss) if, for any $\alpha > 0$, if, for any $\alpha > 0, \{x(\alpha t)\}_{t \in R} \overset{d}{=} \{\alpha^H x(t)\}_{t \in R}$, where $\overset{d}{=}$ stands for the sense of the finite-dimensional distributions.

Definition 2.1 Let f be a real-valued function on $[a, b] \times \Omega$ such that

(I) f is a second order random function;

(II) is $\mathcal{B}\,[a, b] \times \mathcal{F}$ – *measurable.*

If $f(t) = \sum_{i=0}^{m-1} f_i \mathcal{X}[t_i, t_{i+1})(t)$ is a random step-function, then we define

$$\int_a^b f(x) dB_H(t) = \sum_{i=0}^{m-1} f_i [B_H(t_{i+1}) - B_H(t_i)]$$

In general, let $\Delta : a = t_0 < t_1 < \ldots < t_m = b$ be a partion of $[a, b]$. Set $f_i = f(t_i'), |\Delta| = \max_{0 \leq i \leq m-1} |t_{i+1} - t_i|$, $I_m(f, \omega) = \sum_{i=0}^{m-1} f_i [B_H(t_{i+1}) - B_H(t_i)]$, where $t_i' \in [t_i, t_{i+1}]$.

If $I_m(f, \omega)$ converges to a random variable $I(f, \omega)$ in quadratic mean (q.m) as $|\Delta| \to 0$, then we define

$$\int_a^b f(x) dB_H(t) = I(f, \omega)$$

We denote the set of functions such that $\int_a^b f(x) dB_H(t)$ exists by $\varepsilon\,(H)$.

Theorem 1: *Let $f(t, \omega)$ be a real-valued function defined on $[a, b] \times \Omega$ such that*

(C.1) both (I) and (II) hold;

(C.2) for each $\omega \in \Omega, f(t,\omega)$ *is absolutely continuous on* $[a, b]$ *and* $\int_a^b E[f_t(t,\omega)]^2\, dt < \infty$, *where* f_t *denotes the partial derivative of f with respect to t;*

(C.3) the process $f = (f(t, \omega))_{t\in[a,b]}$ and $B_H = (B_H(t))_{t\in[a,b]}$ are independent, then $f \in \varepsilon(H)$.

Note that for each $t \in [a, b]$, $f(., \omega)$ and $B_H(., \omega)$ are $\mathcal{F}$-measurable. When f is a deterministic function of time t, condition (C.3) is naturally satisfied. This is the situation in which we will use this theorem in the next section.

Theorem 2: *Let* $a_l(t,\omega)$, $b_l(t,\omega)$ *be* $\mathcal{B}[a, b] \times \mathcal{F}$*-measurable such that* $\int_a^b a_l\, |(t,\omega)|^2\, dt < +\infty$ *for almost every w and* $b_l(t,\omega)$ *satisfies (C.1)-(C.3) with* $H = H_i$ *in Theorem 1(l=1,2). Set*

$$x_l(t) - x_l(a) = \int_a^t a_l(\tau,\omega)d\tau + \int_a^t b_l(\tau,\omega)dB_{H_i}(\tau) \tag{2.2}$$

Let $f: [a, b] \times R \times R \to R$. *Assume that* $f_{x_1^p x_2^q}$ *and* $f_{x_1^p x_2^q t}(p,q = 0,\ldots,n)$ *are all continuous, where* $f_{x_1^p x_2^q} = \partial^{p+q} f / \partial x_1^p \partial x_2^q, f_{x_1^p x_2^q t} = \partial f_{x_1^p x_2^q} / \partial t$. *Set* $Y(t) = f(t, x_1(t), x_2(t))$. *Assume that* $1/n < H_1 \le 1/(n-1)$ *and* $H_1 \le H_2 < 1$. *Then for any given* $n \ge 2$, *we have*

$$Y(t) - Y(s)$$

$$\int_s^t [f_\tau(\tau, x_1(\tau), x_2(\tau)) + a_1(\tau) f_{x_1}(\tau, x_1(\tau), x_2(\tau)) + a_2(\tau) f_{x_2}(\tau, x_1(\tau), x_2(\tau))]d\tau$$

$$+ \lim_{\substack{|\Delta|\to 0 \\ m\to+\infty}} \sum_{i=0}^{m-1} \sum_{j=1}^{n-1} \frac{1}{j!} \sum_{p=0}^{j} \times C_j^p f_{x_1^p x_2^{j-p}}(t_i, x_1(t_i), x_2(t_i) b_1^p(t_i) b_2^{j-p}(t_i)[B_{H_1}(t_i)]^p [B_{H_2}(t_i)]^{j-p}$$

$$= \int_s^t [f_\tau(\tau, x_1(\tau), x_2(\tau)) + a_1(\tau) f_{x_1}(\tau, x_1(\tau), x_2(\tau)) + a_2(\tau) f_{x_2}(\tau, x_1(\tau), x_2(\tau))]d\tau$$

$$+ \lim_{\substack{|\Delta|\to 0 \\ m\to+\infty}} \sum_{i=0}^{m-1} \sum_{j=1}^{n-1} \frac{1}{j!} \sum_{\substack{p=0 \\ pH_1+(j-p)H_2 \le 1}}^{j} \times C_j^p f_{x_1^p x_2^{j-p}}(t_i, x_1(t_i), x_2(t_i) b_1^p(t_i) b_2^{j-p}(t_i)$$

$$\times [B_{H1}(t_{i+1}) - B_{H_1}(t_i)]^p [B_{H_2}(t_{i+1}) - B_{H_2}(t_i)]^{j-p}$$

in the sense of probability, where $[s, t] \subseteq [a, b]$.

Proof: See Appendix A.

From Theorem 2 and by using Taylor's formula, we also have:

Corollary 1: *Under the condition of Theorem 2,*

$$Y(t+\Delta t)-Y(t)=[f_t(t,x_1(t),x_2(t))+a_1(t)f_{x_1}(t,x_1(t),x_2(t))+a_2(t)f_{x_2}(t,x_1(t),x_2(t))]\Delta t$$

$$+\sum_{j=1}^{n-1}\frac{1}{j!}\sum_{\substack{p=0\\ pH_1+(j-p)H_2\le 1}}^{j} C_j^p f_{x_1^p x_2^{j-p}}(t,x_1(t),x_2(t))b_1^p(t)b_2^{j-p}(t)$$

$$\times[B_{H_1}(t+\Delta t)-B_{H_1}(t)]^p[B_{H_2}(t+\Delta t)-B_{H_2}(t)]^{j-p}+o(|\Delta t|).$$

Similar to Theorem 2, we also have the generalized conclusion:

Theorem 3: *Let* $a_l(t,\omega)$, $b_l(t,\omega)$ be $\mathcal{B}[a, b] \times \mathcal{F}$*-measurable such that* $\int_a^b a_l\,|(t,\omega)|^2\,dt < +\infty$ *for almost every w and* $b_l(t,\omega)$ *satisfies (C.1)-(C.3) with* $H = H_i$ *in Theorem 1(l=1, . . ., k). Set*

$$x_l(t)-x_l(a)=\int_a^t a_l(\tau,\omega)d\tau+\int_a^t b_l(\tau,\omega)dB_{H_i}(\tau)$$

