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INCOMING LOCAL EXPONENT FOR A TWO-CYCLE BICOLOUR HAMILTONIAN DIGRAPH WITH A DIFFERENCE OF $4n + 1$

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Abstract. A bicolour digraph is a directed graph with arcs in two colours, red and black. Let m and h be nonnegative integers representing the number of red arcs and black arcs, respectively. The incoming local exponent of a vertex v_x on a bicolour digraph is the smallest positive integer $m+h$ over all pairs of nonnegative integers (m, h) such that for every vertex in v_g there is a walk from v_g to v_x consisting of m red arcs and h black arcs. We discuss incoming local exponents for a Hamiltonian bicolour digraph with two cycles of lengths n and $5n + 1$. We also present the primitivity of this digraph, as well as a formula for the incoming local exponents at its vertices.

1. Introduction

A digraph D is defined as a pair of sets (V, A), wherein V is a non-empty set of vertices, and A is a set of arcs connecting a pair of vertices. A bicolour digraph $D^{(2)}$ is presented in two colours, in this case, red and black. Let m and h be nonnegative integers representing the number of red and black arcs, respectively. A walk from a vertex v_f to a vertex v_g where there are m red arcs and h black arcs is called an (m, h)-walk and is denoted by $v_f \overset{(m,h)}{\longrightarrow} v_g$. For a walk Z in $D^{(2)}$, the number of red arcs in Z is denoted by $s(Z)$, and the number of black arcs in Z is denoted by $t(Z)$. The length of walk Z is $\ell(Z) = s(Z) + t(Z)$. The composition of W is presented in the form of a column matrix $\begin{bmatrix} s(Z) \\ t(Z) \end{bmatrix}$. The distance from vertex v_f to vertex v_g, denoted by $\delta(v_f, v_g)$, is the shortest length of $v_f \to v_g$ path. If there are nonnegative integers m and h such that for every pair of vertices v_f and v_g in $D^{(2)}$, we have a $v_f \overset{(m,h)}{\longrightarrow} v_g$ walk and a $v_g \overset{(m,h)}{\longrightarrow} v_f$ walk, then the bicolour digraph is primitive [1]. The exponent of $D^{(2)}$ is the smallest positive integer $m + h$ over all pairs of nonnegative integers m and h [10]. The incoming local exponent of a vertex v_x, denoted by $\exp\text{in}(v_x, D^{(2)})$, is the smallest positive

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integer $m_x + h_x$ over all pairs of nonnegative integers (m_x, h_x) such that for every vertex v_g in $D^{(2)}$, there is a $v_g \overset{(m_x,h_x)}{\longrightarrow} v_x$ walk [12].

Research on the exponents of bicolour digraphs, especially those for digraphs having two cycles with different lengths, is divided into two main areas, namely, research involving a difference some natural number j, and difference $(q-1)n+1$, for $q \geq 2$, with q also a natural number. The first type of research has been extensive for $j = 1$ and includes work done by Gao and Shao [2], Suwilo and Shader [14], Huang and Liu [3], Suwilo [12], Suwilo [13], and Mardiningsih et al. [7]. The difference $j = 2$ has been investigated by Syahmarani and Suwilo [16] and Suwilo and Syafrianty [15]. Mardiningsih et al. [8] considered $j = 3$. The second type of research, when the difference between two cycles is $(q-1)n+1$, for $q = 2$, has been conducted by Luo [4] and Sumardi and Suwilo [11]. Luo [5] and Prasetyo et al. [9] investigated the exponents for $q = 3$, while Luo [6] examined them for $q = 4$.

We discuss the incoming local exponents for a Hamiltonian bicolour digraph for $q = 5$. The lengths of the two cycles are n and $5n+1$, respectively, for n common vertices. In section 2, we present the primitivity of the bicolour digraph. In section 3, it is discussed the results of previous studies that are useful for determining the bounds of incoming local exponents in bicolour digraphs. Sections 4 and 5 present the results and conclusions, respectively.

2. Primitivity

Suppose that $D^{(2)}$ is a bicolour digraph and $Q = \{Q_1, Q_2, \ldots, Q_r\}$ is the set of all cycles in $D^{(2)}$. A cycle matrix M in a bicolour digraph $D^{(2)}$ is a 2 x r matrix such that the ith column, for $i = 1, 2, \ldots, r$, is the composition of the ith cycle. The form of the cycle matrix is $M = \begin{bmatrix} s(Q_1) & s(Q_2) & \ldots & s(Q_r) \\ t(Q_1) & t(Q_2) & \ldots & t(Q_r) \end{bmatrix}$. If the rank of the cycle matrix M is 1, the content of M is defined to be 0; otherwise, the content of M is the greatest common divisor of the determinants of all 2 x 2 submatrices of M. Fornasini and Valcher [1] state that a bicolour digraph is primitive if and only if the content of M is 1.

Corollary 2.1. *Let $D^{(2)}$ be a strongly connected bicolour digraph with two cycles of length n and $5n+1$. If $D^{(2)}$ is primitive, then the cycle matrix* $M = \begin{bmatrix} 1 & 5 \\ n-1 & 5n-4 \end{bmatrix}$ *or* $M = \begin{bmatrix} n-1 & 5n-4 \\ 1 & 5 \end{bmatrix}$.

Proof. The cycle matrix form of a bicolour digraph $D^{(2)}$ with two cycles is $M = \begin{bmatrix} s_1 & s_2 \\ n & 5n+1 \end{bmatrix}$ for some $0 \leq s_1 \leq n$ and $0 \leq s_2 \leq 5n+1$. Since $D^{(2)}$ is primitive, the determinant of M is ± 1. If $\det(M) = 1$, then $(5s_1 - s_2)\,n + s_1 = 1$. Since $0 \leq s_2 \leq 5n+1$, we have $5s_1 - s_2 = 0$. Hence, $s_1 = 1$, and $s_2 = 5$. So, $M = \begin{bmatrix} 1 & 5 \\ n-1 & 5n-4 \end{bmatrix}$. If $\det(M) = -1$, then $(s_2 - 5s_1)\,n - s_1 = 1$. Since $0 \leq s_2 \leq 5n+1$, we have $s_2 - 5s_1 = 1$. Consequently, $s_1 = n-1$, and $s_2 = 5n-4$. Thus, $M = \begin{bmatrix} n-1 & 5n-4 \\ 1 & 5 \end{bmatrix}$. □

Changing all arc colours from red to black and vice versa does not change an incoming local exponent. Without loss of generality, we assume that the cycle matrix of $D^{(2)}$ is $M = \begin{bmatrix} 1 & 5 \\ n-1 & 5n-4 \end{bmatrix}$. Hence, $D^{(2)}$ has five or six red arcs.

3. Bounds for the Incoming Local Exponents of Bicolour Digraphs

We start with some results that will be useful in obtaining the upper and lower bounds of incoming local exponents.

Proposition 3.1. [12] *Given a two-cycle bicolour digraph $D^{(2)}$ and any vertex v_x positioned on both cycles in $D^{(2)}$, if for some nonnegative integers m and h, there is a path P_{v_y,v_x} from v_y to v_x such that the system*

$$M\mathbf{u} + \begin{bmatrix} s(P_{v_y,v_x}) \\ t(P_{v_y,v_x}) \end{bmatrix} = \begin{bmatrix} m \\ h \end{bmatrix}$$

has a nonnegative integer solution, then $\mathrm{expin}(v_x, D^{(2)}) \leq m + h$.

