Mathematical Methods, Physical Methods and Simulation Science and Technology
Vol. 2 (January-December, 2018)

Fuzzy Platonic Spaces as A Model for Quantum Physics

M.S. EL NASCHIE

We start from the geometry of classical platonic solids and their generalization to four dimensional polytopes. Subsequently it is shown how the so obtained relationships are related to high energy particle physics. In particular the topology of a fuzzy Dodecahedron is used to give information about the elementary particles content of the standard model of high energy physics.

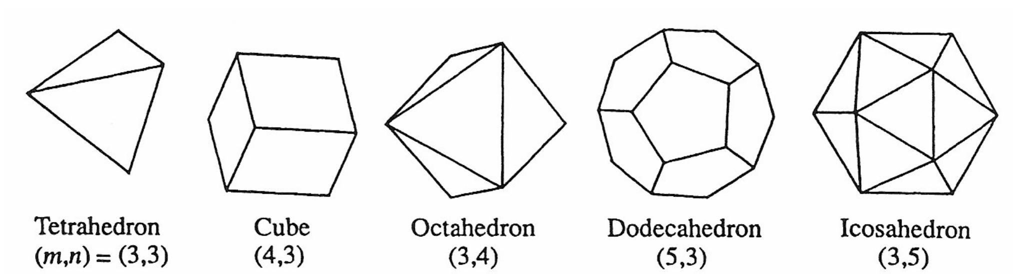
INTRODUCTION

It was around 1919 when a very young school boy at an impressionable age, called Werner Heisenberg read for the first time Plato's dialogue "Tomaeus" where a philosophical proposition was advanced putting the five regular platonic solids in relation to the building blocks of the universe [1]. It was in the late sixties when the author came across Heisenberg's writing on the subject which made a deep impression on him and in particular the role of symmetry in particle physics [2-4]. None the less, in all these past years, the author was reasonably convinced that the platonic solids themselves are of marginal importance being merely the carrier of the over powering idea of intrinsic space symmetry. It was only relatively recently that a direct interest in the topology of the platonic solids was rekindled due to work on fuzzy hyperbolic and Kähler manifolds such as M_F^4 and $K(E^{(\infty)})$ discussed in various relatively recent publications [5,6]. A second source of inspiration was undoubtedly the work of Luminet as well as Iovane [8].

In the present work we will start by reviewing basic properties of the topology of the platonic solids and establishing some connections with particle physics. Subsequently we look into the generalization to four dimensional polytopes and possible connections with super strings and P-Brane theories. Finally the relation between the sharp and fuzzy forms of the platonic theory is explored following F. John's idea of adding and subtracting "small terms" in order to simplify the field equations without affecting the accuracy of predictions [9,10].

1. The Topology of the Platonic Solids

We start by recalling the basic combinatorial fact of the original five platonic solids fixing their topology (Fig. 1). For the sake of completeness we have included in Table (1) two more solids, namely the Dihedron and its dual.



The Platonic polyhedra.

Figure 1: The Five Classical Platonic Solids

Table 1
Combinatorial Facts about the Regular Polyhedron

				Number of		
	m	*n*	*Vertices*	*Edges*	*Faces*	
Dihedron	k	2	k	k	2	(k ≥ 3)
Dual dihedron	2	k	2	k	k	(k ≥ 3)
Tetrahedron	3	3	4	6	4	
Cube	4	3	8	12	6	
Octahedron	3	4	6	12	8	
Dodecahedron	5	3	20	30	12	
Icosahedron	3	5	12	30	30	

Let us calculate first the Euler characteristic which is given by [11-13]

$$\begin{aligned}\chi &= \text{Number of faces} + \text{number of vertices} - \text{number of edges}\\ &= F + V - E\end{aligned}$$

It is evident that in all cases, the value is that of a topological sphere, namely $\chi = 2$. For example in the case of the Dodecahedron we find following Table (1) that:

$$\begin{aligned}\chi &= 12 + 20 - 30\\ &= 32 - 30\\ &= 2\end{aligned}$$

as should be. The same is naturally true for the dual topology of the Icosahedron.