Let f: $[a,b]\times\underbrace{R\times\ldots\times R}_{k}\to R.$ *Assume that* $f_{x_1^{p_1}\ldots x_k^{p_k}}$ *and* $f_{x_1^{p_1}\ldots x_k^{p_k}t}\,(p_1,\ldots,pk=0,\ldots,n)$ *are all continuous, where* $f_{x_1^{p_1}\ldots x_k^{p_k}}=\partial^{p_1+\ldots+p_k}f/\partial x_1^{p_1}\ldots\partial x_k^{p_k}$, $f_{x_1^{p_1}\ldots x_k^{p_k}t}=\partial f_{x_1^{p_1}\ldots x_k^{p_k}}/\partial t.$ Set $Y(t)$ $= f(t,x_1(t),\ldots,x_k(t))$. *Assume that* $1/n < H_1 \le 1/(n-1)$ *and* $H_1 \le H_2 \le \ldots \le H_k < 1$. *Then for any given* $n \ge 2$, *we have*

$$Y(t)-Y(s)=\int_s^t[f_\tau(\tau,x_1(\tau),\ldots,x_k(\tau))+\sum_{l=1}^{k}a_l(\tau)f_{xl}(\tau,x_1(\tau),\ldots,x_k(\tau))]d\tau$$

$$+\lim_{\substack{|\Delta|\to 0\\ m\to+\infty}}\sum_{i=0}^{m-1}\sum_{j=1}^{n-1}\frac{1}{j!}\sum_{p_1+\ldots+p_k=j}^{p_1,\ldots,p_k\ge 0}\frac{j!}{p_1!\ldots p_k!}$$

$$\times f_{x_1^{p_1}\ldots x_k^{p_k}}(t_i,x_1(t_i),\ldots,x_k(t_i))[B_{H_1}(t_{i+1})-B_{H_1}(t_i)]^{p_1}\ldots[B_{H_k}(t_{i+1})-B_{H_k}(t_i)]^{p_k}$$

$$= \int_s^t [f_\tau(\tau, x_1(\tau), \ldots, x_k(\tau)) + \sum_{l=1}^{k} a_l(\tau) f_{x_l}(\tau, x_1(\tau), \ldots, x_k(\tau))] d\tau$$

$$+ \lim_{\substack{|\Delta|\to 0 \\ m\to+\infty}} \sum_{i=0}^{m-1} \sum_{j=1}^{n-1} \sum_{p_1+\ldots+p_k=j}^{p_1,\ldots,p_k\geq 0} \frac{b_1^{p_1}(t_i)\ldots b_k^{p_k}(t_i)}{p_1!\ldots p_k!}$$

$$\times f_{x_1^{p_1}\ldots x_k^{p_k}}(t_i, x_1(t_i), \ldots, x_k(t_i))[B_{H_1}(t_{i+1}) - B_{H_1}(t_i))]^{p_1} \ldots [B_{H_k}(t_{i+1}) - B_{H_k}(t_i)]^{pk}$$

$$= \int_s^t [f_\tau(\tau, x_1(\tau), \ldots, x_k(\tau)) + \sum_{l=1}^{k} a_l(\tau) f_{x_l}(\tau, x_1(\tau), \ldots, x_k(\tau))] d\tau$$

$$+ \lim_{\substack{|\Delta|\to 0 \\ m\to+\infty}} \sum_{i=0}^{m-1} \sum_{j=1}^{n-1} \sum_{\substack{p_1+\ldots+p_k=j \\ p_1H_1+\ldots p_kH_k\leq 1}}^{p_1,\ldots,p_k\geq 0} \frac{b_1^{p_1}(t_i)\ldots b_k^{p_k}(t_i)}{p_1!\ldots p_k!}$$

$$\times f_{x_1^{p_1}\ldots x_k^{p_k}}(t_i, x_1(t_i), \ldots, x_k(t_i))[B_{H_1}(t_{i+1}) - B_{H_1}(t_i))]^{p_1} \ldots [B_{H_k}(t_{i+1}) - B_{H_k}(t_i)]^{pk}$$

in the sense of probability, where $[s, t] \subseteq [a, b]$.

Proposition 1: [18] *Suppose* $H \in (0, 1)$, *then*

$$P\left(\lim_{\Delta t\to 0+} \sup \frac{\Delta B_H}{(\Delta t)^H \sqrt{\log|\log \Delta t|}} = \sqrt{V_H}\right) = 1,$$

where $V_H = \Gamma(2 - 2H) \cos(\pi H) / \pi H(1 - 2H) > 0$.

3. MODEL OF RAINBOW OPTION PRICING FOR $\frac{1}{2} < H < 1$

In this section, we will give a fractional version of the Black-Scholes formula of rainbow option pricing in case of the Hurst exponent $H \in (\frac{1}{2}, 1)$. Considering the option written on a stock I and stock II, suppose that:

(A.1) There are no-transaction costs, and it is possible to hedge the risk of a portfolio in markets.

(A.2) The value process $D(t)$ of the riskless bond is given by

$$dD(t) = r(t)\, D(t)dt, \tag{3.1}$$

where $r(t)$ is the risk-free interest rate, which is a deterministic function of t. The log-Price S of stock I and stock II satisfy

$$dS_i = \zeta_i(t)dt + \eta_i(t)dB_{H_i},\ \ H_i \in (\tfrac{1}{2},1)(i=1,2), H_1 \le H_2 \tag{3.2}$$

respectively, where $u_i(t)$ and $\sigma_i(t)(i = 1,2)$ are deterministic functions of time t and satisfy the conditions of Theorem 2.

(A.2) Let $C\ (t, S_1, S_2)$ and $P\ (t, S_1, S_2)$ be the values at time t of European call and European put written on the stock I and stock II with expiration date T and exercise price X_1, X_2 respectively. Suppose $C\ (t, S_1, S_2)$ and $P\ (t, S_1, S_2)$ satisfy the terminal conditions

$$\begin{cases} C(T,S_1^T,S_2^T) = \max(e^{S_1^T} - X_1, e^{S_2^T} - X_2, 0)\,; \\ C(t,-\infty,-\infty) = 0 \qquad\qquad\qquad\qquad ; \\ \lim_{S_i\to+\infty} \dfrac{C(t,S_1,S_2)}{e^{S_i}} = 0\ \ (i=1,2). \end{cases} \tag{3.3}$$

and

$$\begin{cases} P(T,S_1^T,S_2^T) = \max(X_1 - e^{S_1^T}, X_2 - e^{S_2^T}, 0)\,; \\ P(t,-\infty,-\infty) = \max(X_1, X_2)e^{-r(T-t)} \qquad ; \\ \lim_{S_i\to+\infty} \dfrac{P(t,S_1,S_2)}{e^{S_i}} = 0\ \ (i=1,2). \end{cases} \tag{3.4}$$

(A.4) There exists a portfolio involving stock I, stock II, and rainbow option written on them such that this portfolio itselft is riskless. That is, the value of the riskless bond $D(t)$ can be exactly replicated by a 'self-financing' investment strategy involving stock I, stock II and the rainbow option written on both of them, i.e., when we take the number of unit of the rainbow option written on stock I and stock II as one, this 'self-financing' strategy satisfies the following conditions:

$$ND(t) = C(t,S_1,S_2) + N_1 e^{S_1} + N_2 e^{S_2} \tag{3.5}$$

and

$$N\Delta D(t) = \Delta C(t,S_1,S_2)N_1\Delta e^{S_1} + N_2\Delta e^{S_2} + o(\Delta t) \tag{3.6}$$

where $S_i = S_i(t, \omega)$ $(i = 1,2)$ denotes the log-price of stock I and stock II respectively, and N and $N_i (i = 1,2)$ denote the number of units of the riskless, stock I and stock II at date t, respectively.