Lemma 3.2. [12] *Given a primitive two-cycle bicolour digraph $D^{(2)}$ and any vertex v_y on $D^{(2)}$ with the incoming local exponent* $\mathrm{expin}(v_y, D^{(2)})$*, then for every $x = 1, 2, \ldots, 5n+1$, it follows that* $\mathrm{expin}(v_x, D^{(2)}) \leq \mathrm{expin}(v_y, D^{(2)}) + \delta(v_y, v_x)$.

Lemma 3.3. [7] *Given a primitive bicolour digraph $D^{(2)}$ with two cycles, namely, Q_1 and Q_2, with cycle matrix $M = \begin{bmatrix} s(Q_1) & s(Q_2) \\ t(Q_1) & t(Q_2) \end{bmatrix}$ and that* $\det(M) = 1$*, if* $\mathrm{expin}(v_x, D^{(2)})$ *is generated via the (m_x, h_x)-walk, then*

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} t(Q_2)s(P_{v_y,v_x}) - s(Q_2)t(P_{v_y,v_x}) \\ s(Q_1)t(P_{v_w,v_x}) - t(Q_1)s(P_{v_w,v_x}) \end{bmatrix}$$

for some paths P_{v_y,v_x} and P_{v_w,v_x}.

4. Results

This paper discusses the incoming local exponents in a two-cycle Hamiltonian bicolour digraph with cycle length difference $4n+1$. The first cycle has length n, and the second cycle has length $5n+1$, while the number of common vertices is n. The first cycle is $v_1 \to v_2 \to v_3 \to \cdots \to v_{n-1} \to v_n \to v_1$, which is denoted by Q_1. The second cycle is $v_1 \to v_2 \to v_3 \to \cdots \to v_{n-1} \to v_n \to v_{n+1} \cdots \to v_{5n} \to v_{5n+1}$, which is represented by Q_2. Based on Corollary 2.1, the primitive bicolour digraph results in five or six red arcs.

Suppose the five red arcs in $D^{(2)}$ are $v_a \to v_{a+1}$, $v_b \to v_{b+1}$, $v_c \to v_{c+1}$, $v_d \to v_{d+1}$, and $v_e \to v_{e+1}$ for $1 \leq a \leq n-1$ and $n \leq b < c < d < e \leq 5n+1$. Let the six red arcs in $D^{(2)}$ be $v_n \to v_1$, $v_a \to v_{a+1}$, $v_b \to v_{b+1}$, $v_c \to v_{c+1}$, $v_d \to v_{d+1}$, and $v_e \to v_{e+1}$ for $n \leq a < b < c < d < e \leq 5n+1$. The distance from vertex v_{a+1} to vertex v_1 in Q_1 is denoted by $\delta_{1,1} = \delta(v_{a+1}, v_1)$, whereas the distance from vertex v_{a+1} to vertex v_1 in Q_2 is denoted by $\delta_{1,2} = \delta(v_{a+1}, v_1)$. Let δ_2, δ_3, δ_4, and δ_5 represent the distance from vertex v_{b+1} to vertex v_1, the distance from vertex v_{c+1} to vertex v_1, the distance from vertex v_{d+1} to vertex v_1, and the distance from vertex v_{e+1} to vertex v_1, respectively.

Theorem 4.1. *Given $D^{(2)}$, a two-cycle primitive bicolour digraph with cycle lengths of n and $5n+1$, if $D^{(2)}$ has four or five consecutive red arcs in Q_2, then for every $x = 1, 2, ..., 5n+1$,* $\text{expin}(v_x, D^{(2)}) =$

$$\begin{cases} 25n^2 + 5n\left(\delta_5 - \delta_{1,2}\right) + \delta_5 + \delta\left(v_1, v_x\right), & \text{for } \delta_{1,2} - \delta_2 \leq n \\ 20n^2 - 16n + \delta_5 + \delta\left(v_1, v_x\right), & \text{for } n < \delta_{1,2} - \delta_2 < 4n - 1 \\ 20n^2 - 16n + 5n(\delta_{1,1} - \delta_5) + \delta_{1,1} + \delta\left(v_1, v_x\right), & \text{for } \delta_{1,2} - \delta_2 \geq 4n - 1. \end{cases}$$

Proof. Suppose the $\text{expin}(v_x, D^{(2)})$ value for each $x = 1, 2, ..., 5n+1$ is generated from the (m_x, h_x)-walk. The proof of Theorem 4.1 is presented in the following three cases.

Case 1. (for $\delta_{1,2} - \delta_2 \leq n$)

First, we need to show that $\text{expin}(v_x, D^{(2)}) \geq 25n^2 + 5n\left(\delta_5 - \delta_{1,2}\right) + \delta_5 + \delta(v_1, v_x)$. Choose paths P_{v_a,v_x} and P_{v_{e+1},v_x} and define $k_1 = t(Q_2)s(P_{v_a,v_x}) - s(Q_2)t(P_{v_a,v_x})$ and $k_2 = s(Q_1)t(P_{v_{e+1},v_x}) - t(Q_1)s(P_{v_{e+1},v_x})$. We consider six subcases.

The vertex v_x is positioned on path $v_1 \to v_a$. Using path P_{v_a,v_x}, we get path $(5, \delta_{1,2} - 4 + \delta(v_1, v_x))$, resulting in $k_1 = 25n - 5(\delta_{1,2} + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(0, \delta_5 + \delta(v_1, v_x))$, leading to $k_2 = \delta_5 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 25n - 5(\delta_{1,2} - \delta_5) \\ 25n^2 + 5n(\delta_5 - \delta_{1,2}) - 25n + 5\delta_{1,2} - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\text{expin}(v_x, D^{(2)}) \geq 25n^2 + 5n\left(\delta_5 - \delta_{1,2}\right) + \delta_5 + \delta(v_1, v_x) \tag{4.1}$$

for every vertex v_x positioned on the path $v_1 \to v_a$.

The vertex v_x is positioned on path $v_{a+1} \to v_b$. Using path P_{v_a,v_x}, we get path $(1, \delta_{1,2} - 5n - 1 + \delta(v_1, v_x))$, thus arriving at $k_1 = 30n + 1 - 5(\delta_{1,2} + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(1, \delta_5 - 1 + \delta(v_1, v_x))$, leaving us with $k_2 = \delta_5 - n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 25n + 1 - 5(\delta_{1,2} - \delta_5) \\ 25n^2 + 5n(\delta_5 - \delta_{1,2}) - 25n - 1 + 5\delta_{1,2} - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\text{expin}(v_x, D^{(2)}) \geq 25n^2 + 5n\left(\delta_5 - \delta_{1,2}\right) + \delta_5 + \delta(v_1, v_x) \tag{4.2}$$

for every vertex v_x positioned on the path $v_{a+1} \to v_b$.

The vertex v_x is positioned on path $v_{b+1} \to v_c$. Using path P_{v_a,v_x}, we get path $(2, \delta_{1,2} - 5n - 2 + \delta(v_1, v_x))$, resulting in $k_1 = 35n + 2 - 5(\delta_{1,2} + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(2, \delta_5 - 2 + \delta(v_1, v_x))$, meaning that $k_2 = \delta_5 - 2n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 25n + 2 - 5(\delta_{1,2} - \delta_5) \\ 25n^2 + 5n(\delta_5 - \delta_{1,2}) - 25n - 2 + 5\delta_{1,2} - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\text{expin}(v_x, D^{(2)}) \geq 25n^2 + 5n\left(\delta_5 - \delta_{1,2}\right) + \delta_5 + \delta(v_1, v_x) \tag{4.3}$$

for every vertex v_x positioned on the path $v_{b+1} \to v_c$.

