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New Generalized Method to Construct New Non-travelling Waves Solutions and Travelling Waves Solutions of K-D Equations

YU-JIE REN, M. S. EL NASCHIE &
HONG-QING ZHANG

With the aid of computerized symbolic computation, we obtain new types of general solution of a first-order nonlinear ordinary differential equation with six degree and devise a new generalized method and its algorithm, which can be used to construct more new exact solutions of general nonlinear differential equations. The (2+1)-dimensional K-D equation is chosen to illustrate our algorithm such that more families of new exact solutions are obtained, which contain non- travelling waves solutions and travelling waves solutions. This algorithm can also be applied to other nonlinear differential equations.

1. INTRODUCTION

In recent years, the nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, biology, etc. Many methods were developed for finding the exact solutions of NPDEs, such as generalized F-expansion method [1], inverse scattering method [2], Bäcklund transformation [3], Darboux transformation [4, 5], Hirota bilinear method [6], algebrogeometric method [7], tanh-function method [8], the sin-cosine method [9], algebraic method [10 -13], and so on.

In this paper, with the aid of the symbolic computation system-Maple, we obtain new types of general solution of a first-order nonlinear ordinary differential equation with six degree by considering some special cases. At the same time, we develop a new generalized method by means of the solutions of this equation and more general transformations than exiting [10 -13], which exceeds the applicability of the existing algebraic method in obtaining a series of exact solutions of NPDEs. The obtained solutions may include non- travelling waves solutions and travelling waves solutions. The algorithm, the solutions of this equation, and its applications are demonstrated later.

Our paper is organized as follows. In the following Section 2, we give new types of general solution of a firstorder nonlinear ordinary differential equation with six degree. In Section 3, we summarize the new generalized method. In Section 4, we apply the generalized method to construct new non-travelling waves solutions of the (2+1)-dimensional Konopelchenko-Dubrovsky (K-D) equation in[14] and bring out many solutions. In Section 5, we apply the generalized method to construct new travelling waves solutions of the (2+1)-dimensional K-D equation. Conclusions will be presented in finally.

2. SOME TYPES OF GENERAL SOLUTION OF A FIRST-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION WITH SIX DEGREE

In this section, with the aid of the symbolic computation system-Maple, we will give some types of general solution of a first-order nonlinear ordinary differential equation with six degree. We introduce the following ODE

$$\frac{d\phi(\xi)}{d\xi} = \epsilon\sqrt{c_0 + c_1\phi(\xi) + c_2\phi^2(\xi) + c_3\phi^3(\xi) + c_4\phi^4(\xi) + c_5\phi^5(\xi) + c_6\phi^6(\xi)}. \quad (2.1)$$

where $\epsilon = \pm 1$. By considering the different values of c_0, c_1, c_2, c_3, c_4, c_5 and c_6, we find that Eq. (2.1) admits many kinds of general solutions which are listed as follows.

Case 1. If $c_0 = c_1 = c_3 = c_5 = 0$, $4\,c_2c_6 - c_4^2 = 0$, Eq.(2.1) possesses:

the general solutions of hyperbolic type when $c_2 > 0$,

$$\phi_{11}(\xi) = \epsilon\sqrt{\frac{2c_2}{e^{-2\sqrt{c_2}(\xi - C_1)} - c_4}} = \epsilon\sqrt{\frac{2c_2\left(1+\tanh\left(\sqrt{c_2}(\xi - C_1)\right)\right)}{1 - c_4 - (1+c_4)\tanh\left(\sqrt{c_2}(\xi - C_1)\right)}}, \quad (2.2)$$

and

$$\phi_{12}(\xi) = \epsilon\sqrt{\frac{2c_2}{e^{2\sqrt{c_2}(\xi - C_1)} - c_4}} = \epsilon\sqrt{\frac{2c_2\left(1+\tanh\left(\sqrt{c_2}(\xi - C_1)\right)\right)}{1 - c_4 + (1+c_4)\tanh\left(\sqrt{c_2}(\xi - C_1)\right)}}, \quad (2.3)$$

the general solutions of triangular type when $c_2 < 0$,

$$\phi_9(\xi) = \epsilon\sqrt{\frac{2c_2}{e^{-2i\sqrt{-c_2}(\xi - C_1)} - c_4}} = \epsilon\sqrt{\frac{2c_2\left(1+\tan\left(i\sqrt{-c_2}(\xi - C_1)\right)\right)}{1 - c_4 - (1+c_4)\tan\left(i\sqrt{-c_2}(\xi - C_1)\right)}}, \quad (2.4)$$

and

$$\phi_{10}(\xi)=\epsilon\sqrt{\frac{2c_2}{e^{2i\sqrt{-c_2}(\xi-C_1)}}-c_4}=\epsilon\sqrt{\frac{2c_2\left(1-\tan\left(i\sqrt{-c_2}(\xi-C_1)\right)\right)}{1-c_4+(1+c_4)\tan\left(i\sqrt{-c_2}(\xi-C_1)\right)}}, \tag{2.5}$$

where $\varepsilon=\pm1, i=\sqrt{-1},$ and C_1 is an arbitrary constant.

Case 2. If $c_0=c_1=c_3=c_5=0$, $4c_2c_6-c_4^2>0$, Eq. (2.1) possesses:

the general solutions of hyperbolic type

$$\phi_1(\xi)=\varepsilon\sqrt{\frac{2c_2\left(c_4-\sqrt{4c_2c_6-c_4^2}\sinh\left(2\sqrt{c_2}(\xi-C_1)\right)\right)}{4c_2c_6\sinh^2\left(2\sqrt{c_2}(\xi-C_1)\right)-c_4^2\cosh^2\left(2\sqrt{c_2}(\xi-C_1)\right)}}, c_2>0, c_4>0 \tag{2.6}$$

and

$$\phi_2(\xi)=\varepsilon\sqrt{\frac{2c_2\left(c_4+\sqrt{4c_2c_6-c_4^2}\sinh\left(2\sqrt{c_2}(\xi-C_1)\right)\right)}{4c_2c_6\sinh^2\left(2\sqrt{c_2}(\xi-C_1)\right)-c_4^2\cosh^2\left(2\sqrt{c_2}(\xi-C_1)\right)}}, c_2>0, c_4<0 \tag{2.7}$$

where $\varepsilon=\pm1$ and C_1 is an arbitrary constant.

the general solutions of triangular type

$$\phi_3(\xi)=\varepsilon\sqrt{\frac{2c_2\left(-c_4+i\sqrt{4c_2c_6-c_4^2}\sin(f(\xi))\right)}{4c_2c_6\sin^2(f(\xi))+c_4^2\cos^2(f(\xi))}}, c_2>0, c_4\cos(f(\xi))>0, \tag{2.8}$$

and

$$\phi_4(\xi)=\varepsilon\sqrt{\frac{2c_2\left(-c_4-i\sqrt{4c_2c_6-c_4^2}\sin(f(\xi))\right)}{4c_2c_6\sin^2(f(\xi))+c_4^2\cos^2(f(\xi))}}, c_2<0, c_4\cos(f(\xi))<0 \tag{2.9}$$

where $\varepsilon=\pm1, i=\sqrt{-1}, f(\xi)=\sqrt{-c_2}(\xi-C_1),$ here C_1 is an arbitrary constant.