In a previous work we showed that adding all the elements of the Dodecahedron or for that matter its dual form, then we obtain a lower bound on the number of elementary particles of the standard model. In other words we have [11-13]

$$\begin{aligned} N(SM) &= F = V + E \\ &= 12 + 20 + 30 \\ &= 62 \end{aligned}$$

particles. It is important to interpret this number of elements in what we have termed the solution of certain equations for the three-orbit case as shown in Table (2). There we see that the total number of elements is 2 (k + 1), 14, 26 and 62 for the Dihedron, the Tetrahedron, the Octahedron and the Dodecahedron respectively. For the Icosahedron we find the same value of the Dodecahedron.

Equally important to note is our conjecture which is similar to that made in the standard model as well as string and other similar theories that the number of elementary particles in the standard model is accurately approximated by the dimension (order) of the relevant symmetry group as given in Table (3). Thus we may write

Table 2
Solutions of the Equation for the Three-orbit Case

					Number of elements in the orbit			
Case	*mp*	*mq*	*mr*	*n*	*p*	*q*	*r*	
(i)	2	2	k	2k	k	k	2	(k ≥ 2)
(ii)	2	3	3	12	6	4	4	
(iii)	2	3	4	24	12	8	6	
(iv)	2	3	5	60	30	20	12	

Table 3
Rotational Symmetries of the Regular Polyhedron

	Number of axes of orders:					*Order of the*
	2	*3*	*4*	*5*	*k*	*symmetry group*
Dihedron	k	—	—	—	1	2k
Dual dihedron	k	—	—	—	1	2k
Tetrahedron	3	4	—	—	—	12
Cube	6	4	3	—	—	24
Octahedron	6	4	3	—	—	24
Dodecahdron	15	10	—	6	—	60
Icosahedron	15	10	—	6	—	60

$$\begin{aligned} N(SM) &\simeq \text{Dim (sym. Dode.)} \\ &\simeq 60\,. \end{aligned}$$

This is the number of elementary particles believed to have already been discovered and consequently the difference between N(SM) = 62 and N(SM) = 60 may be interpreted as the two most elusive particles of quantum physics, namely the graviton and the Higgs [6,14].

2. The Transfinite–Fuzzy Topology of Certain Platonic Solids

We normally think of the platonic bodies, their geometry and topology as the epitome of precision and exactness which leaves no space what so ever for the notion of fuzziness. However let us look at Table (4) to see how the geometry of these solids is fixed. A mere glance at this Table suffices to make clear that in the case of the Dodecahedron and its dual form, the corner points and consequently the entire shape cannot be fixed with infinite exactness. This is true even when we disregard the ever present problem of physical measurement because of the involvement of the golden mean $\phi = (\sqrt{5} - 1) / 2$. This number, as is well known, is the most irrational number, i.e. the least accurately approximated using a rational continued expansion. At the same time we know that $d_c^{(0)} = \phi = (\sqrt{5} - 1) / 2$ is the backbone Hausdorff dimension of E-infinity and plays a central role there [5,6,17]. For this reason it is feasible to think, when appropriate, of the Dodecahedron not as a sharp geometrical-topological shape but as a dynamically fluctuating fractal-like shape. Proceeding in this way one could utilize the procedure first advanced by the Courant Institute mathematician, the American Prof. F. John of adding or subtracting small terms to simplify the equations of curved elastic surfaces without affecting the accuracy of the end result [9,10]. Further extension of this method in E-infinity theory is more rooted into the compactification procedure of modular spaces [6].