The vertex v_x is positioned on path $v_{c+1} \to v_d$. Using path P_{v_a,v_x}, we get path $(3, \delta_{1,2} - 5n - 3 + \delta(v_1, v_x))$, leading to $k_1 = 40n + 3 - 5(\delta_{1,2} + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(3, \delta_5 - 3 + \delta(v_1, v_x))$, thus arriving at $k_2 = \delta_5 - 3n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 25n + 3 - 5(\delta_{1,2} - \delta_5) \\ 25n^2 + 5n(\delta_5 - \delta_{1,2}) - 25n - 3 + 5\delta_{1,2} - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\operatorname{expin}(v_x, D^{(2)}) \geq 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5 + \delta(v_1, v_x) \tag{4.4}$$

for every vertex v_x positioned on the path $v_{c+1} \to v_d$.

The vertex v_x is positioned on path $v_{d+1} \to v_e$. Using path P_{v_a,v_x}, we get path $(4, \delta_{1,2} - 5n - 4 + \delta(v_1, v_x))$, meaning that $k_1 = 45n + 4 - 5(\delta_{1,2} + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(4, \delta_5 - 4 + \delta(v_1, v_x))$, resulting in $k_2 = \delta_5 - 4n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 25n + 4 - 5(\delta_{1,2} - \delta_5) \\ 25n^2 + 5n(\delta_5 - \delta_{1,2}) - 25n - 4 + 5\delta_{1,2} - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\operatorname{expin}(v_x, D^{(2)}) \geq 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5 + \delta(v_1, v_x) \tag{4.5}$$

for every vertex v_x positioned on the path $v_{d+1} \to v_e$.

The vertex v_x is positioned on path $v_{e+1} \to v_{5n+1}$. Using path P_{v_a,v_x}, we get path $(5, \delta_{1,2} - 5n - 5 + \delta(v_1, v_x))$, thus arriving at $k_1 = 50n + 5 - 5(\delta_{1,2} + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(0, \delta_5 - 5n - 1 + \delta(v_1, v_x))$, leading to $k_2 = \delta_5 - 5n - 1 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 25n - 5(\delta_{1,2} - \delta_5) \\ 25n^2 + 5n(\delta_5 - \delta_{1,2}) - 30n - 1 + 5\delta_{1,2} - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Let $p_1 = 25n - 5(\delta_{1,2} - \delta_5)$ and $p_2 = 25n^2 + 5n(\delta_5 - \delta_{1,2}) - 30n - 1 + 5\delta_{1,2} - 4\delta_5 + \delta(v_1, v_x)$. We consider the walk (p_1, p_2) from v_{e+1} to v_x. Note that path P_{v_{e+1},v_x} is $(0, \delta_5 - 5n - 1 + \delta(v_1, v_x))$ and that solving the system $M\mathbf{u} + \begin{bmatrix} s(P_{v_{e+1},v_x}) \\ t(P_{v_{e+1},v_x}) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ leads to $u_1 = 25n - 5\delta_{1,2} + 5\delta_5$ and $u_2 = 0$. Because the path P_{v_{e+1},v_x} lies totally on cycle Q_2, there is no (p_1, p_2)-walk from v_{e+1} to v_x. Therefore, $\operatorname{expin}(v_x, D^{(2)}) > p_1 + p_2$. The shortest walk from v_{e+1} to v_x with minimal p_1 red arcs and minimal p_2 red arcs is a $(p_1 + s(Q_2), p_2 + t(Q_2))$-walk. Since $s(Q_2) + t(Q_2) = 5n + 1$, we get

$$\begin{aligned} \operatorname{expin}(v_x, D^{(2)}) &\geq p_1 + p_2 + s(Q_2) + t(Q_2) \\ &= 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5 + \delta(v_1, v_x) \end{aligned} \tag{4.6}$$

for every vertex v_x positioned on the path $v_{e+1} \to v_{5n+1}$.

From (4.1), (4.2), (4.3), (4.4), (4.5), and (4.6), we conclude that $\text{expin}(v_x, D^{(2)}) \geq 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5 + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n+1$.

Next, we will show that $\text{expin}(v_x, D^{(2)}) \leq 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5 + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 5n+1$. First, we will show that $\text{expin}(v_1, D^{(2)}) = 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5$ and then utilize Lemma 3.2 to ensure that $\text{expin}(v_x, D^{(2)}) \leq 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5 + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n+1$.

From (4.1), we have $\text{expin}(v_1, D^{(2)}) \geq 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5$. Next, it is necessary to prove that $\text{expin}(v_1, D^{(2)}) \leq 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5$ for every v_y and that for $y = 1, 2, ..., 5n+1$, the system

$$M\mathbf{u} + \begin{bmatrix} s(P_{v_y,v_1}) \\ t(P_{v_y,v_1}) \end{bmatrix} = \begin{bmatrix} 25n - 5(\delta_{1,2} - \delta_5) \\ 25n^2 + 5n(\delta_5 - \delta_{1,2}) - 25n + 5\delta_{1,2} - 4\delta_5 \end{bmatrix} \tag{4.7}$$

has a nonnegative integer solution for the path P_{v_y,v_1}.

The solution of system (4.7) is $u_1 = 25n - 5\delta_{1,2} - (5n-4)s(P_{v_y,v_1}) + 5t(P_{v_y,v_1})$ and $u_2 = \delta_5 - (1-n)s(P_{v_y,d_1}) - t(P_{v_y,v_1})$. If v_y is positioned on the $v_1 \to v_a$ path, then there is a $(5, 5n-4-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 25n - 5(\delta_{1,2} + \delta(v_1, v_y)) \geq 0$ since $\delta_{1,2} + \delta(v_1, v_y) \leq 5n$ for $n \geq 3$ and $u_2 = \delta_5 + \delta(v_1, v_y) - 1 \geq 7$ since $\delta_5 + \delta(v_1, v_y) \geq 2n+1$ for $n \geq 3$. If v_y is positioned on the $v_{a+1} \to v_b$ path, then there is a $(4, 5n-3-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 30n+1-5(\delta_{1,2}+\delta(v_1, v_y)) \geq 11$ since $\delta_{1,2} + \delta(v_1, v_y) \leq 5n+1$ for $n \geq 3$ and $u_2 = \delta_5 + \delta(v_1, v_y) - n - 1 \geq 4$ since $\delta_5 + \delta(v_1, v_y) \geq 2n+2$ for $n \geq 3$. If v_y is positioned on the $v_{b+1} \to v_c$ path, then there is a $(3, 5n-2-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 35n+2-5(\delta_{1,2}+\delta(v_1, v_y)) \geq 12$ since $\delta_{1,2}+\delta(v_1, v_y) \leq 6n+1$ for $n \geq 3$ and $u_2 = \delta_5+\delta(v_1, v_y)-2n-1 \geq 6$ since $\delta_5+\delta(v_1, v_y) \geq 4n+1$ for $n \geq 3$. If v_y is positioned on the $v_{c+1} \to v_d$ path, then there is a $(2, 5n-1-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 40n+3-5(\delta_{1,2}+\delta(v_1, v_y)) \geq 23$ since $\delta_{1,2} + \delta(v_1, v_y) \leq 6n+2$ for $n \geq 3$ and $u_2 = \delta_5 + \delta(v_1, v_y) - 3n - 1 \geq 4$ since $\delta_5 + \delta(v_1, v_y) \geq 5n-1$ for $n \geq 3$. If v_y is positioned on the $v_{d+1} \to v_e$ path, then there is a $(1, 5n-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 45n+4-5(\delta_{1,2}+\delta(v_1, v_y)) \geq 14$ since $\delta_{1,2}+\delta(v_1, v_y) \leq 7n$ for $n \geq 3$ and $u_2 = \delta_5+\delta(v_1, v_y)-4n-1 \geq 2$ since $\delta_5+\delta(v_1, v_y) \geq 5n$ for $n \geq 3$. If v_y is positioned on the $v_{e+1} \to v_{5n+1}$ path, then there is a $(0, 5n+1-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 50n+5-5(\delta_{1,2}+\delta(v_1, v_y)) \geq 5$ since $\delta_{1,2} + \delta(v_1, v_y) \leq 10n$ for $n \geq 3$ and $u_2 = \delta_5 + \delta(v_1, v_y) - 5n - 1 \geq 0$ since $\delta_5 + \delta(v_1, v_y) \geq 5n+1$ for $n \geq 3$.