Case 3. If $c_0=c_1=c_3=c_5=0$, $4c_2c_6-c_4<0$, Eq.(2.1) possesses:

the general solutions of hyperbolic type

$$\phi_5(\xi)=\varepsilon\sqrt{\frac{2c_2\left(c_4+i\sqrt{c_4^2-4c_2c_6}\sinh\left(2\sqrt{c_2}(\xi-C_1)\right)\right)}{4c_2c_6\sinh^2\left(2\sqrt{c_2}(\xi-C_1)\right)-c_4^2\cosh^2\left(2\sqrt{c_2}(\xi-C_1)\right)}}, c_2>0, c_4>0 \tag{2.10}$$

and

$$\phi_6(\xi)=\varepsilon\sqrt{\frac{2c_2\left(c_4-i\sqrt{c_4^2-4c_2c_6}\sinh\left(2\sqrt{c_2}(\xi-C_1)\right)\right)}{4c_2c_6\sinh^2\left(2\sqrt{c_2}(\xi-C_1)\right)-c_4^2\cosh^2\left(2\sqrt{c_2}(\xi-C_1)\right)}},c_2>0,c_4<0 \qquad (2.11)$$

where $\varepsilon=\pm1, i=\sqrt{-1}$ and C_1 is an arbitrary constant.

the general solutions of hyperbolic type

$$\phi_5(\xi)=\varepsilon\sqrt{\frac{2c_2\left(c_4+i\sqrt{c_4^2-4c_2c_6}\sinh\left(2\sqrt{c_2}(\xi-C_1)\right)\right)}{4c_2c_6\sinh^2\left(2\sqrt{c_2}(\xi-C_1))-c_4^2\cosh^2\left(2\sqrt{c_2}(\xi-C_1)\right)\right)}},c_2>0,c_4>0 \qquad (2.12)$$

and

$$\phi_6(\xi)=\varepsilon\sqrt{\frac{2c_2\left(c_4-i\sqrt{c_4^2-4c_2c_6}\sinh\left(2\sqrt{c_2}(\xi-C_1)\right)\right)}{4c_2c_6\sinh^2\left(2\sqrt{c_2}(\xi-C_1)\right)-c_4^2\cosh^2\left(2\sqrt{c_2}(\xi-C_1)\right)}},c_2>0,c_4<0 \qquad (2.13)$$

where $\varepsilon=\pm1, i=\sqrt{-1},$ and C_1 is an arbitrary constant.

the general solutions of triangular type

$$\phi_7(\xi)=\varepsilon\sqrt{\frac{2c_2\left(-c_4-\sqrt{c_4^2-4c_2c_6}\sinh(f(\xi))\right)}{4c_2c_6\sin^2(f(\xi))+c_4^2\cos^2(f(\xi))}},c_2<0,c_4\cos(f)(\xi))>0, \qquad (2.14)$$

and

$$\phi_8(\xi)=\varepsilon\sqrt{\frac{2c_2\left(-c_4+\sqrt{c_4^2-4c_2c_6}\sin(f(\xi))\right)}{4c_2c_6\sin^2(f(\xi))+c_4^2\cos^2(f(\xi))}},c_2<0,c_4\cos(f)(\xi))<0 \qquad (2.15)$$

where $\varepsilon=\pm1$ and $f(\xi)=2\sqrt{-c_2}(\xi-C_1),$ here C_1 is an arbitrary constant.

Remark 2 When $c_5=c_6=0$, Eq.(2.1) admits the solutions provided in[10 -13]. We do not list the solutions here in order to avoid unnecessary repetition.

3. SUMMARY OF THE GENERALIZED METHOD

In this section, we outline the main steps of our method which is called generalized method.The key idea of our method is to take full advantage of Eq.(2.1) and its solutions to seek more types of new solutions of a wide class of NPDEs in mathematical physics, which simply proceeds as follows:

Step 1. For a given NPDEs, with some physical fields $u_i(t, x_1, x_2, \cdots, x_m)$, $(i = 1, 2, \cdots, n)$ in m+1 independent variables $t, x_1, x_2, \cdots, x_m$,

$$F_j(u_1,\cdots,u_n,u_{1,t},\cdots,u_{n,t},u_{1,x_1},\cdots,u_{n,x_m},u_{1,tt},\cdots,u_{n,tt}u_{1,tx_1}\cdots u_{n,tx_m},\cdots)=0, \qquad (3.1)$$

where $j = 1, 2, \cdots, n$. By using the more general transformation

$$u_i(t, x_1, x_2, \cdots, x_m) = U_i(\xi),\ \xi = \alpha(x_2, x_3, \cdots, x_m, t)\, x_1 + \beta(x_2, x_3, \cdots, x_m, t), \quad (3.2)$$

where $\alpha(x_2, x_3, \cdots, x_m, t) \neq 0$ and $(x_2, x_3, \cdots, x_m, t)$ are functions to be determined later.For example, when $n = 2$, we may take $\xi = \alpha(x_2, t)\, x_1 + \beta(x_2, t)$, here $\alpha(x_2, t)$ and $\beta(x_2, t)$ are undetermined functions.Then Eqs.(3.1) is reduced to nonlinear differential equations

$$G_j(U_1,\cdots,U_nU_1'\cdots,U_nU_1''\cdots,U_n''\cdots)=0, \qquad (3.3)$$

where $G_j(j = 1, 2, \cdots, n)$ are all polynomials of $U_i(i = 1, 2, \cdots, n)$,α,β and their derivatives. If G_k of them is not a polynomial of $U_i(i = 1, 2, \cdots, n)$,α,β and their derivatives,then we may use new variable $v_i(\xi)(i = 1, 2, \cdots, n)$ which makes G_k become a polynomial of $v_i(\xi)$, α, β and their derivatives. Otherwise the following transformation will fail to seek solutions of Eqs.(3.1).

Step 2. We introduce a new variable $\phi(\xi)$ which is a solution of the following ODE

$$\frac{d\phi(\xi)}{d\xi}=\epsilon\sqrt{c_0+c_1\phi(\xi)+c_2\phi^2(\xi)+c_3\phi^3(\xi)+c_4\phi^4(\xi)+\cdots+c_r\phi^r(\xi)}. \qquad (3.4)$$

Then the derivatives with respect to the variable ξ become the derivatives with respect to the variable ϕ.

Step 3. By using the new variable ϕ, we expand the solution of Eqs.(3.1) in the forms:

$$u_i=a_{i,0}(X)+\sum_{k=1}^{p_i}\Big(a_{i,k}(X)\phi^k(\xi(X))+b_{i,k}(X)\phi^{-k}(\xi(X))\Big). \qquad (3.5)$$

where $X = (t, x_1, x_2, \cdots, x_m)$, $\xi = \xi(X)$, $a_{i,0}(X)$, $a_{i,k}(X)$, $b_{i,k}(X)(i = 1, 2, \cdots, n; k = 1, 2, \cdots, p_i)$ are all differentiable functions of X to be determined later.

Step 4. In order to determine p_i ($i = 1, 2, \cdots, n$) and r, we may substitute (3.4) into (3.3) and balance the highest derivative term with the nonlinear terms in Eqs. (3.3) by using the derivatives with respect to the variable ϕ, we can obtain a relation for p_i and r, from which the different possible values of p_i and r can be obtained.These values lead to the series expansions of the solutions for Eqs.(3.1).

Step 5. Substituting (3.5) into the given Eqs.(3.1) and collecting coefficients of polynomials of ϕ^k, ϕ^{-k}, and $\phi^i\phi^{-k}\sqrt{\sum_{j=1}^{r} c_j\phi^j(\xi)}$, with the aid of Maple, then setting each coefficient to zero, we will get a system of overdetermined partial differential equations with respect to $\alpha(X)$, $\beta(X)$, $a_{i,0}(X)$, $a_{i,k}(X)$, $b_{i,k}(X)(i = 1, 2, \cdots, n; k = 1, 2, \cdots, p_i)$ and $c_j(j = 1, 2, \cdots, r)$.