Since it is possible to hedge the risk of a portfolio, let's consider how to do hedge the risk of a portfolio. Like the Black-Scholes case, we can form a riskless portfolio involving two stocks and a rainbow option written on them. First, from corollary 1, we have

$$\Delta e^{S_1} = \zeta_1(t)e^{S_1}\Delta t + \eta_1(t)e^{S_1}\Delta B_{H_1} + o(\Delta t),$$

$$\Delta e^{S_2} = \zeta_2(t)e^{S_2}\Delta t + \eta_2(t)e^{S_2}\Delta B_{H_2} + o(\Delta t)$$

and

$$\Delta C(t,S_1,S_2) = (\tfrac{\partial C}{\partial t} + \zeta_1(t)\tfrac{\partial C}{\partial S_1} + \zeta_2(t)\tfrac{\partial C}{\partial S_2})\Delta t + (\tfrac{\partial C}{\partial S_1}\eta_1(t)\Delta B_{H_1} + \tfrac{\partial C}{\partial S_2}\eta_2(t)\Delta B_{H_2}) + o(\Delta t)$$

Now we derive the rainbow option pricing formula. Applying Corollary 1 to $\Delta e^{S_1}, \Delta e^{S_1}$ *and* $\Delta C(t,S_1,S_2)$, we see that it follows from (3.1), (3.2), (3.5) and (3.6) that

$$\begin{aligned} & r(t)\Big(C(t,S_1,S_2) + N_1 e^{S_1}, N_2 e^{S_2}\Big)\Delta t \\ & = \left((\frac{\partial C}{\partial t} + \zeta_1(t)\frac{\partial C}{\partial S_1} + \zeta_2(t)\frac{\partial C}{\partial S_2}) + N_1\zeta_1(t)e^{S_1} + N_2\zeta_2(t)e^{S_2} \right)\Delta t \\ & + \left(\frac{\partial C}{\partial S_1}\eta_1(t) + N_1\eta_1(t)e^{S_1} \right)\Delta B_{H_1} + \left(\frac{\partial C}{\partial S_2}\eta_2(t) + N_2\eta_2(t)e^{S_2} \right)\Delta B_{H_2} + o(\Delta t). \end{aligned} \tag{3.7}$$

Dividing both sides of (3.7) orderly by $(\Delta t)^{H_1}\sqrt{\log|\log \Delta t|}, (\Delta t)^{H_2}\sqrt{\log|\log \Delta t|}, \Delta t$ and let $\Delta t \to +0$, we get from proposition 1 that

$$\eta_1(t)\frac{\partial C}{\partial S_1} + N_1\eta_1(t)e^{S_1} = 0 \tag{3.8}$$

$$\eta_2(t)\frac{\partial C}{\partial S_2} + N_2\eta_2(t)e^{S_2} = 0 \tag{3.9}$$

$$r(t)\Big(C(t,S_1,S_2) + N_1e^{S_1}, N_2e^{S_2}\Big) = (\frac{\partial C}{\partial t} + \zeta_1(t)\frac{\partial C}{\partial S_1} + \zeta_2(t)\frac{\partial C}{\partial S_2}) + N_1\zeta_1(t)e^{S_1} + N_2\zeta_2(t)e^{S_2} \tag{3.10}$$

Solving (3.8) and (3.9), we obtain

$$N_1 = -\frac{\partial C}{\partial S_1}e^{-s_1} \tag{3.11}$$

$$N_2 = -\frac{\partial C}{\partial S_2}e^{-s_2} \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we get the equation for European call $C(t, S_1, S_2)$ written on stock I and stock II:

$$r(t)C(t,S_1,S_2) = \frac{\partial C}{\partial t} + r(t)(\frac{\partial C}{\partial S_1} + \frac{\partial C}{\partial S_2}) \tag{3.13}$$

Solving (3.13) by using terminal condition (3.3), we obtain

$$r(t)C(t,S_1,S_2) = \max(0, e^{S_1} - X_1e^{-\tilde{r}(t)(T-t)}, e^{S_2} - X_2e^{-\tilde{r}(t)(T-t)}) \tag{3.14}$$

where

$$\tilde{r}(t) = \frac{1}{T-t}\int_t^T r(\tau)d\tau.$$

Remarks 1: From put-call relationship, we have the corresponding equations to (3.13)

$$r(t)P(t,S_1,S_2) = \frac{\partial P}{\partial t} + r(t)(\frac{\partial C}{\partial S_1} + \frac{\partial C}{\partial S_2})(\tfrac{1}{2} < H_1, H_2 < 1).$$

Remarks 2: Similarly, we can obtain that if $C(t,S_1,S_2,...,S_k)(k \geq 1)$ is the values at time t of European all written on stock 1, Stock 2, . . .,stock k with expiration date T and exercise $X_1, X_2, ..., X_k$ respectively and suppose $C(t,S_1,S_2,...,S_k)(k \geq 1)$ satisfies the terminal conditios

$$\begin{cases} C(T,S_1^T,S_2^T,\ldots,S_k) = \max(e^{S_1^T} - X_1, e^{S_2^T} - X_2, \ldots, e^{S_k^T} - X_k, 0) \ ; \\ C(t,-\infty,-\infty,\ldots,-\infty) = 0 \\ \lim_{S_i \to +\infty} \frac{C(t,S_1,S_2\ldots,S_k)}{e^{S_i}} = 1 \ \ (i = 1,2,\ldots,k) \end{cases} \quad (3.15)$$

we can get the equation:

$$r(t)C(t,S_1,S_2,\ldots,S_k) = \frac{\partial C}{\partial t} + r(t)(\frac{\partial C}{\partial S_1} + \frac{\partial C}{\partial S_2} + \ldots + \frac{\partial C}{\partial S_k}). \quad (3.16)$$

Solving the equation (3.16) by using the terminal conditions (3.15), we can obtain:

$$C(t,S_1,S_2,\ldots,S_k) = \max(e^{S_1} - X_1 e^{-\tilde{r}(t)(T-t)}, e^{S_2} - X_2 e^{-\tilde{r}(t)(T-t)}, \ldots, e^{S_k} - X_k e^{-\tilde{r}(t)(T-t)}, 0) \quad (3.17)$$

where

$$\tilde{r}(t) = \frac{1}{T-t}\int_t^T r(\tau)d\tau.$$

Remark 3: From (3.14), we know that the price of rainbow option with terminal conditions is closely related to the maximum of stock I and stock II but not only related to stock I or stock II.

Remark 4: The terminal conditions (3.3) are simple, so we can get the analytic solution to the equation (3.13). In another words, Maybe it's very difficult for us to obtain tha analytic solution when the terminal conditions is complicated.

Remark 5: If $1/n < H \leq 1/(n-1)$ $(n \geq 3)$, the equations which is deduced for getting rainbow option pricing formulas become fairly complicated. Under that condition, it's very difficult for us to obtain the solution even when the terminal conditions are simple.

4. CONCLUSION

In this paper, we develop a fractional version of the Black-Scholes model for valuing rainbow options with Hurst exponents in (1/2, 1). The formula on pricing rainbow option with terminal conditions (3.3) and (3.4) are obtained, and also we deduce a more common version in **Remark 2.** . More than those, we notice that:

1. As in the Black-Scholes model the 'Delta neutral' makes a portfolio a perfect hedge when $H \in (1/2, 1)$. It is also the same perfect thing for the rainbow option with $H \in (1/2, 1)$.