Therefore, for every $y = 1, 2, ..., 5n+1$, the system (4.7) has a nonnegative integer solution. Proposition 3.1 then ensures that for every $y = 1, 2, ..., 5n+1$, there is a $v_y \overset{(m,h)}{\longrightarrow} v_1$ walk with $m = 25n - 5(\delta_{1,2} - \delta_5)$ and $h = 25n^2 + 5n(\delta_5 - \delta_{1,2}) - 25n + 5\delta_{1,2} - 4\delta_5$. Consequently, $\text{expin}(v_1, D^{(2)}) \leq 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5$. So, $\text{expin}(v_1, D^{(2)}) = 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5$. Using Lemma 3.2, we conclude that $\text{expin}(v_x, D^{(2)}) \leq 25n^2 + 5n\,(\delta_5 - \delta_{1,2}) + \delta_5 + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n+1$.

Case 2. (for $n < \delta_{1,2} - \delta_2 < 4n-1$)

First, we need to show that $\exp\mathrm{in}(v_x, D^{(2)}) \geq 20n^2 - 16n + \delta_5 + \delta(v_1, v_x)$. Choose paths P_{v_b,v_x} and P_{v_{e+1},v_x} and define $k_1 = t(Q_2)s(P_{v_b,v_x}) - s(Q_2)t(P_{v_b,v_x})$ and $k_2 = s(Q_1)t(P_{v_{e+1},v_x}) - t(Q_1)s(P_{v_{e+1},v_x})$. We consider six sub-cases.

The vertex v_x is positioned on path $v_1 \to v_a$. Using path P_{v_b,v_x}, we get path $(4, \delta_5 + \delta(v_1, v_x))$, thus arriving at $k_1 = 20n - 16 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(0, \delta_5 + \delta(v_1, v_x))$, resulting in $k_2 = \delta_5 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 20n-16 \\ 20n^2 - 36n + 16 + \delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\exp\mathrm{in}(v_x, D^{(2)}) \geq 20n^2 - 16n + \delta_5 + \delta(v_1, v_x) \tag{4.8}$$

for every vertex v_x positioned on the path $v_1 \to v_a$.

The vertex v_x is positioned on path $v_{a+1} \to v_b$. Using path P_{v_b,v_x}, we get path $(5, \delta_5 - 1 + \delta(v_1, v_x))$, leading to $k_1 = 25n - 15 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(1, \delta_5 - 1 + \delta(v_1, v_x))$, arriving at $k_2 = \delta_5 - n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 20n-15 \\ 20n^2 - 36n + 15 + \delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\exp\mathrm{in}(v_x, D^{(2)}) \geq 20n^2 - 16n + \delta_5 + \delta(v_1, v_x) \tag{4.9}$$

for every vertex v_x positioned on the path $v_{a+1} \to v_b$.

The vertex v_x is positioned on path $v_{b+1} \to v_c$. Using path P_{v_b,v_x}, we get path $(1, \delta_5 - 5n + 2 + \delta(v_1, v_x))$, leading to $k_1 = 30n - 14 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(2, \delta_5 - 2 + \delta(v_1, v_x))$, thus ending up with $k_2 = \delta_5 - 2n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 20n-14 \\ 20n^2 - 36n + 14 + \delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\exp\mathrm{in}(v_x, D^{(2)}) \geq 20n^2 - 16n + \delta_5 + \delta(v_1, v_x) \tag{4.10}$$

for every vertex v_x positioned on the path $v_{b+1} \to v_c$.

The vertex v_x is positioned on path $v_{c+1} \to v_d$. Using path P_{v_b,v_x}, we get path $(2, \delta_5 - 5n + 1 + \delta(v_1, v_x))$, thus arriving at $k_1 = 35n - 13 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(3, \delta_5 - 3 + \delta(v_1, v_x))$, ending up with $k_2 = \delta_5 - 3n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 20n-13 \\ 20n^2 - 36n + 13 + \delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\exp\mathrm{in}(v_x, D^{(2)}) \geq 20n^2 - 16n + \delta_5 + \delta(v_1, v_x) \tag{4.11}$$

for every vertex v_x positioned on the path $v_{c+1} \to v_d$.

The vertex v_x is positioned on path $v_{d+1} \to v_e$. Using path P_{v_b,v_x}, we get path $(3, \delta_5 - 5n + \delta(v_1, v_x))$, resulting in $k_1 = 40n - 12 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(4, \delta_5 - 4 + \delta(v_1, v_x))$, leading to $k_2 = \delta_5 - 4n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 20n-12 \\ 20n^2 - 36n + 12 + \delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,
$$\operatorname{expin}(v_x, D^{(2)}) \geq 20n^2 - 16n + \delta_5 + \delta(v_1, v_x) \tag{4.12}$$
for every vertex v_x positioned on the path $v_{d+1} \to v_e$.

The vertex v_x is positioned on path $v_{e+1} \to v_{5n+1}$. Using path P_{v_b,v_x}, we get path $(4, \delta_5 - 5n - 1 + \delta(v_1, v_x))$, arriving at $k_1 = 45n - 11 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(0, \delta_5 - 5n - 1 + \delta(v_1, v_x))$, leading to $k_2 = \delta_5 - 5n - 1 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get
$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$
$$\begin{bmatrix} 20n - 16 \\ 20n^2 - 41n + 15 + \delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$
Let $p_1 = 20n - 16$ and $p_2 = 20n^2 - 41n + 15 + \delta_5 + \delta(v_1, v_x)$. We consider the walk (p_1, p_2) from v_{e+1} to v_x. Note that path P_{v_{e+1},v_x} is $(0, \delta_5 - 5n - 1 + \delta(v_1, v_x))$ and that solving the system $M\mathbf{u} + \begin{bmatrix} s(P_{v_{e+1},v_x}) \\ t(P_{v_{e+1},v_x}) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ results in $u_1 = 20n - 16$ and $u_2 = 0$. Because the path P_{v_{e+1},v_x} lies totally on cycle Q_2, there is no (p_1, p_2)-walk from v_{e+1} to v_x. Therefore, $\operatorname{expin}(v_x, D^{(2)}) > p_1 + p_2$. The shortest walk from v_{e+1} to v_x with minimal p_1 red arcs and minimal p_2 red arcs is a $(p_1 + s(Q_2), p_2 + t(Q_2))$-walk. Since $s(Q_2) + t(Q_2) = 5n + 1$, we get
$$\begin{aligned} \operatorname{expin}(v_x, D^{(2)}) &\geq p_1 + p_2 + s(Q_2) + t(Q_2) \\ &= 20n^2 - 16n + \delta_5 + \delta(v_1, v_x) \end{aligned} \tag{4.13}$$
for every vertex v_x positioned on the path $v_{e+1} \to v_{5n+1}$.