Step 6. Solving the over-determined partial differential equations with Maple, then we can determine $\alpha(X)$, $\beta(X)$, $a_{i,0}(X)$, $a_{i,k}(\mathrm{X})$, $b_{i,k}(X)(i = 1, 2, \cdots, n; k = 1, 2, \cdots, p_i)$ and $c_j(j = 1, 2, \cdots, r)$.

Step 7. From the constants $c_j(j = 0, 1, ..., r)$ obtained in Step 6 to Eq.(3.4), and then we can obtain all the possible solutions.

Remark 3.1 When $c_5 = c_6 = 0$ and $b_{i,k} = 0$, Eq.(2.1) and ansatz (3.5) just become one used in our previous method [10-12]. However, if $c_5 \neq 0$ or $c_6 \neq 0$, we may obtain the solutions that cannot be found by using the methods [10-12]. It should be pointed out that there is no any method to find all solutions of NPDEs. But our method can be used to find more solutions of NPDEs.

Remark 3.2 By the description above, we can find that our method is more general than the method in [10- 13]. We have improved the method [10-13] in five aspects: First, we extend the ODE with four degree(1.5) into the ODE with six degree(2.1) and get its new general solutions. Second, we change the solution of Eqs.(3.1) into more general solution (3.5) and get more types of new rational solutions and irrational solutions. Third, we replace the traveling wave transformation (1.2) in [10-13] by the more general transformation (3.2). Fourth, what we suppose the coefficients of the ansätz (3.2) and (3.5) are undetermined functions, but the coefficients of the transformation (1.4) in [10] are all constants. Fifth, we present a more general algebra method than the method given [10-13] which is called the generalized method to find more types of exact solutions of nonlinear differential equations based upon the solutions of the ODE (2.1). This can get more general solutions of the NPDEs than those by the method in [10-13].

4. THE APPLICATION OF OUR METHOD AND NEW NON- TRAVELLING WAVES SOLUTIONS OF THE (2+1)-DIMENSIONAL K-D EQUATIONS

In this section, we will make use of new generalized method and symbolic computation to find new non-travelling waves solutions of the (2+1)-dimensional K-D equations in [14], the solutions we find are those which we have never seen before within our knowledge.

The (2+1)-dimensional K-D equations in [14] read:

$$\begin{cases} 2u_t - 2u_{xxx} - 12\alpha u u_x + 3\beta^2 u^2 u_x - 6v_y + 6\beta u_x v = 0 \\ u_y = v_x \end{cases} \tag{4.1}$$

we firstly take the following general transformation

$$u(t, x, y) = u(\xi),\ v(t, x, y) = v(\xi), \xi = p(x) + q(y, t),$$

where $p(x)$ and $q(y, t)$ are functions to be determined later.

By using the new variable $\phi = \phi(\xi)$ which is a solution of the following ODE

$$\frac{d\phi(\xi)}{d\xi} = \epsilon\sqrt{c_0 + c_1\phi(\xi) + c_2\phi^2(\xi) + c_3\phi^3(\xi) + c_4\phi^4(\xi) + c_5\phi^5(\xi) + c_6\phi^6(\xi)}. \tag{4.2}$$

we expand the solution of Eqs.(4.1) in the forms:

$$u = a_0(X) + \sum_{i=1}^{n}\left(a_i(X)\phi^i(\xi(X)) + b_i(X)\phi^{-i}(\xi(X))\right). \tag{4.3a}$$

$$v = b_0(X) + \sum_{j=1}^{m}\left(d_j(X)\phi^j(\xi(X)) + e_j(X)\phi^{-j}(\xi(X))\right). \tag{4.3b}$$

where $X = (t, x, y)$, $a_0(X)$, $a_i(X)$, $b_0(X)$, $b_i(X)$, $d_j(X)$, $e_j(X)(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ are all differentiable functions of X to be determined later.

Substituting (4.3) into (4.1) along with (4.2) and balance the highest derivative term with the nonlinear terms in Eqs.(4.1) by using the derivatives with respect to the variable ϕ, we can determine the parameter $n = 2$ and $m = 2$ in (4.3) and lead to:

$$u = a_0(y, t) + a_1(y, t)\phi(\xi(X)) + a_2(y, t)\phi^2(\xi(X)) + a_3(y, t)\phi^{-1}(\xi(X)) + a_4(y, t)\phi^{-2}(\xi(X))$$
$$v = b_0(y, t) + b_1(y, t)\phi(\xi(X)) + b_2(y, t)\phi^2(\xi(X)) + b_3(y, t)\phi^{-1}(\xi(X)) + b_4(y, t)\phi^{-2}(\xi(X)). \tag{4.4}$$

By substituting (4.4) into the given Eqs.(4.1) along with (4.2) and collecting coefficients of polynomials of ϕ^k and $\phi^{\tau}\sqrt{\sum_{j=1}^{r} c_j \phi^j(\xi)}$, with the aid of Maple, then setting each coefficient to zero, we will get a system of over-determined partial differential equations with respect to $a_0(X)$, $a_1(X)$, $a_2(X)$, $b_0(X)$, $b_1(X)$, $b_2(X)$), c_2, c_4, c_6, $p(y, t)$ and $q(y, t)$ as follows:

$$9\beta^2\epsilon\left(\frac{d}{dx}p(x)\right)(a_3(y,t))^2 a_2(y,t)c_6 + 15\beta^2\epsilon\left(\frac{d}{dx}p(x)\right)(a_3(y,t))^2 a_1(y,t)c_4 + 18\beta^2\epsilon\left(\frac{d}{dx}p(x)\right)a_0(y,t)a_3(y,t)$$

$$a_1(y,t)c_6 + 6a_2(y,t)\,\epsilon^3\left(\frac{d}{dx}p(x)\right)^3 c_6{}^2 - 42a_1(y,t)\,\epsilon^3\left(\frac{d}{dx}p(x)\right)^3 c_4c_6 + 12\beta\epsilon\left(\frac{d}{dx}p(x)\right)a_3(y,t)b_1(y,t)c_6$$

$$+3\beta^2\epsilon\left(\frac{d}{dx}p(x)\right)(a_1(y,t))^3 c_6 + 6\beta\epsilon\left(\frac{d}{dx}p(x)\right)a_1(y,t)b_3(y,t)c_6 - 36\alpha\epsilon\left(\frac{d}{dx}p(x)\right)a_1(y,t)a_3(y,t)c_6 = 0, \tag{4.5}$$

because the over-determined partial differential equations are so many, just part of them are shown here for convenience. Solving the over-determined partial differential equations with Maple, then we have the following solutions.

Case 1

$$b_3 = \frac{C_1\left(4\beta^2C_1c_4 + 8\alpha c_6 - 4\beta^2F_1(t)c_6\right)}{\beta c_6}, b_0 = \frac{y}{3}\frac{dF_1(t)}{dt} + F_2(t), a_2 = 0,$$

$$a_0 = F_1(t), a_4 = 0, a_3 = C_1, a_1 = 0, b_4 = 0, b_2 = 0, p = \frac{C_1\beta x}{4\epsilon\sqrt{c_6}} + c_2, b_1 = 0, \tag{4.6}$$

$$q = -f\,\frac{C_1(\Theta\beta^4 - 48\beta^3F_2(t)\,c_6{}^2\,\Xi\beta^2 + 192c_6{}^2\alpha^2)}{64c_6{}^{5/2}\beta}dt - C_3 - 1/4C_1\epsilon y\left(-2\alpha + F_1(t)\beta^2\right)\sqrt{c_6} + \Psi,$$

where $\Theta = 24(F_1(t))^2c_6{}^2 + 4C_1{}^2c_6c_2 - 24C_1c_4F_1(t)c_6 + 3C_1{}^2c_4{}^2, \Psi = \frac{C_1{}^2y\beta^2c_4}{16\epsilon\, c_6{}^{3/2}}, \Xi = 48C_1c_6c_4\alpha - 96\alpha F_1(t)$

$c_6{}^2, F_1, F_2$ are arbitrary functions of t, and C_1, C_2, C_3 are arbitrary constants.