2. In the Black-Scholes model, we must acquire the parameters such as the volatility of the stock price and the risk-free interest which cannot be observed, But in our model, When $H \in (1/2, 1)$, only $\tilde{r}(t)$ needs to be estimated.
3. Compared with the option formula by using historical volatility, the rainbow option with $H \in (1/2, 1)$ is more simple.
4. It's very obvious that the solution (3.14) and (3.17) have nothing to with implied volatility and historical volatility.
5. Like [19], we can easily know our model is also no-arbitrage.

APPENDIX A

Proof of Theorem 2

Let $\Delta : s = t_0 < t_1 < \cdots < t_m = t$ be a partition of $[s,t].|\Delta| = \max_{0\le i\le m-1} |t_{i+1} - t_i|$, By using Taylor's formula, we have

$$Y(t) - Y(s) = \sum_{i=0}^{m-1} [f(t_{i+1}, x_1(t_{i+1}), x_2(t_{i+1})) - f(t_i, x_1(t_i), x_2(t_i))]$$

$$= \sum_{i=0}^{m-1} \Big\{ f(t_{i+1}, x_1(t_i), x_2(t_{i+1})) - f(t_i, x_1(t_i), x_2(t_i))$$

$$+ \sum_{j=1}^{n-1} \frac{1}{j!} \sum_{p=0}^{j} C_j^p f_{x_1^p x_2^{j-p}} (t_{i+1}, x_1(t_i), x_2(t_i))(x_1(t_{i+1}) - x_1(t_i))^p (x_2(t_{i+1}) - x_2(t_i))^{j-p}$$

$$+ \frac{1}{n!} \sum_{p=0}^{j} C_n^p f_{x_1^p x_2^{n-p}} (t_{i+1}, \xi_{1i}, \xi_{2i})(x_1(t_{i+1}) - x_1(t_i))^p (x_2(t_{i+1}) - x_2(t_i))^{n-p} \Big\}$$

$$= \sum_{i=0}^{m-1} f_t(\tau_i, x_1(t_i), x_2(t_i)) \Delta t$$

$$+ \sum_{i=0}^{m-1} \sum_{j=1}^{n-1} \frac{1}{j!} \sum_{p=0}^{j} C_j^p f_{x_1^p x_2^{j-p}} (t_i, x_1(t_i), x_2(t_i))(x_1(t_{i+1}) - x_1(t_i))^p (x_2(t_{i+1}) - x_2(t_i))^{j-p}$$

$$+ \sum_{i=0}^{m-1} \sum_{j=1}^{n-1} \frac{1}{j!} \sum_{p=0}^{j} C_j^p f_{x_1^p x_2^{j-p}} (\tau_i, x_1(t_i), x_2(t_i))(x_1(t_{i+1}) - x_1(t_i))^p (x_2(t_{i+1}) - x_2(t_i))^{j-p} \Delta t_i$$

$$+\sum_{i=0}^{m-1}\frac{1}{n!}\sum_{p=0}^{n}C_n^p f_{x_1^p x_2^{n-p}}(t_i,\xi_{1i},\xi_{2i})(x_1(t_{i+1})-x_1(t_i))^p(x_2(t_{i+1})-x_2(t_i))^{n-p}$$

$$+\sum_{i=0}^{m-1}\frac{1}{n!}\sum_{p=0}^{n}C_n^p f_{x_1^p x_2^{n-p}t}(\tau_i,\xi_{1i},\xi_{2i})(x_1(t_{i+1})-x_1(t_i))^p(x_2(t_{i+1})-x_2(t_i))^{n-p}\Delta t_i$$

$$=V_0(\omega)+V_1(\omega)+V_2(\omega)+V_3(\omega)+V_4(\omega) \qquad \text{(A.1)}$$

where $\Delta t_i = t_{i+1} - t_i$ and $t_i \le \tau_i \le t_{i+1} (i = 0, \ldots, m-1)$, and $\xi_{li} = \xi_{li}(\omega)$ is between $x_l(t_i,\omega)$ and $x_l(t_{i+1},\omega)(l = 1, 2)$. Clearly, we note that

$$V_0(\omega)\sum_{i=0}^{m-1} f_t(\tau_i,x_1(t_i),x_2(t_i))\Delta t \xrightarrow{a.s.} \int_s^t f_\tau(\tau,x_1(\tau),x_2(\tau))\Delta t \text{ as } |\Delta|\to 0. \qquad \text{(A.2)}$$

Since almost every sample function of $\{x_l(t)\}(l = 1, 2)$ is continuous, we may write $\xi_{li}(\omega) = x_l(\sigma_i(\omega))$ a.s., where $t_i \le \sigma_i(\omega) \le t_{i+1}$. From (2.1) and (A.1) we have

$V_1(\omega) + V_3(\omega)$

$$=\sum_{i=0}^{m-1}\sum_{j=1}^{n-1}\frac{1}{j!}\sum_{p=0}^{j}C_j^p f_{x_1^p x_2^{j-p}}(t_i,x_1(t_i),x_2(t_i))\left(\int_{t_i}^{t_{i+1}} a_1(\tau,\omega)\,d\tau+\int_{t_i}^{t_{i+1}} b_1(\tau,\omega)\,dB_{H_1}(\tau)\right)^p$$

$$\times\left(\int_{t_i}^{t_{i+1}} a_2(\tau,\omega)\,d\tau+\int_{t_i}^{t_{i+1}} b_2(\tau,\omega)\,dB_{H_2}(\tau)\right)^{j-p}$$

$$+\sum_{i=0}^{m-1}\frac{1}{n!}\sum_{p=0}^{n}C_n^p f_{x_1^p x_2^{n-p}}(t_i,\xi_{1i},\xi_{2i})\left(\int_{t_i}^{t_{i+1}} a_1(\tau,\omega)\,d\tau+\int_{t_i}^{t_{i+1}} b_1(\tau,\omega)\,dB_{H_1}(\tau)\right)^p$$

$$\times\left(\int_{t_i}^{t_{i+1}} a_2(\tau,\omega)\,d\tau+\int_{t_i}^{t_{i+1}} b_2(\tau,\omega)\,dB_{H_2}(\tau)\right)^{n-p}$$

$$=\sum_{i=0}^{m-1}\sum_{j=1}^{n-1}\frac{1}{j!}\sum_{p=0}^{j}C_j^p f_{x_1^p x_2^{j-p}}(t_i,x_1(t_i),x_2(t_i))$$

$$\times\left(b_1(t_i)(B_{H_1}(t_{i+1})-B_{H_1}(t_i))+\int_{t_i}^{t_{i+1}}[a_1(\tau)+b_1'(\tau)(B_{H_1}(t_{i+1})-B_{H_1}(\tau))]d\tau\right)^p$$

$$\times\left(b_2(t_i)(B_{H_2}(t_{i+1})-B_{H_2}(t_i))+\int_{t_i}^{t_{i+1}}[a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))]d\tau\right)^{j-p}$$

$$+\sum_{i=0}^{m-1}\frac{1}{n!}\sum_{p=0}^{n}C_n^p f_{x_1^p x_2^{n-p}}(t_i,\xi_{1i},\xi_{2i})$$

$$\times\left(b_1(t_i)(B_{H_1}(t_{i+1})-B_{H_1}(t_i))+\int_{t_i}^{t_{i+1}}[a_1(\tau)+b_1'(\tau)(B_{H_1}(t_{i+1})-B_{H_1}(\tau))]d\tau\right)^p$$

$$\times\left(b_2(t_i)(B_{H_2}(t_{i+1})-B_{H_2}(t_i))+\int_{t_i}^{t_{i+1}}[a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))]d\tau\right)^{n-p}$$