From (4.8), (4.9), (4.10), (4.11), (4.12), and (4.13), we conclude that expin $(v_x, D^{(2)}) \geq 20n^2 - 16n + \delta_5 + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n + 1$.

Next, we will show that $\operatorname{expin}(v_x, D^{(2)}) \leq 20n^2 - 16n + \delta_5 + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n+1$. First, we will show that $\operatorname{expin}(v_1, D^{(2)}) = 20n^2 - 16n + \delta_5$ and then utilize Lemma 3.2 to ensure that $\operatorname{expin}(v_x, D^{(2)}) \leq 20n^2 - 16n + \delta_5 + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n + 1$.

From (4.8), we have $\operatorname{expin}(v_1, D^{(2)}) \geq 20n^2 - 16n + \delta_5$. Next, it is necessary to prove that $\operatorname{expin}(v_1, D^{(2)}) \leq 20n^2 - 16n + \delta_5$ for every v_y and that for $y = 1, 2, ..., 5n + 1$, the system
$$\begin{aligned} & M\mathbf{u} + \begin{bmatrix} s(P_{v_y,v_1}) \\ t(P_{v_y,v_1}) \end{bmatrix} \\ &= \begin{bmatrix} 20n - 16 \\ 20n^2 - 36n + 16 + \delta_5 \end{bmatrix} \end{aligned} \tag{4.14}$$
has a nonnegative integer solution for the path P_{v_y,v_1}.

The solution of system (4.14) is $u_1 = 20n - 16 - 5\delta_5 - (5n-4)s(P_{v_y,v_1}) + 5t(P_{v_y,v_1})$ and $u_2 = \delta_5 - (1-n)s(P_{v_y,d_1}) - t(P_{v_y,v_1})$. If v_y is positioned on the $v_1 \to v_a$ path, then there is a $(5, 5n - 4 - \delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 20n - 16 - 5(\delta_5 + \delta(v_1, v_y)) \geq 4$ since $\delta_5 + \delta(v_1, v_y) \leq 3n - 1$ for $n \geq 3$ and $u_2 = \delta_5 + \delta(v_1, v_y) - 1 \geq 0$ since $\delta_5 + \delta(v_1, v_y) \geq n - 2$ for $n \geq 3$. If v_y is positioned on the $v_{a+1} \to v_b$ path, then there is a $(4, 5n - 3 - \delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 25n - 15 - 5(\delta_5 + \delta(v_1, v_y)) \geq 0$ since $\delta_5 + \delta(v_1, v_y) \leq 4n$ for $n \geq 3$ and $u_2 = \delta_5 + \delta(v_1, v_y) - n - 1 \geq 0$ since

$\delta_5+\delta(v_1,v_y)\geq n+1$ for $n\geq 3$. If v_y is positioned on the $v_{b+1}\to v_c$ path, then there is a $(3,5n-2-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=30n-14-5(\delta_5+\delta(v_1,v_y))\geq 11$ since $\delta_5+\delta(v_1,v_y)\leq 4n+1$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-2n-1\geq 6$ since $\delta_5+\delta(v_1,v_y)\geq 4n+1$ for $n\geq 3$. If v_y is positioned on the $v_{c+1}\to v_d$ path, then there is a $(2,5n-1-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=35n-13-5(\delta_5+\delta(v_1,v_y))\geq 22$ since $\delta_5+\delta(v_1,v_y)\leq 5n-1$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-3n-1\geq 4$ since $\delta_5+\delta(v_1,v_y)\geq 5n-1$ for $n\geq 3$. If v_y is positioned on the $v_{d+1}\to v_e$ path, then there is a $(1,5n-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=40n-12-5(\delta_5+\delta(v_1,v_y))\geq 33$ since $\delta_5+\delta(v_1,v_y)\leq 5n$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-4n-1\geq 2$ since $\delta_5+\delta(v_1,v_y)\geq 5n$ for $n\geq 3$. If v_y is positioned on the $v_{e+1}\to v_{5n+1}$ path, then there is a $(0,5n+1-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=45n-11-5(\delta_5+\delta(v_1,v_y))\geq 9$ since $\delta_5+\delta(v_1,v_y)\leq 8n-1$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-5n-1\geq 0$ since $\delta_5+\delta(v_1,v_y)\geq 5n+1$ for $n\geq 3$.

Therefore, for every $y=1,2,...,5n+1$, the system (4.14) has a nonnegative integer solution. Proposition 3.1 ensures that for every $y=1,2,...,5n+1$, there is a $v_y\overset{(m,h)}{\longrightarrow}v_1$ walk with $m=20n-16$ and $h=20n^2-36n+16+\delta_5$. Consequently, $\text{expin}(v_1,D^{(2)})\leq 20n^2-16n+\delta_5$. So, $\text{expin}(v_1,D^{(2)})=20n^2-16n+\delta_5$. By Lemma 3.2, we conclude that expin $(v_x,D^{(2)})\leq 20n^2-16n+\delta_5+\delta(v_1,v_x)$ for $x=1,2,...,5n+1$.

Case 3. (for $\delta_{1,2}-\delta_2\geq 4n-1$)

First, we need to show that $\text{expin}(v_x,D^{(2)})\geq 20n^2-16n+5n(\delta_{1,1}-\delta_5)+\delta_{1,1}+\delta\,(v_1,v_x)$. Choose paths P_{v_b,v_x} and P_{v_{a+1},v_x} and define $k_1=t(Q_2)s(P_{v_b,v_x})-s(Q_2)t(P_{v_b,v_x})$ and $k_2=s(Q_1)t(P_{v_{a+1},v_x})-t(Q_1)s(P_{v_{a+1},v_x})$. We consider six subcases.

The vertex v_x is positioned on path $v_1\to v_a$. Using path P_{v_b,v_x}, we get path $(4,\delta_5+\delta(v_1,v_x))$, resulting in $k_1=20n-16-5(\delta_5+\delta(v_1,v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0,\delta_{1,1}+\delta(v_1,v_x))$, arriving at $k_2=\delta_{1,1}+\delta(v_1,v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix}\geq M\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$=\begin{bmatrix} 20n-16-5(d_5-d_{11}) \\ 20n^2+5n(d_{11}-d_5)-36n+16+5\delta_5-4d_{11}+\delta(v_1,v_x) \end{bmatrix}.$$

Hence,

$$\text{expin}(v_x,D^{(2)})\geq 20n^2+5n(d_{11}-d_5)-16n+\delta_{1,1}+\delta\,(v_1,v_x) \qquad (4.15)$$

for every vertex v_x positioned on the path $v_1\to v_a$.

The vertex v_x is positioned on path $v_{a+1}\to v_b$. Using path P_{v_b,v_x}, we get path $(5,\delta_5-1+\delta(v_1,v_x))$, ending up with $k_1=25n-15-5(\delta_5+\delta(v_1,v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0,\delta_{1,1}-n+\delta(v_1,v_x))$, resulting in $k_2=\delta_{1,1}-n+\delta(v_1,v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix}\geq M\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 20n - 15 - 5(d_5 - d_{11}) \\ 20n^2 + 5n(d_{11} - d_5) - 36n + 15 + 5\delta_5 - 4d_{11} + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\text{expin}(v_x, D^{(2)}) \geq 20n^2 + 5n(d_{11} - d_5) - 16n + \delta_{1,1} + \delta(v_1, v_x) \tag{4.16}$$

for every vertex v_x positioned on the path $v_{a+1} \to v_b$.