Case 2

$$q = F_4(t), p = C_1, a_0 = F_1(t), b_0 = 1/3\left(\frac{d}{dt}F_1(t)\right)y + F_2(t), \tag{4.7}$$

$$b_2 = F_3(t),\ a_2 = 0,\ a_4 = 0,\ a_1 = 0,\ a_3 = 0,\ b_4 = 0,\ b_1 = 0,\ b_3 = 0,$$

where F_1, F_2, F_3, F_4 are arbitrary functions of t, and C_1 is arbitrary constant.

Case 3

$$q = F_5(t), p = C_1, b_3 = F_2(t), b_0 = 1/3\left(\frac{d\_F1(t)}{dt}\right)y + F_3(t), \tag{4.8}$$

$$a_0 = F_1(t),\ b_2 = F_4(t),\ a_2 = 0,\ a_4 = 0,\ a_1 = 0,\ a_3 = 0,\ b_4 = 0,\ b_1 = 0,$$

where F_1, F_2, F_3, F_4 are arbitrary functions of t, and C_1 is arbitrary constant.

Case 4

$$q = F_7(t), b_3 = F_3(t), b_0 = 1/3\left(\frac{dF_1(t)}{dt}\right)y + F_4(t), b_1 = F_5(t) \tag{4.9}$$

$$b_2 = F_6(t),\ b_4 = F_2(t),\ p = C_1,\ a_0 = F_1(t),\ a_2 = 0,\ a_4 = 0,\ a_1 = 0,\ a_3 = 0,$$

where F_1, F_2, F_3, F_4, F_5, F_7 are arbitrary functions of t, and C_1 is arbitrary constant.

Case 5

$$q = F_6(t), p = C_1, b_0 = 1/3\left(\frac{dF_1(t)}{dt}\right)y + F_3(t),, a_0 = F_1(t), a_2 = 0, a_4 = 0,$$

(4.10)

$$b_1 = F_4(t),\ b_3 = F_2(t),\ b_2 = F_5(t),\ a_1 = 0,\ a_3 = 0,\ b_4 = 0,$$

where F_1, F_2, F_3, F_4, F_5, F_6 are arbitrary functions of t, and C_1 is arbitrary constant.

Case 6

$$a_0 = F_1(t), a_4 = 0, b_3 = y/3\left(\frac{dF_3(t)}{dt}\right) + F_4(t), b_1 = \frac{\left(\frac{d\_F2(t)}{dt}\right)c_6 y}{3c_4} + F_6(t), \tag{4.11}$$

$$b_0 = y/3\left(\frac{dF_1(t)}{dt}\right) + F_5(t), b_2 = y/3\left(\frac{dF_2(t)}{dt}\right)F_7(t), a_1 = \frac{F_2(t)c_6}{c_4},$$

$$a_3 = F_3(t),\ b_4 = -1/2\ \beta\ (F_2(t))^2,\ q = C_2,\ a_2 = F_2(t),\ p = C_1,$$

where F_1, F_2, F_4, F_5, F_6, F_7 are arbitrary functions of t, and C_2 is arbitrary constant.

Because the solutions are so many, just part of them are shown here for convenience. So we get the general forms of solutions of equations (4.1):

$$\begin{cases} u(t,x,y) = a_0(y,t) + a_1(y,t)\phi(p(x) + q(y,t) + a_2(y,t)\phi^2(p(x) + q(y,t)) \\ \qquad + a_3(y,t)\phi^{-1}(p(x) + q(y,t) + a_4(y,t)\phi^{-2}(p(x) + q(y,t)), \\ v(t,x,y) = b_0(y,t) + b_1(y,t)\phi(p(x)) + q(y,t) + b_2(y,t)\phi^2(p(x) + q(y,t)) \\ \qquad + b_3(y,t)\phi^{-1}(p(x) + q(y,t) + b_4(y,t)\phi^{-2}(p(x) + q(y,t)), \end{cases} \tag{4.12}$$

where $q(y, t)$, a_i, $b_i (i = 0, 1, 2, 3, 4)$ and $p(x)$ satisfy (4.6) - (4.11) respectively.

Type 1. When $c_0 = c_1 = c_3 = c_5 = 0$, $4\,c_2c_6 - c_4^2 = 0$, corresponding (4.2), we can get the general solutions of real number type when $c_2 > 0$,

$$\phi(p(x) + q(y,t) = \epsilon\sqrt{\frac{2c_2\left(1 \pm \tanh\left(\sqrt{c_2}\,(p(x) + q(y,t) - C)\right)\right)}{1 - c_4 \mp (1 + c_4)\tanh\left(\sqrt{c_2}\,(p(x) + q(y,t) - C)\right)}}, \tag{4.13}$$

and the general solutions of complex number type when $c_2 < 0$,

$$\phi(p(x) + q(y,t) = \epsilon\sqrt{\frac{2c_2(1 \pm \tan(i\sqrt{-c_2}\,(p(x) + q(y,t) - C)))}{1 - c_4 \mp (1 + c_4)\tan(i\sqrt{-c_2}\,(p(x) + q(y,t) - C))}}, \tag{4.14}$$

where $\varepsilon = \pm 1, i = \sqrt{-1},$ and C is an arbitrary constant.

Substituting (4.13) and (4.14) into (4.12) respectively, we get the irrational solutions and rational solutions of combined hyperbolic type or triangular type solutions of Eqs.(4.1).

For example, when we select $q\,(y, t)$, $a_i\,(y, t)$, $b_i\,(y, t)$, $(i = 0, 1, 2, 3, 4)$ and $p(x)$ satisfy Case 1, we can easily get the following irrational solutions of combined hyperbolic type or triangular type solutions of Eqs. (4.1):

$$\begin{cases} u_{1,2}(t,x,y) = F_1(t) + C_1\left(\phi\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}} + C_2 + q\right)\right)^2, \\ v_{1,2}(t,x,y) = \frac{y}{3}\left(\frac{dF_1(t)}{dt}\right) + F_2(t) \\ \qquad + \frac{C_1(\beta^2 C_1 c_4 + 8\alpha c_6 - 4\beta^2 F_1(t) c_6)}{4\beta c_6}\left(\phi\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}} + C_2 + q\right)\right)^2, \end{cases} \tag{4.15}$$

where $q = -f\frac{C_1(\Theta\beta^4 - 48\beta^3 F_2(t)c_6{}^2 + \Xi\beta^2 + 192c_6{}^2\alpha^2)}{64c_6{}^{5/2}\beta}dt - C_3 - 1/4\,C_1\epsilon y(-2\alpha + F_1(t)\beta^2)\sqrt{c_6} + \Psi,$ here

$\Theta = 24(F_1(t))^2 c_6{}^2 + 4C_1{}^2 c_6 c_2 - 24C_1 c_4 F_1(t)c_6 + 3C_1{}^2 c_4{}^2, \Psi = \frac{C_1{}^2 y\beta^2 c_4}{16\epsilon c_6{}^{3/2}}, \Xi = 48C_1 c_6 c_4\alpha - 96\alpha F_1(t)c_6{}^2, F_1, F_2$ are arbitrary functions of t, and C_1, C_2 are arbitrary constants.