We estimate $V_2(\omega) + V_4(\omega)$ first. Set

$$L_1(\omega)=\max_{\substack{u,v,w\in[a,b]\\ j=1,\cdots,n\\ 0\le p\le j}}\left\{|x_1(u)|,|x_2(u)|,\left|f_{x_1^p x_2^{j-p}}(u,x_1(v),x_2(w))\right|,\left|f_{x_1^p x_2^{j-p}t}(u,x_1(v),x_2(w))\right|,\right.$$

$$\int_a^b\left|a_1(\tau)+b_1'(\tau)(B_{H_1}(\phi(\tau))-B_{H_1}(\tau))\right|d\tau,|b_1(u)|,|b_2(u)|,$$

$$\left.\int_a^b\left|a_2(\tau)+b_2'(\tau)(B_{H_2}(\phi(\tau))-B_{H_2}(\tau))\right|d\tau,\left|B_{H_1}(u)\right|,\left|B_{H_2}(u)\right|\right\}$$

$$L_2(\omega)=\max_{0\le i\le m-1}\left\{\left|\int_{t_i}^{t_{i+1}}a_1(\tau)d\tau\right|,\left|\int_{t_i}^{t_{i+1}}a_2(\tau)d\tau\right|,|x_1(t_{i+1})-x_1(t_i)|,|x_2(t_{i+1})-x_2(t_i)|,\right.$$

$$\left|B_{H_1}(t_i+1)-B_{H_1}(t_i)\right|,\left|\int_{t_i}^{t_{i+1}}[a_1(\tau)+b_1'(\tau)(B_{H_1}(t_{i+1})-B_{H_1}(\tau))]d\tau\right|,$$

$$\left.\left|B_{H_2}(t_i+1)-B_{H_2}(t_i)\right|,\left|\int_{t_i}^{t_{i+1}}[a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))]d\tau\right|\right\},$$

where $\phi(t) = t_{i+1}$ if $t_i \le t < t_{i+1}$.Then $L_1(\omega)$ is finite for almost every ω and as $L_2(\omega)\overset{a.s.}{\to}0$ as $|\Delta| \to 0$ since $\int_a^t a_l(\tau)d\tau, B_{H_1}(t)$,

$\int_a^t b_1'(\tau)d\tau, B_{H_1}(\tau)d\tau$ and $x_l(t)(l = 1, 2)$ are uniformly continuous on $[a, b]$ for alost every ω. Thus we obtain

$$\left|V_2(\omega)+V_4(\omega)\right|\le\sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{1}{j!}\left(\sum_{p=0}^{j}C_j^p L_1(\omega)L_2^j(\omega)\right)|\Delta t_i|$$

$$= L_1(\omega)\sum_{i=0}^{m-1}\left(\sum_{j=1}^{n}\frac{2^j}{j!}L_2^j(\omega)\right)|\Delta t_i|$$

$$\le L_1(\omega)\left(e^{2L_2(\omega)}-1\right)\sum_{i=0}^{m-1}|\Delta t_i|$$

$$\le L_1(\omega)(t-s)\left(e^{2L_2(\omega)}-1\right)\le L_1(\omega)(b-a)\left(e^{2L_2(\omega)}-1\right)\xrightarrow{a.s.}0 \text{ as } |\Delta|\to 0. \qquad (A.3)$$

To deal with $V_1(\omega) + V_3(\omega)$, we write

$V_1(\omega) + V_3(\omega)$

$$= \sum_{i=0}^{m-1}\Big\{f_{x_1}(t_i,x_1(t_i),x_2(t_i))\Big(b_1(t_i)(B_{H_1}(t_{i+1})-B_{H_1}(t_i))\int_{t_i}^{t_{i+1}}[a_1(\tau)+b_1'(\tau)(B_{H_1}(t_{i+1})-B_{H_1}(\tau))]d\tau\Big)$$

$$+f_{x_2}(t_i,x_1(t_i),x_2(t_i))\Big(b_2(t_i)(B_{H_2}(t_{i+1})-B_{H_2}(t_i))\int_{t_i}^{t_{i+1}}[a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))]d\tau\Big)\Big\}$$

$$+\sum_{i=0}^{m-1}\sum_{j=2}^{n-1}\frac{1}{j!}\Big\{\sum_{p=0}^{j}C_j^p f_{x_1^p x_2^{j-p}}(t_i,x_1(t_i),x_2(t_i))$$

$$\times\Big(b_1(t_i)B_{H_1}(t_{i+1})-B_{H_1}(t_i))+\int_{t_i}^{t_{i+1}}[a_1(\tau)+b_1'(\tau)(B_{H_1}(t_{i+1})-B_{H_1}(\tau))]d\tau\Big)^p$$

$$\times\Big(b_2(t_i)B_{H_2}(t_{i+1})-B_{H_2}(t_i))+\int_{t_i}^{t_{i+1}}[a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))]d\tau\Big)^{j-p}$$

$$+\sum_{i=0}^{m-1}\frac{1}{n!}\Big\{\sum_{p=0}^{n}C_n^p f_{x_1^p x_2^{n-p}}(t_i,\xi_{1i},\xi_{2i})$$

$$\times\Big(b_1(t_i)B_{H_1}(t_{i+1})-B_{H_1}(t_i))+\int_{t_i}^{t_{i+1}}[a_1(\tau)+b_1'(\tau)(B_{H_1}(t_{i+1})-B_{H_1}(\tau))]d\tau\Big)^p$$

$$\times\Big(b_2(t_i)B_{H_2}(t_{i+1})-B_{H_2}(t_i))+\int_{t_i}^{t_{i+1}}[a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))]d\tau\Big)^{n-p}\Big\}$$

$= V_5(\omega) + V_6(\omega) + V_7(\omega)$.

$V_5(\omega)$

$$= \sum_{i=0}^{m-1}\left\{f_{x_1}(t_i, x_1(t_i), x_2(t_i))\int_{t_i}^{t_{i+1}} a_1(\tau)d\tau + f_{x_2}(t_i, x_1(t_i), x_2(t_i))\int_{t_i}^{t_{i+1}} a_2(\tau)d\tau\right\}$$

$$+\sum_{i=0}^{m-1}\left\{f_{x_1}(t_i, x_1(t_i), x_2(t_i))[b_1(t_i)(B_{H_1}(t_{i+1}) - B_{H_1}(t_i))] + f_{x_2}(t_i, x_1(t_i), x_2(t_i))[b_2(t_i)(B_{H_2}(t_{i+1}) - B_{H_2}(t_i))]\right\}$$

$$+\sum_{i=0}^{m-1}\left\{f_{x_1}(t_i, x_1(t_i), x_2(t_i))\int_{t_i}^{t_{i+1}} b_1'(\tau)B_{H_1}(t_{i+1}) - B_{H_1}(\tau)]d\tau\right.$$

$$\left. + f_{x_2}(t_i, x_1(t_i), x_2(t_i))\int_{t_i}^{t_{i+1}} b_2'(\tau)B_{H_2}(t_{i+1}) - B_{H_2}(\tau)]d\tau\right\}$$