The vertex v_x is positioned on path $v_{b+1} \to v_c$. Using path P_{v_b,v_x}, we get path $(1, \delta_5 - 5n + 2 + \delta(v_1, v_x))$, leading to $k_1 = 30n - 14 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} - 2n - 7 + \delta(v_1, v_x))$, arriving at $k_2 = \delta_{1,1} - 2n - 7 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 20n - 49 - 5(d_5 - d_{11}) \\ 20n^2 + 5n(d_{11} - d_5) - 71n + 42 + 5\delta_5 - 4d_{11} + \delta(v_1, v_x) \end{bmatrix}.$$

Let $p_1 = 20n - 49 - 5(d_5 - d_{11})$ and $p_2 = 20n^2 + 5n(d_{11} - d_5) - 71n + 42 + 5\delta_5 - 4d_{11} + \delta(v_1, v_x)$. We consider the walk (p_1, p_2) from v_{a+1} to v_x. Note that path P_{v_{a+1},v_x} is $(0, \delta_{1,1} - 2n - 7 + \delta(v_1, v_x))$ and that solving the system $M\mathbf{u} + \begin{bmatrix} s(P_{v_{a+1},v_x}) \\ t(P_{v_{a+1},v_x}) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ leads to $u_1 = 20n - 49 - 5(d_5 - d_{11})$ and $u_2 = 0$. Because the path P_{v_{a+1},v_x} lies totally on cycle Q_2, there is no (p_1, p_2)-walk from v_{a+1} to v_x. Therefore, $\text{expin}(v_x, D^{(2)}) > p_1 + p_2$. The shortest walk from v_{a+1} to v_x with minimal p_1 red arcs and minimal p_2 red arcs is a $(p_1 + s(Q_2), p_2 + t(Q_2))$-walk. Since $s(Q_2) + t(Q_2) = 5n + 1$, we get

$$\text{expin}(v_x, D^{(2)}) \geq p_1 + p_2 + 7(s(Q_2) + t(Q_2))$$

$$= 20n^2 + 5n(d_{11} - d_5) - 16n + \delta_{1,1} + \delta(v_1, v_x) \tag{4.17}$$

for every vertex v_x positioned on the path $v_{b+1} \to v_c$.

The vertex v_x is positioned on path $v_{c+1} \to v_d$. Using path P_{v_b,v_x}, we get path $(2, \delta_5 - 5n + 1 + \delta(v_1, v_x))$, arriving at $k_1 = 35n - 13 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} - 3n - 4 + \delta(v_1, v_x))$, leading to $k_2 = \delta_{1,1} - 3n - 4 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 20n - 33 - 5(d_5 - d_{11}) \\ 20n^2 + 5n(d_{11} - d_5) - 56n + 29 + 5\delta_5 - 4d_{11} + \delta(v_1, v_x) \end{bmatrix}.$$

Let $p_1 = 20n - 33 - 5(d_5 - d_{11})$ and $p_2 = 20n^2 + 5n(d_{11} - d_5) - 56n + 29 + 5\delta_5 - 4d_{11} + \delta(v_1, v_x)$. We consider the walk (p_1, p_2) from v_{a+1} to v_x. Note that path P_{v_{a+1},v_x} is $(0, \delta_{1,1} - 3n - 4 + \delta(v_1, v_x))$ and that solving the system $M\mathbf{u} + \begin{bmatrix} s(P_{v_{a+1},v_x}) \\ t(P_{v_{a+1},v_x}) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ gives us $u_1 = 20n - 33 - 5(d_5 - d_{11})$ and $u_2 = 0$. Because the path P_{v_{a+1},v_x} lies totally on cycle Q_2, there is no (p_1, p_2)-walk from v_{a+1} to v_x. Therefore, $\text{expin}(v_x, D^{(2)}) > p_1 + p_2$. The shortest walk from v_{a+1} to v_x

with minimal p_1 red arcs and minimal p_2 red arcs is a $(p_1+s(Q_2), p_2+t(Q_2))$-walk. Since $s(Q_2)+t(Q_2) = 5n+1$, we get

$$\begin{aligned} \exp\mathrm{in}(v_x, D^{(2)}) &\geq p_1 + p_2 + 5(s(Q_2)+t(Q_2)) \\ &= 20n^2 + 5n(d_{11}-d_5) - 16n + \delta_{1,1} + \delta(v_1, v_x) \end{aligned} \tag{4.18}$$

for every vertex v_x positioned on the path $v_{c+1} \to v_d$.

The vertex v_x is positioned on path $v_{d+1} \to v_e$. Using path P_{v_b,v_x}, we get path $(3, \delta_5 - 5n + \delta(v_1, v_x))$, ending up with $k_1 = 40n - 12 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} - 4n - 2 + \delta(v_1, v_x))$, resulting in $k_2 = \delta_{1,1} - 4n - 2 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 20n - 22 - 5(d_5 - d_{11}) \\ 20n^2 + 5n(d_{11}-d_5) - 46n + 20 + 5\delta_5 - 4d_{11} + \delta(v_1, v_x) \end{bmatrix}.$$

Let $p_1 = 20n - 22 - 5(d_5 - d_{11})$ and $p_2 = 20n^2 + 5n(d_{11}-d_5) - 46n + 20 + 5\delta_5$ $-4d_{11} + \delta(v_1, v_x)$. We consider the walk (p_1, p_2) from v_{a+1} to v_x. Note that path P_{v_{a+1},v_x} is $(0, \delta_{1,1} - 4n - 2 + \delta(v_1, v_x))$ and that solving the system $M\mathbf{u} + \begin{bmatrix} s(P_{v_{a+1},v_x}) \\ t(P_{v_{a+1},v_x}) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ leads to $u_1 = 20n - 22 - 5(d_5 - d_{11})$ and $u_2 = 0$. Because the path P_{v_{a+1},v_x} lies totally on cycle Q_2, there is no (p_1, p_2)-walk from v_{a+1} to v_x. Therefore, $\exp\mathrm{in}(v_x, D^{(2)}) > p_1 + p_2$. The shortest walk from v_{a+1} to v_x with minimal p_1 red arcs and minimal p_2 red arcs is a $(p_1+s(Q_2), p_2+t(Q_2))$-walk. Since $s(Q_2)+t(Q_2) = 5n+1$, we get

$$\begin{aligned} \exp\mathrm{in}(v_x, D^{(2)}) &\geq p_1 + p_2 + 2(s(Q_2)+t(Q_2)) \\ &= 20n^2 + 5n(d_{11}-d_5) - 16n + \delta_{1,1} + \delta(v_1, v_x) \end{aligned} \tag{4.19}$$

for every vertex v_x positioned on the path $v_{d+1} \to v_e$.

The vertex v_x is positioned on path $v_{e+1} \to v_{5n+1}$. Using path P_{v_b,v_x}, we get path $(4, \delta_5 - 5n - 1 + \delta(v_1, v_x))$, arriving at $k_1 = 45n - 11 - 5(\delta_5 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} - 5n + \delta(v_1, v_x))$, resulting in $k_2 = \delta_{1,1} - 5n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 20n - 11 - 5(d_5 - d_{11}) \\ 20n^2 + 5n(d_{11}-d_5) - 36n + 11 + 5\delta_5 - 4d_{11} + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\exp\mathrm{in}(v_x, D^{(2)}) \geq 20n^2 + 5n(d_{11}-d_5) - 16n + \delta_{1,1} + \delta(v_1, v_x) \tag{4.20}$$

for every vertex v_x positioned on the path $v_{e+1} \to v_{5n+1}$.