Substituting (4.13) and (4.14) into (4.15) respectively, we get the rational solutions of combined hyperbolic type of Eqs.(4.1).

$$\begin{cases} u_{1,2}(t,x,y) = F_1(t) + C_1\left(\frac{2c_2(1\pm\tanh(\sqrt{c_2}(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C))}{1-c_4\mp(1+c_4)\tanh(\sqrt{c_2}(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-c))}\right), \\ v_{1,2}(t,x,y) = \frac{y}{3}\left(\frac{dF_1(t)}{dt}\right) + F_2(t) + \frac{C_1(\beta^2 C_1 c_4 + 8\alpha c_6 - 4\beta^2 F_1(t)c_6)}{4\beta c_6} \\ \qquad \left(\frac{2c_2(1\pm\tanh(\sqrt{c_2}(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C))}{1-c_4\mp(1+c_4)\tanh(\sqrt{-c_2}(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C))}\right), c_2 > 0, \end{cases} \tag{4.16}$$

and rational solutions of combined triangular type of Eqs.(4.1):

$$\begin{cases} u_{3,4}(t,x,y) = F_1(t) + C_1\left(\frac{2c_2(1\pm\tan(i\sqrt{-c_2}(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C))}{1-c_4\mp(1+c_4)\tan(i\sqrt{-c_2}(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-c))}\right), \\ v_{3,4}(t,x,y) = \frac{y}{3}\left(\frac{dF_1(t)}{dt}\right) + F_2(t) + \frac{C_1(\beta^2 C_1 c_4 + 8\alpha c_6 - 4\beta^2 F_1(t)c_6)}{4\beta c_6} \\ \qquad \left(\frac{2c_2(1\pm\tan(i\sqrt{-c_2}(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C))}{1-c_4\mp(1+c_4)\tan(i\sqrt{-c_2}(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C))}\right), c_2 > 0, \end{cases} \tag{4.17}$$

where C is a arbitrary constant.

Type 2. When $c_0 = c_1 = c_3 = c_5 = 0$, $4c_2c_6 - c_4 > 0$, corresponding (4.2), we can get the general solutions of real number type when $c_2 > 0$,

$$\phi(\xi) = \varepsilon\sqrt{\frac{2c_2\left(c_4 \pm \sqrt{-c_4^{\,2} + 4c_2c_6}\,\sinh\left(2\sqrt{c_2}\,(\xi - C)\right)\right)}{4c_2c_6\left(\sinh\left(2\sqrt{c_2}\,(\xi - C)\right)\right) - c_4^{\,2}\left(\cosh\left(2\sqrt{c_2}\,(\xi - C)\right)\right)^2}}, \tag{4.18}$$

and the general solutions of complex number type when c2 < 0,

$$\phi(\xi) = \varepsilon\sqrt{\frac{2c_2\left(c_4 \pm i\sqrt{-c_4^{\,2} + 4c_2c_6}\,\sin(2\sqrt{-c_2}\,(\xi - C))\right)}{-4c_6\sin(2\sqrt{-c_2}\,(\xi - C))^2 c_2 - c_4^{\,2}(\cos(2\sqrt{c_2}\,(\xi - C)))^2}}, \tag{4.19}$$

where $\xi = p(y,t)x - q(y,t), \varepsilon = \pm 1, i = \sqrt{-1},$ and C is an arbitrary constant.

Substituting (4.18) and (4.19) into (4.12) respectively, we get the irrational solutions and rational solutions of combined hyperbolic type or triangular type solutions of Eqs.(4.1).

For example,when we select $q\,(y, t)$, $a_i\,(y, t)$, $b_i\,(y, t)$, $(i = 0, 1, 2, 3, 4)$ and $p(y, t)$ satisfy Case 1, we can easily get the following irrational solutions of combined hyperbolic type or triangular type of Eqs.(4.1): we get the rational solutions of combined hyperbolic type of Eqs.(4.1).

$$\begin{cases} u_{5,6}(t,x,y) = F_1(t) + C_1\left(\dfrac{2c_2\left(c_4 \pm \sqrt{-c_4^{\,2}+4c_2c_6}\,\sinh\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)}{4c_2c_6\left(\sinh\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)^2 - c_4^{\,2}\left(\cosh\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)^2}\right), \\ v_{5,6}(t,x,y) = \frac{y}{3}\left(\frac{dF_1(t)}{dt}\right) + F_2(t) + \dfrac{C_1\left(\beta^2 C_1 c_4 + 8\alpha c_6 - 4\beta^2 F_1(t) c_6\right)}{4\beta c_6} \\ \qquad \left(\dfrac{2c_2\left(c_4 \pm \sqrt{-c_4^{\,2}+4c_2c_6}\,\sinh\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)}{4c_2c_6\left(\sinh\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)^2 - c_4^{\,2}\left(\cosh\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)^2}\right), c_2 > 0, \end{cases} \tag{4.20}$$

and rational solutions of combined triangular type of Eqs.(4.1):

$$\begin{cases} u_{7,8}(t,x,y)=F_1(t)+C_1\left(\dfrac{2c_2\left(c_4\pm i\sqrt{-c_4{}^2+4c_2c_6}\,\sin\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)}{-4c_6\left(\sin\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)^2c_2-c_4{}^2\left(\cos\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)^2}\right), \\ v_{7,8}(t,x,y)=\frac{y}{3}\left(\frac{dF_1(t)}{dt}\right)+F_2(t)+\dfrac{C_1\left(\beta^2C_1c_4+8\alpha c_6-4\beta^2F_1(t)c_6\right)}{4\beta c_6} \\ \qquad\left(\dfrac{2c_2\left(c_4\pm i\sqrt{-c_4{}^2+4c_2c_6}\,\sin\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)}{-4c_6\left(\sin\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)^2c_2-c_4{}^2\left(\cos\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)^2}\right), c_2<0, \end{cases}$$

(4.21)

where $q=-\int\dfrac{C_1(\Theta\beta^4-48\beta^3F_2(t)c_6{}^2+\Xi\beta^2+192c_6{}^2\alpha^2)}{64c_6{}^{5/2}\beta}dt-C_3-1/4\,C_1\epsilon\,y(-2\alpha+F_1(t)\beta^2)\sqrt{c_6}+\Psi,$

here $\Theta=24(F_1(t))^2c_6{}^2+4C_1{}^2c_6c_2-24C_1c_4F_1(t)c_6+3C_1{}^2c_4{}^2, \Psi=\frac{C_1{}^2y\beta^2c_4}{16\epsilon c_6{}^{3/2}}, \Xi=48C_1c_6c_4\alpha-96\alpha F_1(t)c_6{}^2, F_1, F_2$ are arbitrary functions of t, and C_1, C_2, C are arbitrary constants.

Type 3. When $c_0=c_1=c_3=c_5=0$, $4c_2c_6-c_4{}^2<0$, corresponding (4.2),we can get two the general solutions of hyperbolic type

$$\phi(\xi)=\varepsilon\sqrt{\frac{2c_2\left(c_4\pm i\sqrt{c_4{}^2-4c_2c_6}\,\sinh\left(2\sqrt{c_2}(\xi-C_1)\right)\right)}{4c_2c_6\sinh^2\left(2\sqrt{c_2}(\xi-C_1)\right)-c_4{}^2\cosh^2\left(2\sqrt{c_2}(\xi-C_1)\right)^2}}, c_2>0, \quad (4.22)$$

and the general solutions of triangular type

$$\phi(\xi)=\varepsilon\sqrt{\frac{2c_2\left(-c_4\pm\sqrt{c_4{}^2-4c_2c_6}\,\sin\left(2\sqrt{-c_2}(\xi-C)\right)\right)}{4c_2c_6\sin^2\left(2\sqrt{-c_2}(\xi-C)\right)+c_4{}^2\cos^2\left(2\sqrt{-c_2}(\xi-C)\right)^2}}, c_2<0, \quad (4.23)$$

where $\varepsilon = \pm 1, i = \sqrt{-1}, \xi = F_1(t)x - q(y,t),$ and C is an arbitrary constant.