$= V_{51}(\omega) + V_{52}(\omega) + V_{53}(\omega).$

$V_6(\omega)$

$$= \sum_{i=0}^{m-1}\sum_{j=2}^{n-1}\frac{1}{j!}\left\{\sum_{p=0}^{j} C_j^p f_{x_1^p x_2^{j-p}}(t_i, x_1(t_i), x_2(t_i))b_2^{j-p}(t_i)\left(B_{H_1}(t_{i+1}) - B_{H_1}(t_i)\right)^p \left(B_{H_2}(t_{i+1}) - B_{H_2}(t_i)\right)^{j-p}\right\}$$

$$+\sum_{i=0}^{m-1}\sum_{j=2}^{n-1}\frac{1}{j!}\sum_{p=0}^{j} C_j^p f_{x_1^p x_2^{j-p}}(t_i, x_1(t_i), x_2(t_i))\left(b_1(t_i)B_{H_1}(t_{i+1}) - B_{H_1}(t_i)\right)^p$$

$$\times\left.\left(\sum_{q=0}^{j-p-1} C_{j-p}^q \left(b_2(t_i)(B_{H_2}(t_{i+1}) - B_{H_2}(t_i))\right)^q \left(\int_{t_i}^{t_{i+1}}[a_2(\tau) + b_2'(\tau)(B_{H_2}(t_{i+1}) - B_{H_2}(\tau))]d\tau\right)^{j-p-q}\right)\right\}$$

$$+\sum_{i=0}^{m-1}\sum_{j=2}^{n-1}\frac{1}{j!}\left\{\sum_{p=0}^{j} C_j^p f_{x_1^p x_2^{j-p}}(t_i, x_1(t_i), x_2(t_i))\left(b_2(t_i)(B_{H_2}(t_{i+1}) - B_{H_2}(t_i))\right)^{j-p}\right.$$

$$\times\left.\left(\sum_{k=0}^{p-1} C_p^k \left(b_1(t_i)(B_{H_1}(t_{i+1}) - B_{H_1}(t_i))\right)^k \left(\int_{t_i}^{t_{i+1}}[a_1(\tau) + b_1'(\tau)(B_{H_1}(t_{i+1}) - B_{H_1}(\tau))]d\tau\right)^{p-k}\right)\right\}$$

$$+\sum_{i=0}^{m-1}\sum_{j=2}^{n-1}\frac{1}{j!}\left\{\sum_{p=0}^{j} C_j^p f_{x_1^p x_2^{j-p}}(t_i, x_1(t_i), x_2(t_i))\right.$$

$$\times\left(\left[\sum_{k=0}^{p-1} C_p^k \left(b_1(t_i)(B_{H_1}(t_{i+1}) - B_{H_1}(t_i))\right)^k \left(\int_{t_i}^{t_{i+1}}[a_1(\tau) + b_1'(\tau)(B_{H_1}(t_{i+1}) - B_{H_1}(\tau))]d\tau\right)^{p-k}\right]\right.$$

$$\times\left.\left.\left[\sum_{q=0}^{j-p-1} C_{j-p}^q \left(b_2(t_i)(B_{H_2}(t_{i+1}) - B_{H_2}(t_i))\right)^q \left(\int_{t_i}^{t_{i+1}}[a_2(\tau) + b_2'(\tau)(B_{H_2}(t_{i+1}) - B_{H_2}(\tau))]d\tau\right)^{j-p-q}\right]\right)\right\}$$

$= V_{61}(\omega) + V_{62}(\omega) + V_{63}(\omega) + V_{64}(\omega).$

$V_7(\omega)$

$$=\sum_{i=0}^{m-1}\frac{1}{n!}\left\{\sum_{p=0}^{n}C_n^p f_{x_1^p x_2^{n-p}}(t_i,\xi_{1i},\xi_{2i})b_1^p(t_i)b_2^{n-p}(t_i)\left(B_{H_1}(t_{i+1})-B_{H_1}(t_i)\right)^p\left(B_{H_2}(t_{i+1})-B_{H_2}(t_i)\right)^{n-p}\right\}$$

$$+\sum_{i=0}^{m-1}\frac{1}{n!}\left\{\sum_{p=0}^{n}C_n^p f_{x_1^p x_2^{n-p}}(t_i,\xi_{1i},\xi_{2i})\left(b_1(t_i)(B_{H_1}(t_{i+1})-B_{H_1}(t_i))\right)^p\right.$$

$$\left.\times\left(\sum_{q=0}^{n-p-1}C_{n-p}^q\left(b_2(t_i)(B_{H_2}(t_{i+1})-B_{H_2}(t_i))\right)^q\left(\int_{t_i}^{t_{i+1}}[a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))]d\tau\right)^{n-p-q}\right)\right\}$$

$$+\sum_{i=0}^{m-1}\frac{1}{n!}\left\{\sum_{p=0}^{n}C_n^p f_{x_1^p x_2^{n-p}}(t_i,\xi_{1i},\xi_{2i})\left(b_2(t_i)(B_{H_2}(t_{i+1})-B_{H_2}(t_i))\right)^{n-p}\right.$$

$$\left.\times\left(\sum_{k=0}^{p-1}C_p^k\left(b_1(t_i)(B_{H_1}(t_{i+1})-B_{H_1}(t_i))\right)^k\left(\int_{t_i}^{t_{i+1}}[a_1(\tau)+b_1'(\tau)(B_{H_1}(t_{i+1})-B_{H_1}(\tau))]d\tau\right)^{p-k}\right)\right\}$$

$$+\sum_{i=0}^{m-1}\frac{1}{n!}\left\{\sum_{p=0}^{n}C_n^p f_{x_1^p x_2^{n-p}}(t_i,\xi_{1i},\xi_{2i})\right.$$

$$\times\left(\left[\sum_{k=0}^{p-1}C_p^k\left(b_1(t_i)(B_{H_1}(t_{i+1})-B_{H_1}(t_i))\right)^k\left(\int_{t_i}^{t_{i+1}}[a_1(\tau)+b_1'(\tau)(B_{H_1}(t_{i+1})-B_{H_1}(\tau))]d\tau\right)^{p-k}\right]\right.$$

$$\left.\left.\times\left[\sum_{q=0}^{n-p-1}C_{n-p}^q\left(b_2(t_i)(B_{H_2}(t_{i+1})-B_{H_2}(t_i))\right)^q\left(\int_{t_i}^{t_{i+1}}[a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))]d\tau\right)^{n-p-q}\right]\right)\right\}$$

$= V_{71}(\omega) + V_{72}(\omega) + V_{73}(\omega) + V_{74}(\omega).$

Since $B_{H_1}(t)$ and $f_{x_l}(t_i, x_1(t), x_2(t))$ are uniformly continuous on $[a, b]$ a.s., we can conclude that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $|B_{H_l}(t_{i+1}) - B_{H_l}(t_i)| < \varepsilon$ and $|f_{x_l}(t_i, x_1(t_i), x_2(t_i)) - f_{x_l}(t, x_1(t), x_2(t))| < \varepsilon$ and $t \in [t_i + t_{i+1})$ and $|\Delta| < \delta$ (l =1, 2). Thus

$$\left|\sum_{i=0}^{m-1}f_{x_l}(t_i,x_1(t_i),x_2(t_i))\int_{t_i}^{t_{i+1}}b_l'(\tau)[B_{H_l}(t_{i+1})-B_{H_1}(\tau)]d\tau\right|$$

$$\le \varepsilon L_1(\omega)\int_s^t|b_l'(\tau)|\,d\tau \le \varepsilon L_1(\omega)\int_s^t|b_l'(\tau)|\,d\tau,$$

and

$$\left|\sum_{i=0}^{m-1} f_{x_l}(t_i, x_1(t_i), x_2(t_i)) \int_{t_i}^{t_{i+1}} a_l(\tau) d\tau - \int_s^t f_{x_l}(\tau, x_1(\tau), x_2(\tau)) a_l(\tau) d\tau\right|$$

$$\leq \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} | f_{x_l}(t_i, x_1(t_i), x_2(t_i)) - f_{x_l}(\tau, x_1(\tau), x_2(\tau)) \| a_l(\tau) | d\tau$$

$$\leq \varepsilon \int_s^t | a_l(\tau) | d\tau \leq \varepsilon \int_a^b | a_l(\tau) | d\tau \leq \varepsilon \left(\frac{1}{2} \int_a^b | a_l(\tau) |^2 \, d\tau + \frac{1}{2}(b-a) \right).$$