Thus the dimension of the compactified Klein curve Dim $\Gamma(7) = 336$ increases to Dim $\Gamma_c(7) = 336 + 16k \cong 339$ where $k = \phi^3 (1 - \phi^3) = 0.18033989$ and ϕ is the golden mean. Similarly the Euler characteristic for K3 Kähler manifold increases from $\chi = 24$ to $\chi = 26 + k = 26.18033989$ in the case of a fuzzy K3 [5,6]. Proceeding in the same vain of the above, we can write for the fuzzy version of the classical Dodecahedron that

$$\begin{aligned} \langle \chi_F \rangle &= \langle F \rangle + \langle V \rangle - \langle E \rangle \\ &= (13) + (22 + k) - (33 + 2k) \\ &= 2 - k \simeq 2 \end{aligned}$$

Table 4
The Coordinates of the Five Platonic Solids

Polyhedron	*Coordinates*
tetrahedron	(1,1,1), (1,-1,-1), (-1,-1,1), (-1,1,-1)
hexahedron	(1,1,1), (1,1,-1), (1,-1,1), (-1,1,1), (1,-1,-1), (-1,1-1), (-1,-1,1), (-1,-1,-1)
octahedron	(1,0,0), (0,0,1), (0,1,0), (-1,0,0), (0,-1,0), (0,0-1)
dodecahedron	$(0, \pm\phi^{-1}, \pm\phi)$, $(\pm\phi^{-1}, \pm\phi, 0)$, $(\pm\phi, 0, \pm\phi^{-1})$, $(\pm1, \pm1, \pm1)$,
icosahedron	$(1,0,\phi)$, $(1,0,-\phi)$, $(-1,0,\phi)$, $(-1,0,-\phi)$, $(0,\phi,1)$, $(0,\phi,-1)$, $(0,-\phi,1)$, $(0,-\phi,-1)$, $(\phi,1,0)$, $(\phi,-1,0)$, $(-\phi,1,0)$, $(-\phi,-1,0)$

Table 5
Combinatorial Facts of the 4 Dimensional Polytopes

Name	*Schafli symbol*	N_0	N_1	N_2	N_3	*g*
5-cell, α_4 . .	(3, 3, 3)	5	10	10	5	120
16-cell, β_4 . .	(3, 3, 4)	8	24	32	16	384
Tessaract, γ_4 .	(4, 3, 3)	16	32	24	8	
24-cell . .	(3, 4, 3)	24	96	96	24	1152
600-cell . .	(3, 3, 5)	120	720	1200	600	14400
120-cell . .	(5, 3, 3)	600	1200	720	120	

For < N(S) > on the other hand, one finds

$$\begin{aligned} < N(SM) > &= 68 + 3k \\ &= \bar{\alpha}_o / 2 \end{aligned}$$

as in the exact E-infinity solution [5,6]. Note that in the low dimensional theory there is no straight forward mechanical way of adjusting the value using the transfinite correction scheme. However the situation is clearer when considering higher dimensional topology as will become evident later on.

3. Generalization to Higher Dimensionalities

Let us consider the evaluation of the topology of a point to a cube as shown in Fig. (2). It is thus not difficult to see that a systematic induction leads to what we may label four dimensional cube γ_4 as shown at the extreme right of the figure (2) and which has the following values:

> 16 vertices, 23 edges, 24 faces in addition to 8 cells which do not exist at all in the three dimensional case.