From (4.15), (4.16), (4.17), (4.18), (4.19), and (4.20), we conclude that expin $(v_x, D^{(2)}) \geq 20n^2 + 5n(d_{11}-d_5) - 16n + \delta_{1,1} + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n+1$.

Next, we will show that $\exp\mathrm{in}(v_x, D^{(2)}) \leq 20n^2 + 5n(d_{11}-d_5) - 16n + \delta_{1,1} + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n+1$. First, we will show that $\exp\mathrm{in}(v_1, D^{(2)}) = 20n^2 + 5n(d_{11}-d_5) - 16n + \delta_{1,1}$ and then utilize Lemma 3.2 to ensure that $\exp\mathrm{in}(v_x, D^{(2)})$ $\leq 20n^2 + 5n(d_{11}-d_5) - 16n + \delta_{1,1} + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n+1$.

From (4.15), we have $\exp\text{in}(v_1, D^{(2)}) \geq 20n^2 + 5n(d_{11} - d_5) - 16n + \delta_{1,1}$. Next, it is necessary to prove that $\exp\text{in}(v_1, D^{(2)}) \leq 20n^2 + 5n(d_{11} - d_5) - 16n + \delta_{1,1}$ for every v_y and that for $y = 1, 2, ..., 5n + 1$, the system

$$M\mathbf{u} + \begin{bmatrix} s(P_{v_y,v_1}) \\ t(P_{v_y,v_1}) \end{bmatrix}$$

$$= \begin{bmatrix} 20n - 16 - 5(d_5 - d_{11}) \\ 20n^2 + 5n(d_{11} - d_5) - 36n + 16 + 5\delta_5 - 4d_{11} \end{bmatrix} \tag{4.21}$$

has a nonnegative integer solution for the path P_{v_y,v_1}.

The solution of system (4.21) is $u_1 = 20n - 16 - 5\delta_5 - (5n-4)s(P_{v_y,v_1}) + 5t(P_{v_y,v_1})$ and $u_2 = \delta_{1,1} - (1-n)s(P_{v_y,d_1}) - t(P_{v_y,v_1})$. If v_y is positioned on the $v_1 \to v_a$ path, then there is a $(5, 5n - 4 - \delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 20n - 16 - 5(\delta_5 + \delta(v_1, v_y)) \geq 39$ since $\delta_5 + \delta(v_1, v_y) \leq n - 2$ for $n \geq 3$ and $u_2 = \delta_{1,1} + \delta(v_1, v_y) - 1 \geq 1$ since $\delta_{1,1} + \delta(v_1, v_y) \geq n - 1$ for $n \geq 3$. If v_y is positioned on the $v_{a+1} \to v_b$ path, then there is a $(4, 5n - 3 - \delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 25n - 15 - 5(\delta_5 + \delta(v_1, v_y)) \geq 0$ since $\delta_5 + \delta(v_1, v_y) \leq 4n$ for $n \geq 3$ and $u_2 = \delta_{1,1} + \delta(v_1, v_y) - n - 1 \geq 0$ since $\delta_{1,1} + \delta(v_1, v_y) \geq n + 1$ for $n \geq 3$. If v_y is positioned on the $v_{b+1} \to v_c$ path, then there is a $(3, 5n - 2 - \delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 30n - 14 - 5(\delta_5 + \delta(v_1, v_y)) \geq 11$ since $\delta_5 + \delta(v_1, v_y) \leq 4n + 1$ for $n \geq 3$ and $u_2 = \delta_{1,1} + \delta(v_1, v_y) - 2n - 1 \geq 7$ since $\delta_{1,1} + \delta(v_1, v_y) \geq 4n + 2$ for $n \geq 3$. If v_y is positioned on the $v_{c+1} \to v_d$ path, then there is a $(2, 5n - 1 - \delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 35n - 13 - 5(\delta_5 + \delta(v_1, v_y)) \geq 22$ since $\delta_5 + \delta(v_1, v_y) \leq 5n - 1$ for $n \geq 3$ and $u_2 = \delta_{1,1} + \delta(v_1, v_y) - 3n - 1 \geq 5$ since $\delta_{1,1} + \delta(v_1, v_y) \geq 5n$ for $n \geq 3$. If v_y is positioned on the $v_{d+1} \to v_e$ path, then there is a $(1, 5n - \delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 40n - 12 - 5(\delta_5 + \delta(v_1, v_y)) \geq 33$ since $\delta_5 + \delta(v_1, v_y) \leq 5n$ for $n \geq 3$ and $u_2 = \delta_{1,1} + \delta(v_1, v_y) - 4n - 1 \geq 3$ since $\delta_{1,1} + \delta(v_1, v_y) \geq 5n + 1$ for $n \geq 3$. If v_y is positioned on the $v_{e+1} \to v_{5n+1}$ path, then there is a $(0, 5n+1-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 45n - 11 - 5(\delta_5 + \delta(v_1, v_y)) \geq 44$ since $\delta_5 + \delta(v_1, v_y) \leq 5n + 1$ for $n \geq 3$ and $u_2 = \delta_{1,1} + \delta(v_1, v_y) - 5n - 1 \geq 1$ since $\delta_{1,1} + \delta(v_1, v_y) \geq 5n + 2$ for $n \geq 3$.

Therefore, for every $y = 1, 2, ..., 5n + 1$, the system (4.21) has a nonnegative integer solution. Proposition 3.1 ensures that for every $y = 1, 2, ..., 5n + 1$, there is a $v_y \xrightarrow{(m,h)} v_1$ walk with $m = 20n - 16 - 5(d_5 - d_{11})$ and $h = 20n^2 + 5n(d_{11} - d_5)$ $-36n + 16 + 5\delta_5 - 4d_{11}$. Consequently, $\exp\text{in}(v_1, D^{(2)}) \leq 20n^2 + 5n(d_{11} - d_5) - 16n + \delta_{1,1}$. So, $\exp\text{in}(v_1, D^{(2)}) = 20n^2 + 5n(d_{11} - d_5) - 16n + \delta_{1,1}$. By Lemma 3.2, we conclude that expin $(v_x, D^{(2)}) \leq 20n^2 + 5n(d_{11} - d_5) - 16n + \delta_{1,1} + \delta(v_1, v_x)$ for every $x = 1, 2, ..., 5n + 1$. □

Theorem 4.2. *Suppose that $D^{(2)}$ is a two-cycle primitive bicolour digraph with cycle lengths n and $5n+1$. If $D^{(2)}$ has four or five red arcs in alternating orders of one in Q_2, then for every $x = 1, 2, ..., 5n+1$,* $\mathrm{expin}(v_x, D^{(2)}) =$

$$\begin{cases} 25n^2 + 5n\left(\delta_5 - \delta_{1,2}\right) + \delta_5 + \delta\left(v_1, v_x\right), & \text{for } \delta_{1,2} - \delta_2 \le n \\ 20n^2 - n + 5n(\delta_5 - \delta_2) + \delta_5 + \delta\left(v_1, v_x\right), & \text{for } n < \delta_{1,2} - \delta_2 < 3n \\ 20n^2 - n + 5n(\delta_{1,1} - \delta_2) + \delta_{1,1} + \delta\left(v_1, v_x\right), & \text{for } \delta_{1,2} - \delta_2 \ge 3n. \end{cases}$$

Proof. Suppose the $\mathrm{expin}(v_x, D^{(2)})$ value for each $x = 1, 2, ..., 5n+1$ is generated from a (m_x, h_x)-walk. The proof of Theorem 4.2 is presented in the following 3 cases.