So

$$V_{53} \xrightarrow{a.s.} 0, \tag{A.4}$$

and

$$V_{51} \xrightarrow{a.s.} \int_s^t (a_1(\tau) f_{x_1}(\tau, x_1(\omega), x_2(\tau)) + a_2(\tau) f_{x_2}(\tau, x_1(\omega), x_2(\tau))) \, d\tau \text{ as } |\Delta| \to 0. \tag{A.5}$$

Next,

$$|V_{62}(\omega) + V_{72}(\omega)|$$

$$\leq \sum_{i=0}^{m-1} \sum_{j=2}^{n} \frac{1}{j!} \left\{ \sum_{p=0}^{j} C_j^p L_1^{p+1}(\omega) L_2^p(\omega) \left[\sum_{q=0}^{j-p-1} C_{j-p}^q L_1^q(\omega) L_2^{q+j-p-q-1}(\omega) \right.\right.$$

$$\left.\left. \times \int_{t_i}^{t_{i+1}} | a_2(\tau) + b_2'(\tau)(B_{H_2}(t_{i+1}) - B_{H_2}(\tau)) | d\tau \right] \right\}$$

$$\leq \sum_{j=2}^{n} \frac{L_2^{j-1}(\omega)}{j!} \left\{ \sum_{p=0}^{j} C_j^p L_1^{p+1}(\omega) \sum_{q=0}^{j-p-1} C_{j-p}^q L_1^q(\omega) \int_a^b | a_2(\tau) + b_2'(\tau)(B_{H_2}(t_{i+1}) - B_{H_2}(\tau)) | d\tau \right\}$$

$$\leq \sum_{j=2}^{n} \frac{L_2^{j-1}(\omega)}{j!} \sum_{p=0}^{j} C_j^p L_1^{p+2}(\omega) \left((L_1(\omega)+1)^{j-p} - L_1^{j-p}(\omega) \right)$$

$$= L_1^2(\omega) \sum_{j=2}^{n} \frac{L_2^{j-1}(\omega)}{j!} \left((2L_1(\omega)+1)^j - (2L_1(\omega))^j \right)$$

$$\leq 2L_1^2(\omega)(2L_1(\omega)+1) \sum_{j=1}^{n-1} \frac{\left(L_2(\omega)(2L_1(\omega)+1) \right)^j}{j!}$$

$$\leq 2L_1^2(\omega)(2L_1(\omega)+1)\left(e^{L_2(\omega)(2L_1(\omega)+1)} - 1 \right) \xrightarrow{a.s.} 0 \text{ as } |\Delta| \to 0. \tag{A.6}$$

$$|V_{62}(\omega) + V_{72}(\omega)|$$

$$\leq \sum_{i=0}^{m-1}\sum_{j=2}^{n}\frac{1}{j!}\left\{\sum_{p=0}^{j}C_j^p L_1^{p+1}(\omega)L_2^p(\omega)\left[\sum_{q=0}^{j-p-1}C_{j-p}^q L_1^q(\omega)L_2^{q+j-p-q-1}(\omega)\right.\right.$$

$$\left.\left.\times\int_{t_i}^{t_{i+1}}\left|a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))\right|d\tau\right]\right\}$$

$$\leq \sum_{j=2}^{n}\frac{L_2^{j-1}(\omega)}{j!}\left\{\sum_{p=0}^{j}C_j^p L_1^{p+1}(\omega)\sum_{q=0}^{j-p-1}C_{j-p}^q L_1^q(\omega)\int_a^b\left|a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))\right|d\tau\right\}$$

$$\leq \sum_{j=2}^{n}\frac{L_2^{j-1}(\omega)}{j!}\sum_{p=0}^{j}C_j^p L_1^{p+2}(\omega)\left((L_1(\omega)+1)^{j-p}-L_1^{j-p}(\omega)\right)$$

$$= L_1^2(\omega)\sum_{j=2}^{n}\frac{L_2^{j-1}(\omega)}{j!}\left((2L_1(\omega)+1)^j-(2L_1(\omega))^j\right)$$

$$\leq 2L_1^2(\omega)\left(2L_1(\omega)+1\right)\sum_{j=1}^{n-1}\frac{\left(L_2(\omega)(2L_1(\omega)+1)\right)^j}{j!}$$

$$\leq 2L_1^2(\omega)\left(2L_1(\omega)+1\right)\left(e^{L_2(\omega)(2L_1(\omega)+1)}-1\right)\overset{a.s.}{\to}0 \text{ as } |\Delta|\to 0. \quad (A.6)$$

Similarly,

$$|V_{63}(\omega)+V_{73}(\omega)|\leq 2L_1^2(\omega)\left(2L_1(\omega)+1\right)\left(e^{L_2(\omega)(2L_1(\omega)+1)}-1\right)\overset{a.s.}{\to}0 \text{ as } |\Delta|\to 0. \quad (A.7)$$

Now we estimate $V_{64}(w) + V_{74}(w)$

$$|V_{64}(\omega) + V_{74}(\omega)|$$

$$\leq \sum_{i=0}^{m-1}\sum_{j=2}^{n}\frac{1}{j!}\left\{\sum_{p=0}^{j}C_j^p L_1(\omega)\left(\left[\sum_{k=0}^{p-1}C_p^k L_1^k(\omega)L_2^{k+p-k}(\omega)\right]\right.\right.$$

$$\left.\left.\times\left[\sum_{q=0}^{j-p-1}C_{j-p}^q L_1^q(\omega)L_2^{q+j-p-q-1}(\omega)\int_{t_i}^{t_{i+1}}\left|a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))d\tau\right|\right]\right)\right\}$$

$$\leq \sum_{j=2}^{n}\frac{L_2^{j-1}(\omega)}{j!}\left\{\sum_{p=0}^{j}C_j^p\left(\left[\sum_{k=0}^{p-1}C_p^k L_1^{k+1}(\omega)\right]\left[\sum_{q=0}^{j-p-1}C_{j-p}^q L_1^q(\omega)\right]\right)\int_a^b\left|a_2(\tau)+b_2'(\tau)(B_{H_2}(t_{i+1})-B_{H_2}(\tau))\right|d\tau\right\}$$

$$\sum_{j=2}^{n}\frac{L_2^{j-1}(\omega)}{j!}\left\{\sum_{p=0}^{j}C_j^p\left((L_1(\omega)+1)^p-L_1^p(\omega)\right)\left((L_1(\omega)+1)^{j-p}-L_1^{j-p}(\omega)\right)\right\}$$

$$\leq L_1^2(\omega)\sum_{j=2}^{n}\frac{L_2^{j-1}(\omega)}{j!}\left\{4\sum_{p=0}^{j}C_j^p(L_1(\omega)+1)^{p+j-p}\right\}$$

$$\leq 8L_1^2(\omega)\left(L_1(\omega)+1\right)\sum_{j=1}^{n-1}\frac{\left(2L_2(\omega)(L_1(\omega)+1)\right)^j}{j!}$$

$$\leq 8L_1^2(\omega)\left(L_1(\omega)+1\right)\left(e^{2L_2(\omega)(L_1(\omega)+1)}-1\right)\xrightarrow{a.s.}0 \text{ as } |\Delta|\to 0. \tag{A.8}$$