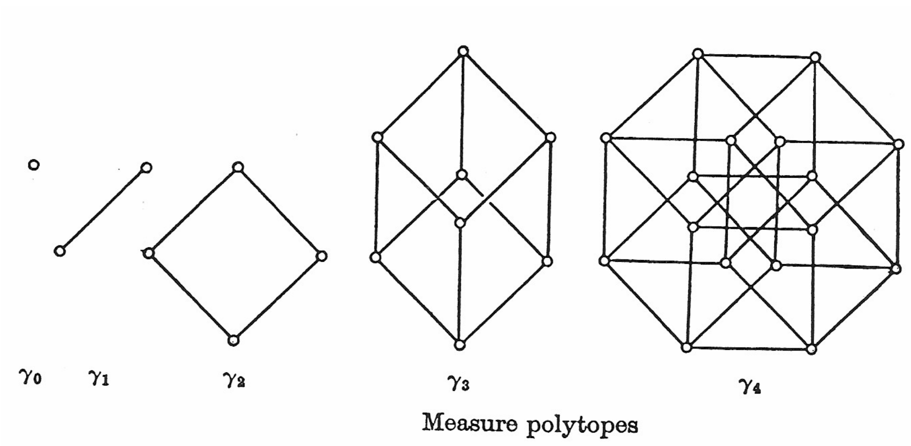


Figure 2: The Evolution of Topology from a Point to a 4-D cube

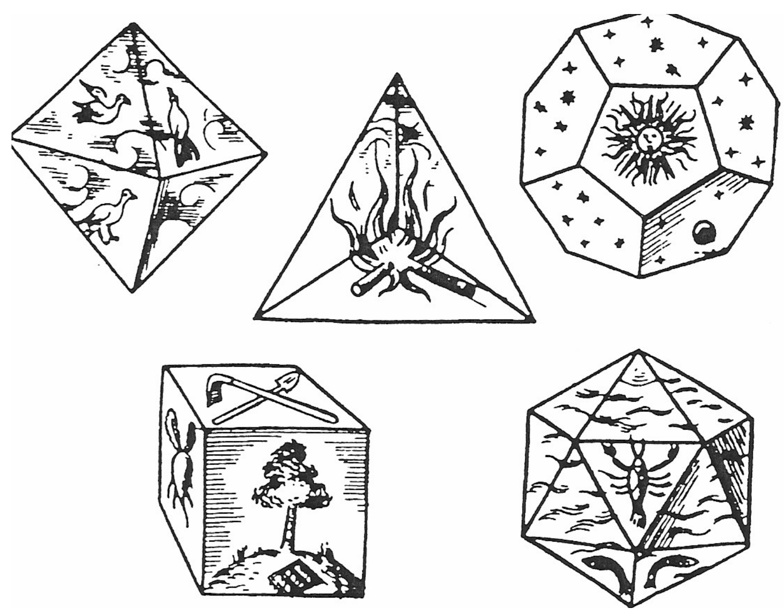


Figure 3: "Tomaeus" Correspondence between the Platonic Solids and the Elements, the Earth and the Universe

This is once more shown in Table (5) where the solid is labelled following Coxeter Tessaract γ_4. Using the Schlafli symbol it is described as a { 4,3,3 } polytope. Finally, following Coxeter we label all combinatorial values as N_i. In the case of γ_4 we have thus [16]

$$N_o = V = 16$$

$$N_1 + E = 32$$

$$N_2 = F = 24$$

and

$$N_3 = \text{cells} = 8$$

Next we discuss the critical dimension $D^{(10)} = 10$ of string theory before showing how we can use four dimensional polytopes in high energy "spacetime" particle physics [11-15].

4. The Role of the Critical String Dimension $D^{(10)} = 10$ and E-infinity Theory

It is well known that a transfinite string dimensional hierarchy may be generated from the E-infinity inverse electro magnetic constant $\bar{\alpha}_o = 137.082039325$ as follows [6]:

$$(\bar{\alpha}_o / 2)(\phi^2) = 26 + k \simeq 26$$

$$(\text{“})(\phi^3) = 16 + k \simeq 10$$

$$(\text{“})(\phi^4) = 10$$

$$(\text{“})(\phi^5) = 6 + k \simeq 6$$

$$(\text{“})(\phi^6) = 4 - k \simeq 4$$

To show that $D^{(10)} = 10$ is indeed a critical dimension in E-infinity, we must examine closely two things. To start with we recall that for the Feigenbaum period doubling to chaos, the parameter λ of [14]

$$\chi_{n+1} = \lambda \chi_n (1 - \chi_n)$$

is given for hyperbolic behaviour by [14]

$$4 < \lambda \leq 4 + \phi^3$$

In other words we have an analogy between the Feigenbaum map and zooming in E-infinity spacetime because of

$$n_t = 4 < \lambda \leq \sim < n > = 4 + \phi^3$$

The criticality of $D^{(10)} = 10$ becomes obvious from the above when we look at the convergence of $< n >$ towards $\sim < n > = 4 + \phi^3$ passing by $< n > = n_t = 4$. Thus writing the series of $\sim < n >$ explicitly we see that [17]

$$< n > = \sum_{n=0}^{n=9} (n)(\phi)^n$$

$$= 3.951669397 < 4 = n_t$$

while

$$< n > = \sum_{0}^{n=10} (n)(\phi)^n$$

$$= 4.070069955 \simeq n_t$$

Consequently $n = D^{(10)} = 10$ must be the critical dimension taking spacetime to a hyperbolic fractal geometry [14].