Substituting (4.22) and (4.23) into (4.12) respectively, we get the irrational solutions and rational solutions of combined hyperbolic type or triangular type solutions of Eqs.(4.1).

For example, when we select $q(y, t), a_i(y, t), b_i(y, t), (i = 0, 1, 2, 3, 4)$ and $p(y, t)$ satisfy Case 1, we can easily get the following irrational solutions of combined hyperbolic type or triangular type of Eqs.(4.1):

$$\begin{cases} u_{9,10}(t,x,y) = F_1(t) + C_1\left(\dfrac{2c_2\left(c_4 \pm i\sqrt{c_4{}^2-4c_2c_6}\sinh\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)}{4c_2c_6\sinh^2\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)-c_4{}^2\cosh^2\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)}\right), \\ v_{9,10}(t,x,y) = \frac{y}{3}\left(\frac{dF_1(t)}{dt}\right) + F_2(t) + \dfrac{C_1\left(\beta^2C_1c_4+8\alpha c_6-4\beta^2F_1(t)c_6\right)}{4\beta c_6} \\ \qquad \left(\dfrac{2c_2\left(c_4 \pm i\sqrt{c_4{}^2-4c_2c_6}\sinh\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)}{4c_2c_6\sinh^2\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)-c_4{}^2\cosh^2\left(2\sqrt{c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)}\right), c_2 > 0, \end{cases} \tag{4.24}$$

and rational solutions of combined triangular type of Eqs.(4.1):

$$\begin{cases} u_{11,12}(t,x,y) = F_1(t) + C_1\left(\dfrac{2c_2\left(-c_4 \pm \sqrt{c_4{}^2-4c_2c_6}\sin\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)}{4c_2c_6\sin^2\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)+c_4{}^2\cos^2\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)}\right), \\ v_{11,12}(t,x,y) = \frac{y}{3}\left(\frac{dF_1(t)}{dt}\right) + F_2(t) + \dfrac{C_1\left(\beta^2C_1c_4+8\alpha c_6-4\beta^2F_1(t)c_6\right)}{4\beta c_6} \\ \qquad \left(\dfrac{2c_2\left(-c_4 \pm \sqrt{c_4{}^2-4c_2c_6}\sin\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)\right)}{4c_2c_6\sin^2\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)+c_4{}^2\cos^2\left(2\sqrt{-c_2}\left(\frac{C_1\beta x}{4\sqrt{c_6\epsilon}}+q-C\right)\right)}\right), c_2 > 0, \end{cases} \tag{4.25}$$

where $q = -\int \dfrac{C_1(\Theta\beta^4 - 48\beta^3F_2(t)c_6{}^2 + \Xi\beta^2 + 192c_6{}^2\alpha^2)}{64c_6{}^{5/2}\beta}dt - C_3 - 1/4\,C_1\epsilon\, y(-2\alpha + F_1(t)\beta^2)\sqrt{c_6} + \Psi,$

here $\Theta = 24(F_1(t))^2 c_6{}^2 + 4C_1{}^2 c_6 c_2 - 24C_1 c_4 F_1(t) c_6 + 3C_1{}^2 c_4{}^2, \Psi = \frac{C_1{}^2 y\beta^2 c_4}{16\epsilon c_6{}^{3/2}}, \Xi = 48C_1 c_6 c_4 \alpha - 96\alpha F_1(t) c_6{}^2, F_1 F_2$ are arbitrary functions of t, and C_1, C_2, C are arbitrary constants.

Remark 4.1 We may further generalize (4.3) as follows:

$$\begin{cases} u = a_0(X) + \sum_{i=1}^{n}(a_{1,i}(X)\phi^i(p(y,t)x + q(y,t)) + a_{2,i}(X)\phi^{-i}(p(y,t)x + q(y,t)) \\ \qquad + a_{3,i}(X)(R + Q\phi(p(y,t)x + q(y,t)))^{1/2}) \\ v = b_0(X) + \sum_{j=1}^{m}(b_{1,j}(X)\phi^j(p(y,t)x + q(y,t)) + b_{2,j}(X)\phi^{-j}(p(y,t)x + q(y,t)) \\ \qquad + b_{3,i}(X)(R + Q\phi(p(y,t)x + q(y,t)))^{1/2}). \end{cases} \tag{4.26}$$

where R, Q are constants, $X = (t, x, y)$, $p(x)$, $q(y, t)$, $a_0(X)$, $b_0(X)$, $a_{k,i}(X)(k = 0, 1, 2, 3, i = 1, 2, \cdots, n)$ and $b_{k,j}(X)(k = 0, 1, 2, 3, j = 1, 2, \cdots, n)$ are all differentiable functions to be determined later.we can get many new explicit solutions of Eqs.(4.1). And we will further consider this ansätz in another paper.

Remark 4.2 Soliton-like solution is a very important solution in the soliton theory. Among the Soliton-like solutions, the arbitrary functions implies that these solutions have rich local structures. So it is very necessary to study the Soliton-like of NPDEs. If we take the solution of Eqs.(4.1) to be of the form

$$f = h(y,t) + \sum_{i=1}^{n} \exp(\phi(p_i(y,t)x + q_i(y,t)))$$

where $h(y, t)$, $p_i(y, t)$, $q_i(y, t)$, $(i = 1, 2, \cdots, n)$ are differentiable functions to be determined. So we may get many useful soliton-like solution of Eqs.(4.1). This may need to further study.

5. THE APPLICATION OF OUR METHOD AND NEW TRAVELLING WAVES SOLUTIONS OF THE (2+1)-DIMENSIONAL K-D EQUATIONS

In this section, we will make use of new generalized method and symbolic computation to find new travelling waves solutions of the (2+1)-dimensional K-D equations in [14], the solutions we find are those which we have never seen before within our knowledge.

we firstly take the following travelling waves transformation

$$u(t, x, y) = u(\xi),\ v(t, x, y) = v(\xi),\ \xi = px - qy + rt + l, \tag{5.1}$$

where p, q, r and l) are all constants to be determined later.

By using the new variable $\phi = \phi(\xi)$ which is a solution of ODE(4.2). We expand the solution of Eqs.(4.1) in the forms:

$$u = a_0 + \sum_{i=1}^{n} (a_i \phi^i(\xi) + b_i \phi^{-i}(\xi)) \tag{5.2a}$$

$$v = b_0 + \sum_{j=1}^{m} (d_j \phi^j(\xi) + e_j \phi^{-j}(\xi)). \tag{5.2b}$$

where a_0, a_i, b_0, b_i, d_j, $e_j (i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ are all constants to be determined later.