Finally, we note that

$$|V_{71}(\omega)| \leq \sum_{i=0}^{m-1}\frac{1}{n!}\left\{\sum_{p=0}^{n} C_n^p L_a^{1+p+n-p}\left|B_{H_1}(t_{i+1})-B_{H_1}(t_i)\right|^p\left|B_{H_2}(t_{i+1})-B_{H_2}(t_i)\right|^{n-p}\right\}$$

$$\leq L_1^{n+1}(\omega)\frac{2^n}{n!}\sum_{i=0}^{m-1}\sum_{p=0}^{n}\left|B_{H_1}(t_{i+1})-B_{H_1}(t_i)\right|^p\left|B_{H_2}(t_{i+1})-B_{H_2}(t_i)\right|^{n-p}.$$

For any given $\varepsilon > 0$, by **Markov** inequality and Cauchy-Schwarz inequality with respect to expectation, and when $1/n < H_1 < 1/(n-1)$, we have

$$P\left\{\sum_{i=0}^{m-1}\sum_{p=0}^{n}\left|B_{H_1}(t_{i+1})-B_{H_1}(t_i)\right|^p\left|B_{H_2}(t_{i+1})-B_{H_2}(t_i)\right|^{n-p}>\varepsilon\right\}$$

$$\leq \frac{1}{\varepsilon}\sum_{i=0}^{m-1}\sum_{p=0}^{n}E\left[\left|B_{H_1}(t_{i+1})-B_{H_1}(t_i)\right|^p\left|B_{H_2}(t_{i+1})-B_{H_2}(t_i)\right|^{n-p}\right]$$

$$\leq \frac{1}{\varepsilon}\sum_{i=0}^{m-1}\sum_{p=0}^{n}\left(E\left|B_{H_1}(t_{i+1})-B_{H_1}(t_i)\right|^{2p}.E\left|B_{H_2}(t_{i+1})-B_{H_2}(t_i)\right|^{2n-p}\right)^{\frac{1}{2}}$$

$$\leq \frac{M}{\varepsilon}\sum_{i=0}^{m-1}\sum_{p=0}^{n}\left|t_{i+1}-t_i\right|^{pH_1+(n-p)H_2} = \frac{M}{\varepsilon}\sum_{i=0}^{M-1}\sum_{p=0}^{n}\left|t_{i+1}-t_i\right|^{pH_2+(n-p)H_1}$$

$$\leq \frac{M}{\varepsilon}\sum_{p=0}^{n}|\Delta|^{p(H_2-H_1)}.(b-a)|\Delta|^{nH_1-1)}\to 0 \text{ as } |\Delta|\to 0. \tag{A.9}$$

where M is a constant. Therefore, $V_{71}(\omega)\xrightarrow{p}0$ as $|\Delta|\to 0$.

It follows from (A.1)-(A.9) that

$Y(t)-Y(s)=\lim_{\substack{|\Delta|\to 0\\ m\to+\infty}}\left(V_0(\omega)+V_{51}(\omega)+V_{52}(\omega)+V_{61}(\omega)\right)$ in the sense of probability. If $pH_1+(j-p)$ $H_2>1$, we also have

$$\sum_{i=0}^{m-1}\frac{1}{j!}C_j^p f_{x_1^p x_2^{j-p}}(t_i,x_1(t_i),x_2(t_i))b_1^p(t_i)b_2^{j-p}(t_i)[B_{H_1}(t_{i+1})-B_{H_1}(t_i)]^p[B_{H_2}(t_{i+1})-B_{H_2}(t_i)]^{j-p}\xrightarrow{p}0 \text{ as } |\Delta|\to 0.$$

Hence Theorem 2 follows from the above discussion.

REFERENCES

[1] Black F, Scholes Ms. (1973) The pricing of option and corporate liabilities. J. Political Economy; **81**: 367-59.

[2] Berg E, Lyhagen J. (1998) short and long-run dependence in Swedish stock returns. Appl Financial Economics; 8: 435-43.

[3] Lo AW. (1991) Long-term memory in stock market prices. Econometrica; **59**: 1279-313.

[4] Hsieth DA. (1991) Chaos and non-linear dynamics:applications to financial market. J. Finance; **46**: 1839-77.

[5] Huang B, Yang CW. (1995) The fractal structure in multinational stock returns. Appl. Economics Lett; **2**: 67-71.

[6] Lo AW, Machinlay AC. (1988) Stock market prices do not follow random walks: evidence from a simple specification test. Rev Financial Stud; **1**: 41-66.

[7] Elton EJ, Gruber MJ. (1995) Modern portfolio theory and investment analysis. New York: Wiley.

[8] Frennberg P, Hansson B. (1993) Testing the random walk hypothesis on Swedish stock prices: 1919-1990. J Banking Finance; **17**: 175-91.

[9] Fama E, French K. (1988) Permanent and temporary components of of stock prices. J. Political Economy; **96**: 245-73.

[10] Poterba J, Summer L. (1988) Mean reversion in stock returns: evidence and implications. J. Financial Economics; **22**: 27-60.

[11] Mandelbrot BB. (1963) The variation of certain speculative price. J Business Univ Chiago; **36**: 394-419.

[12] Evertze CLG, Berkner K. (1995) Large deviation and self-similarity analysis of curves: DAX stock price. Chaos, Solitions & Fractals; **6**: 121-36.

[13] Evertze CJG. (1995) Self-similarity of high-frequency USD-DEN exchange rates. *In: Proceeding of the First International Conference on High Frequency Date in Finance*, vol. 3. Zurich.

[14] Voss RF. 1/ f noise and fractals in economic time series. In: Encamacão JL, Peitgen H-O, Sakas G, Englert G, editors. Fractal geometry and computer graphics. Berlin: Springer 1992. p. 45-52.

[15] Peter EE. (1994) Fractal market analysis. New York: Wiley.

[16] Takahashi M. Evaluation methods for derivative securities using the Hurst exponent. Master thesis, Tsukuba University [in Japanese].

[17] Cutland N, Kopp P, Willinger W. (1993) stock price returns and the Joseph effect: a fractional version of the Black-Scholes model. In: Proceedings of the Monte Vertia Conference, Ascona, Switzerland.

[18] Ren F-Y, Wang X-T Lang J-R. (2000) A proof for French's empirical formula on option pricing. Chaos, Solitons & Fractals; **12** (2001): 2441-2453

[19] Wang X-T Qiu W-Y, Ren F-Y. (2000) option pricing of fractional version of the Black-Scholes model with Hurst exponent H being in $\left(\frac{1}{3},\frac{1}{2}\right)$. Chaos, Solitons & Fractals;12(2001): 599-608.

[20] Embrechts, P. and Maejma, M. (2002) Selfsmlar Processes. Princeton University Press, Princeton.

[21] Arcones MA. (1995) On the law of the iterated logarithm for Gaussian processes. J Theoret Prob; 8(4): 877-904.)

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