5. Four Dimensional Fuzzy Polytope, Super Strings and P-Branes

We start with the sharp 4-D generalization of the classical Dodecahedron { 5, 3, 3 }. This is the Coxeter 120-cell polytope, the combinatorial value of which is given in Table (5). Proceeding as in the classical case, one finds [6]

$$\sum_{i=0}^{i=3} N_i = N_0 + N_1 + N_2 + N_3$$

$$= 600 + 1200 + 720 + 120$$

$$= 2640.$$

For the dual shape { 3, 3, 5 } we have similarly

$$\sum_{i=0}^{i=3} N_i = 2640$$

Consequently the total is given by

$$N_T = (2) \left(\sum_{i=0}^{i=3} N_i\right)$$

$$= (2)\,(2640)$$

$$= 5280$$

It is clear from the above that N_T is ten copies of the number of states of a theory which contain besides strings, other extended objects, namely membranes and 5-Branes. The total number of states of the basically 11-dimensional theory was given else where by [17]:

$$\mathcal{N}_0 = \binom{11}{1} + \binom{11}{2} + \binom{11}{5}$$

$$= 11 + 55 + 462$$

$$= 528$$

Consequently we see that

$$\mathcal{N}_0 = N_T / D^{(10)}$$

$$= (5280) / 10$$

$$= 528$$

as should be. A subsequent 3 step symmetry breaking leads then to the number of elementary particles which a minimally extended standard model should include, namely [17]

$$N(SM) = (N_0) / D^{(8)}$$

$$= 528 / 8$$

$$= 66.$$

Assuming, as is currently believed, that we have experimentally discovered 60 elementary particles, then one could conclude from the above result that we should discover 6 more particles in the future. If we further exclude the graviton, then the five particles left may well be three neutral and two charged Higgs particles as argued on earlier occasions [17].

From the above model one could conjecture that the points-like particles of the standard model are given by

$$N(SM) = N_0 / D^{(10)}$$

$$= (600) / 10$$

$$= 60.$$

Consequently the graviton and the Higgs may be qualitatively different from the particles we discovered so far and this may explain the difficulties of finding them. The Coxeter number N_1 could thus be interpreted as string-like particles while N_2 and N_3 correspond to membrane and P-Brane respectively [17].

Finally let us attempt to introduce fuzziness to { 5, 3, 3 }. In such a case we surmise that the N_i could be modified as follows:

$$N_o = 600 \quad \Rightarrow \quad (\phi)\,(10)^3 \simeq 620$$

$$N_1 = 1200 \quad \Rightarrow \quad D^{(10)}\,(\bar{\alpha}_o - D^{(9)}) \simeq 1271$$

$$N_2 = 720 \quad \Rightarrow \quad (S_c^{(7)})\,(10)^2 \simeq 723$$

$$N_3 = 120 \quad \Rightarrow \quad \bar{\alpha}_{ew} \simeq 128\,.$$

Consequently

$$\begin{aligned} \mathcal{N}_0 &= (2)\,(620 + 1271 + 723 + 128) \\ &= (2)\,(2742)\,. \end{aligned}$$

Thus

$$\begin{aligned} N(SM) &= \mathcal{N}_0 / [D^{(10)}\,D^{(8)}] \\ &= 68.55 = \bar{\alpha}_o / 2 \end{aligned}$$

particles which is almost the exact E-infinity value [17].