Substituting (5.2) into (4.1) along with (4.2) and balance the highest derivative term with the nonlinear terms in Eqs.(4.1) by using the derivatives with respect to the variable ϕ, we can determine the parameter $n = 2$ and $m = 2$ in (5.2) and lead to:

$$\begin{aligned} u &= a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + a_3\phi^{-1}(\xi) + a_4\phi^{-2}(\xi) \\ v &= b_0 + b_1\phi(\xi) + b_2\phi^2(\xi) + b_3\phi^{-1}(\xi) + b_4\phi^{-2}(\xi). \end{aligned} \tag{5.3}$$

By substituting (5.3) into the given Eqs.(4.1) along with (4.2) and collecting coefficients of polynomials of ϕ^k and $\phi^{\tau}\sqrt{\sum_{j=1}^{r} c_j \phi^j(\xi)}$, with the aid of Maple, then setting each coefficient to zero, we will get a system of over-determined partial differential equations with respect to a_0, a_1, a_2, b_0, b_1, b_2, c_2, c_4, c_6, p, q, r and l as follows:

$$24\,\alpha p a_4^2 c_2 - 12\beta p a_4 b_4 c_2 - 12\beta^2 p a_0 a_4^2 c_2 - 12\beta^2 p a_3^2 a_4 c_2 - 6\beta^2 p a_4^3 c_4, 6\beta^2 p a_2^3 c_6 - 96 a_2 p^3 c_6^2 = 0, \tag{5.4}$$

because the over-determined partial differential equations are so many, just part of them are shown here for convenience. Solving the over-determined partial differential equations with Maple, then we have the following solutions.

Case 1

$$p = p, c_2 = c_2, a_2 = a_2, b_1 = 0, c_6 = 1/16\frac{\beta^2 a_2^{\ 2}}{\epsilon^2 p^2}, b_2 = -\frac{-4\epsilon^2 p^2 c_4 + \beta^2 a_0 a_2 - 2\alpha a_2}{\beta}, a_1 = 0, \tag{5.5}$$

$$c_4 = c_4, r = \frac{p(-6\beta^3 a_2^{\ 2} b_0 + 96 p^4 c_4^{\ 2} - 48 p^2 c_4 \beta^2 a_0 a_2 + 96 p^2 c_4 \alpha a_2 + 3\beta^4 a_0^{\ 2} a_2^{\ 2} - 12\alpha a_0 a_2^{\ 2}\beta^2 + 24\alpha^2 a_2^{\ 2} + 8 a_2^{\ 2} p^2 c_2 \beta^2)}{2\beta^2 a_2^{\ 2}},$$

$$a_3 = 0, b_4 = 0, q = \frac{(-4\epsilon^2 p^2 c_4 + \beta^2 a_0 a_2 - 2\alpha a_2)p}{\beta a_2}, b_3 = 0, l = l, b_0 = b_0, a_0 = a_0, a_4 = 0,$$

where p, l, c_2, c_4, a_0 and b_0 are arbitrary constants.

Case 2

$$c_2 = c_2,\ a_1 = 0,\ a_4 = 0,\ a_3 = 0,\ l = l,\ b_0 = b_0,\ a_0 = a_0,\ b_2 = b_2,\ b_4 = b_4,\ r = r, \quad (5.6)$$
$$c_4 = c_4,\ c_6 = c_6,\ b_1 = b_1,\ b_3 = b_3,\ p = 0,\ q = 0,\ a_2 = 0$$

where r, l, c_2, c_4, c_6, a_0, b_0, b_1, b_1, b_3 and b_4 are arbitrary constants.

Case 3

$$p = p,\ c_2 = c_2,\ a_4 = 0,\ a_3 = 0,\ b_4 = 0,\ b_3 = 0,\ l = l,\ b_0 = b_0,\ a_0 = a_0,\ a_2 = 0,\ a_1 = a_1,$$

$$c_4 = 1/4\frac{\beta^2 a_1{}^2}{\epsilon^2 p^2}, b_2 = 0, q = \frac{(\beta^2 a_0 - 2\alpha)p}{\beta}, b_1 = \frac{a_1(\beta^2 a_0 - 2\alpha)}{\beta}, c_6 = 0,$$

$$r = \frac{p(3\beta^4 a_0{}^2 - 12\alpha a_0 \beta^2 + 24\alpha^2 + 2\epsilon^2 p^2 c_2 \beta^2 - 6\beta^3 b_0)}{2\beta^2}, \quad (5.7)$$

where p, l, c_2, c_4, a_0, , a_1 and b_0 are arbitrary constants.

just part of them are shown here for convenience.

So we get the general forms of solutions of equations (4.1):

$$\begin{cases} u = a_0 + a_1\phi(px - qy + rt + l) + a_2\phi^2(px - qy + rt + l) + a_3\phi^{-1}(px - qy + rt + l) \\ \qquad + a_4\phi^{-2}(px - qy + rt + l) \\ v = b_0 + a_1\phi(px - qy + rt + l) + b_2\phi^2(px - qy + rt + l) + b_3\phi^{-1}(px - qy + rt + l) \\ \qquad + b_4\phi^{-2}(px - qy + rt + l), \end{cases} \quad (5.8)$$

where $q(y, t)$, a_i, $b_i(i = 0, 1, 2, 3, 4)$ and $p(x)$ satisfy (5.5) - (5.7) respectively.

Type 1. When $c_0 = c_1 = c_3 = c_5 = 0$, $4c_2c_6 - c_4{}^2 = 0$, corresponding (4.2), we can get two the general solutions of real number type when $c_2 > 0$,

$$\phi(px - qy + rt + l) = \epsilon\sqrt{\frac{2c_2\left(1 \pm \tanh\left(\sqrt{c_2}(px - qy + rt + l)\right)\right)}{1 - c_4 \mp (1 + c_4)\tanh\left(\sqrt{c_2}(px - qy + rt + l)\right)}}, \quad (5.9)$$

and the general solutions of complex number type when $c_2 < 0$,

$$\phi(px - qy + rt + l) = \epsilon\sqrt{\frac{2c_2\left(1 \pm \tan\left(i\sqrt{-c_2}\,(px - qy + rt + l)\right)\right)}{1 - c_4 \mp (1 + c_4)\tan\left(i\sqrt{-c_2}\,(px - qy + rt + l)\right)}}, \tag{5.10}$$

where $\varepsilon = \pm 1, i = \sqrt{-1},$ and C is an arbitrary constant.

Substituting (5.9) and (5.10) into (5.8) respectively, we get the irrational solutions and rational solutions of combined hyperbolic type or triangular type solutions of Eqs.(4.1).

For example,when we select c_2, c_4, c_6, p, q, r, a_i, b_i, $(i = 0, 1, 2, 3, 4)$ and l satisfy Case 1, we can easily get the following irrational solutions of combined hyperbolic type or triangular type solutions of Eqs. (4.1):

$$\begin{cases} u_{13,14} = a_0 + a_2\left(\phi\left(px + \frac{(4\epsilon^2 p^2 c_4 - \beta^2 a_0 a_2 + 2\alpha a_2)py}{\beta a_2} + r_1 t + l\right)\right)^2 \\ v_{13,14} = b_0 + (4\epsilon^2 p^2 c_4 - \beta^2 a_0 a_2 + 2\alpha a_2)\left(\phi\left(px + \frac{(4\epsilon^2 p^2 c_4 - \beta^2 a_0 a_2 + 2\alpha a_2)py}{\beta a_2} + r_1 t + l\right)\right)^2 \beta^{-1}, \end{cases} \tag{5.11}$$

where $r_1 = \frac{p(-6\beta^3 a_2{}^2 b_0 + 96\epsilon^4 p^4 c_4{}^2 - 48\epsilon^2 p^2 c_4 \beta^2 a_0 a_2 + 96\epsilon^2 p^2 c_4 \alpha a_2 + 3\beta^4 a_0{}^2 a_2{}^2 - 12\alpha a_0 a_2{}^2 \beta^2 + 24\alpha^2 a_2{}^2 + 8a_2{}^2 \epsilon^2 p^2 c_2 \beta^2)}{\beta^2 a_2{}^2}$,

here where p, l, c_2, c_4, a_0 and b_0 are arbitrary constants.