CONCLUSION

Tomaeus of Locri presented a mystical correspondence between four of what later became known as the platonic solids, namely the Tetrahedron, the Octahedron, the Icosahedron and the cube and the four natural elements, fire, air, water and earth. The fifth platonic solid, the Dodecahedron is given by Tomaeus the astonishing interpretation of being the envelope of the entire universe (Fig. 3). While it is understandable that the symmetry of the platonic solids have provided inspiration and even insight for W. Heisenberg in his attempt to understand the quantum structure of matter, it is almost surreal that the Poincaré Dodecahedron should turn out to explain some basic contradictions in the recent WMAP data [7].

Equally astonishing is the fact that the Dodecahedron model in some modified four dimensional cases can provide a useful model for quantum physics supporting several of the fundamental findings of E-infinity, super strings and P-Brane theory.

REFERENCES

[1] W. Heisenberg Der Teil und das Ganze. Piper, Munich (1969).

[2] M.S. El Naschie The Higgs–physical and number theoretical arguments for the necessity of a

triple elementary particle in super symmetric spacetime. Chaos, Solitons & Fractals, **22**, p. 1199-1209 (2004).

[3] M.S. El Naschie Introduction to chaos, information and diffusion in quantum physics. Chaos, Solitons & Fractals, vol. **7**, No. 5, p. vii-x (1996).

[4] P. Weibel, G. Ord and O. Rössler Space time physics and fractality. Festschrift in honour of Mohamed El Naschie. Springer, Vienna & New York (2005).

[5] M.S. El Naschie On two new fuzzy Kähler manifolds, Klein modular space and 't Hooft holographic principles. Chaos, Solitons & Fractals, **29**, p. 876–881 (2006).

[6] M.S. El Naschie E-infinity theory and the mass spectrum of high energy particle physics. Chaos, Solitons & Fractals, **19**, p. 209–236 (2004).

[7] J.P. Luminet A cosmic hall of mirrors. Physics World, p. 23-27 September (2005).

[8] G. Iovane Wave guiding and mirroring effects in stochastic self-similar and Cantorian E-infinity universe. Chaos, Solitons & Fractals, **23**(3), p. 691–700 (2005).

[9] F. John Estimates of the derivatives of the stresses in thin shell and interior shell equations. Comm. Pure and Applied Math., p. 253–267 (1965).

[10] M.S. El Naschie Die Ablitung einer Konsistenten schalen theorie aus dem Dreidimensionalen Kontinuum. Austrian Engineering Magazine (ÖIZ) Vol. **124**, Issue 9, p. 339–344 (1979).

[11] M.S. El Naschie The elementary particles content of quantum spacetime via Feynman graphs and higher dimensional polytopes. Chaos, Solitons & Fractals. (2007). Volume 31, Issue 1, pp. 1-4.

[12] M.S. El Naschie Intermediate prerequisites for E-infinity. Chaos, Solitons & Fractals, Volume 30, pp. 622-628 (2006).

[13] M.S. El Naschie Topics in the mathematical physics of E-infinity theory. Chaos, Solitons & Fractals, Vol. 30, Issue 3, pp. 656-663, (2006).

[14] M.S. El Naschie Elementary prerequisites for E-infinity. Chaos, Solitons & Fractals, Vol. 30, Issue 3, pp. 579-605, (2006).

[15] P. Neumann, G. Stoy and E. Thompson Groups and geometry. Oxford (1994).

[16] H.S.M. Coxeter Regular polytopes. Dover, New York (1973).

[17] M.S. El Naschie Elementary number theory in superstrings, loop quantum mechanics, twistors and E-infinity high energy physics. Chaos, Solitons & Fractals, **27**, p. 297–330 (2006).

M.S. El Naschie
Distinguished Fellow of the Frankfurt Association for
the Advancement of Fundamental Research in Physics
Faculty of Physics, University of Frankfurt, Germany.
Address for Correspondance
P.O. Box 272, Cobham, Surrey KTII' 2FQ, United Kingdom
E-mail: chaossf@.com