Substituting (5.9) and (5.10) into (5.11) respectively, we get the rational solutions of combined hyperbolic type of Eqs.(4.1)when $c_2 > 0$,

$$\begin{cases} u_{13,14} = a_0 + a_2\left(\frac{2c_2\left(1 \pm \tanh\left(\sqrt{c_2}\left(px + \frac{(4\epsilon^2 p^2 c_4 - \beta^2 a_0 a_2 + 2\alpha a_2)py}{\beta a_2} + r_1\ t + l\right)\right)\right)}{1 - c_4 \mp (1 + c_4)\tanh\left(\sqrt{c_2}\left(px + \frac{(4\epsilon^2 p^2 c_4 - \beta^2 a_0 a_2 + 2\alpha a_2)py}{\beta a_2} + r_1\ t + l\right)\right)}\right) \\ v_{13,14} = b_0 + \frac{(4\epsilon^2 p^2 c_4 - \beta^2 a_0 a_2 + 2\alpha a_2)}{\beta}\frac{2c_2\left(1 \pm \tanh\left(\sqrt{c_2}\left(px + \frac{(4\epsilon^2 p^2 c_4 - \beta^2 a_0 a_2 + 2\alpha a_2)py}{\beta a_2} + r_1\ t + l\right)\right)\right)}{1 - c_4 \mp (1 + c_4)\tanh\left(\sqrt{c_2}\left(px + \frac{(4\epsilon^2 p^2 c_4 - \beta^2 a_0 a_2 + 2\alpha a_2)py}{\beta a_2} + r_1\ t + l\right)\right)}, \end{cases} \tag{5.12}$$

and rational solutions of combined triangular type of Eqs.(4.1) when $c_2 < 0$,

$$\begin{cases} u_{15,16} = a_0 + a_2\left(\dfrac{2c_2\left(1\pm\tan\left(i\sqrt{-c_2}\left(px+\frac{(4\epsilon^2p^2c_4-\beta^2a_0a_2+2\alpha a_2)py}{\beta a_2}+r_1\,t+l\right)\right)\right)}{1-c_4\mp(1+c_4)\tan\left(i\sqrt{-c_2}\left(px+\frac{(4\epsilon^2p^2c_4-\beta^2a_0a_2+2\alpha a_2)py}{\beta a_2}+r_1\,t+l\right)\right)}\right), \\ v_{15,16} = b_0 + \dfrac{(4\epsilon^2p^2c_4-\beta^2a_0a_2+2\alpha a_2)}{\beta}\dfrac{2c_2\left(1\pm\tan\left(i\sqrt{-c_2}\left(px+\frac{(4\epsilon^2p^2c_4-\beta^2a_0a_2+2\alpha a_2)py}{\beta a_2}+r_1\,t+l\right)\right)\right)}{1-c_4\mp(1+c_4)\tan\left(i\sqrt{-c_2}\left(px+\frac{(4\epsilon^2p^2c_4-\beta^2a_0a_2+2\alpha a_2)py}{\beta a_2}+r_1\,t+l\right)\right)}, \end{cases}$$

(5.13)

Type 2. When $c_0 = c_1 = c_3 = c_5 = 0$, $4c_2c_6 - c_4^2 > 0$, corresponding (4.2), we can get two the general solutions of real number type when $c_2 > 0$,

$$\phi(px-qy+rt+l)=\epsilon\sqrt{\frac{2c_2\left(c_4\pm\sqrt{-c_4^{\,2}+4c_2c_6}\,\sinh(2\sqrt{c_2}\,(px-qy+rt+l))\right)}{4c_2c_6(\sinh(2\sqrt{c_2}\,(px-qy+rt+l)))^2c_2-c_4^{\,2}(\cosh(2\sqrt{c_2}\,(px-qy+rt+l)))^2}},$$

(5.14)

and the general solutions of complex number type when $c_2 < 0$,

$$\phi(px-qy+rt+l)=\epsilon\sqrt{\frac{2c_2\left(c_4\pm i\sqrt{-c_4^{\,2}+4c_2c_6}\,\sin(2\sqrt{-c_2}\,(px-qy+rt+l))\right)}{-4c_6(\sin(2\sqrt{-c_2}\,(px-qy+rt+l)))^2c_2-c_4^{\,2}(\cos(2\sqrt{-c_2}\,(px-qy+rt+l)))^2}},$$

(5.15)

Substituting (5.14) and (5.15) into (5.8) respectively, we get the irrational solutions and rational solutions of combined hyperbolic type or triangular type solutions of Eqs.(4.1).

Type 3. When $c_0 = c_1 = c_3 = c_5 = 0$, $4c_2c_6 - c_4^2 < 0$, corresponding (4.2), we can get two the general solutions of hyperbolic type when $c_2 > 0$,

$$\phi(px-qy+rt+l)=\epsilon\sqrt{\frac{2c_2\left(c_4\pm i\sqrt{c_4^{\,2}-4c_2c_6}\,\sinh(2\sqrt{c_2}\,(px-qy+rt+l))\right)}{4c_2c_6\sinh^2(2\sqrt{c_2}\,(px-qy+rt+l))-c_4^{\,2}\cosh^2(2\sqrt{-c_2}\,(px-qy+rt+l))}},$$

(5.16)

and the general solutions of triangular type when $c_2 < 0$,

$$\phi(px-qy+rt+l)=\epsilon\sqrt{\frac{2c_2\left(-c_4\pm\sqrt{c_4{}^2-4c_2c_6}\,\sin(2\sqrt{-c_2}\,(px-qy+rt+l))\right)}{4c_2c_6\sin^2(2\sqrt{-c_2}\,(px-qy+rt+l))^2+c_4{}^2\cos^2(2\sqrt{-c_2}\,(px-qy+rt+l)))}}. \tag{5.17}$$

Substituting (5.16) and (5.17) into (5.8) respectively, we get the irrational solutions and rational solutions of combined hyperbolic type or triangular type solutions of Eqs.(4.1).

Remark 5.1 When $c_5 = c_6 = 0$, and c_2, c_4, p, q, r, l, a_i, b_i, $(i = 0, 1, 2, 3, 4)$ satisfy Case 1 and 2, we can easily get the solutions provided in [10-13]. We do not list the solutions here in order to avoid unnecessary repetition.

Remark 5.2 When c_6, c_2, c_4, p, q, r, l, a_i, b_i, $(i = 0, 1, 2, 3, 4)$ satisfy Case 3, we can easily get the solutions provided in[10-13].

6. CONCLUSION AND DISCUSSION

In summary, by giving some types of general solution of a first-order nonlinear ordinary differential equation with six degree and presenting a new generalized method to find more exact solutions of NPDEs, we obtain many types of solutions of (2+1)-dimensional K-D equations. These solutions not only contain those solutions given in [10-13], but also include non- travelling waves solutions, etc. The solutions obtained may be of important significance for the explanation of some practical physical problems. Although the computation is very complicated, we make use of the powerful symbolic computation system - Maple which makes the process simple. The transformation is also used to solve many other NPDEs. In addition, we only consider some special cases of (2.1). For other cases,we need to further consider.

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Yu-Jie Ren[1,2], M.S. El Naschie[3] & Hong-Qing Zhang[1]
[1]Department of Applied Mathematics
Dalian University of Technology
Dalian 116024, P. R. China
[2]Department of Mathematics and Physics
Dalian Institute of Light Industry
Dalian 116034, P. R. China
[3]Frankfurt Institute for the advancement of
Theoretical Physics - Frankfurt, Germany
ryj195535@163.com