Case 1. (for $\delta_{1,2} - \delta_2 \le n$)

The formulas for Case 1 in Theorem 4.1 and Theorem 4.2 are the same. Therefore, the proof for Case 1 Theorem 4.1 works for Theorem 4.2.

Case 2. (for $n < \delta_{1,2} - \delta_2 < 3n$)

First, we need to show that $\mathrm{expin}(v_x, D^{(2)}) \ge 20n^2 - n + 5n(\delta_5 - \delta_2) + \delta_5 + \delta\left(v_1, v_x\right)$. Choose paths P_{v_b,v_x} and P_{v_{e+1},v_x} and define $k_1 = t(Q_2)s(P_{v_b,v_x}) - s(Q_2)t(P_{v_b,v_x})$ and $k_2 = s(Q_1)t(P_{v_{e+1},v_x}) - t(Q_1)s(P_{v_{e+1},v_x})$. We consider six subcases.

The vertex v_x is positioned on path $v_1 \to v_a$. Using path P_{v_b,v_x}, we get path $(4, \delta_2 - 3 + \delta(v_1, v_x))$, ending up with $k_1 = 20n - 1 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(0, \delta_5 + \delta(v_1, v_x))$, leading to $k_2 = \delta_5 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \ge M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 20n - 1 - 5(d_2 - d_5) \\ 20n^2 - 21n + 1 + 5n(d_5 - d_2) + 5d_2 - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\mathrm{expin}(v_x, D^{(2)}) \ge 20n^2 - n + 5n(d_5 - d_2) + \delta_5 + \delta\left(v_1, v_x\right) \tag{4.22}$$

for every vertex v_x positioned on the path $v_1 \to v_a$.

The vertex v_x is positioned on path $v_{a+1} \to v_b$. Using path P_{v_b,v_x}, we get path $(5, \delta_2 - 4 + \delta(v_1, v_x))$, arriving at $k_1 = 25n - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(1, \delta_5 - 1 + \delta(v_1, v_x))$, ending up with $k_2 = \delta_5 - n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \ge M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 20n - 5(d_2 - d_5) \\ 20n^2 - 21n + 5n(d_5 - d_2) + 5d_2 - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\mathrm{expin}(v_x, D^{(2)}) \ge 20n^2 - n + 5n(d_5 - d_2) + \delta_5 + \delta\left(v_1, v_x\right) \tag{4.23}$$

for every vertex v_x positioned on the path $v_{a+1} \to v_b$.

The vertex v_x is positioned on path $v_{b+1} \to v_c$. Using path P_{v_b,v_x}, we get path $(1, \delta_2 - 5n - 1 + \delta(v_1, v_x))$, arriving at $k_1 = 30n + 1 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(2, \delta_5 - 2 + \delta(v_1, v_x))$, resulting in $k_2 = \delta_5 - 2n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 20n + 1 - 5(d_2 - d_5) \\ 20n^2 - 21n - 1 + 5n(d_5 - d_2) + 5d_2 - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\operatorname{expin}(v_x, D^{(2)}) \geq 20n^2 - n + 5n(d_5 - d_2) + \delta_5 + \delta\,(v_1, v_x) \tag{4.24}$$

for every vertex v_x positioned on the path $v_{b+1} \to v_c$.

The vertex v_x is positioned on path $v_{c+1} \to v_d$. Using path P_{v_b,v_x}, we get path $(2, \delta_2 - 5n - 2 + \delta(v_1, v_x))$, leading to $k_1 = 35n + 2 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(3, \delta_5 - 3 + \delta(v_1, v_x))$, ending up with $k_2 = \delta_5 - 3n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 20n + 2 - 5(d_2 - d_5) \\ 20n^2 - 21n - 2 + 5n(d_5 - d_2) + 5d_2 - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\operatorname{expin}(v_x, D^{(2)}) \geq 20n^2 - n + 5n(d_5 - d_2) + \delta_5 + \delta\,(v_1, v_x) \tag{4.25}$$

for every vertex v_x positioned on the path $v_{c+1} \to v_d$.

The vertex v_x is positioned on path $v_{d+1} \to v_e$. Using path P_{v_b,v_x}, we get path $(3, \delta_2 - 5n - 3 + \delta(v_1, v_x))$, resulting in $k_1 = 40n + 3 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(4, \delta_5 - 4 + \delta(v_1, v_x))$, leading to $k_2 = \delta_5 - 4n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 20n + 3 - 5(d_2 - d_5) \\ 20n^2 - 21n - 3 + 5n(d_5 - d_2) + 5d_2 - 4\delta_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\operatorname{expin}(v_x, D^{(2)}) \geq 20n^2 - n + 5n(d_5 - d_2) + \delta_5 + \delta\,(v_1, v_x) \tag{4.26}$$

for every vertex v_x positioned on the path $v_{d+1} \to v_e$.

The vertex v_x is positioned on path $v_{e+1} \to v_{5n+1}$. Using path P_{v_b,v_x}, we get path $(4, \delta_2 - 5n - 4 + \delta(v_1, v_x))$, arriving at $k_1 = 45n + 4 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{e+1},v_x}, we get path $(0, \delta_5 - 5n - 1 + \delta(v_1, v_x))$, ending up with $k_2 = \delta_5 - 5n - 1 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 20n - 1 - 5(d_2 - d_5) \\ 20n^2 + 5n(d_5 - d_2) - 26n + 5\delta_2 - 4d_5 + \delta(v_1, v_x) \end{bmatrix}.$$

Let $p_1 = 20n - 1 - 5(d_2 - d_5)$ and $p_2 = 20n^2 + 5n(d_5 - d_2) - 26n + 5\delta_2 - 4d_5$ $+\delta(v_1, v_x)$. We consider the walk (p_1, p_2) from v_{e+1} to v_x. Note that path P_{v_{e+1},v_x} is $(0, \delta_5 - 5n - 1 + \delta(v_1, v_x))$ and that solving the system $M\mathbf{u} + \begin{bmatrix} s(P_{v_{e+1},v_x}) \\ t(P_{v_{e+1},v_x}) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ results in $u_1 = 20n - 1 - 5(d_2 - d_5)$ and $u_2 = 0$. Because the path P_{v_{e+1},v_x}

lies totally on cycle Q_2, there is no (p_1,p_2)-walk from v_{e+1} to v_x. Therefore, $\text{expin}(v_x,D^{(2)}) > p_1+p_2$. The shortest walk from v_{e+1} to v_x with minimal p_1 red arcs and minimal p_2 red arcs is a $(p_1+s(Q_2),p_2+t(Q_2))$-walk. Since $s(Q_2)+t(Q_2)=5n+1$, we get

$$\begin{aligned}\text{expin}(v_x,D^{(2)}) &\geq p_1+p_2+s(Q_2)+t(Q_2)\\ &= 20n^2-n+5n(d_5-d_2)+\delta_5+\delta\,(v_1,v_x)\end{aligned} \tag{4.27}$$

for every vertex v_x positioned on the path $v_{e+1}\rightarrow v_{5n+1}$.

From (4.22), (4.23), (4.24), (4.25), (4.26), and (4.27), we conclude that expin $(v_x,D^{(2)}) \geq 20n^2-n+5n(d_5-d_2)+\delta_5+\delta\,(v_1,v_x)$ for $x=1,2,...,5n+1$.

Next, we will show that $\text{expin}(v_x,D^{(2)}) \leq 20n^2-n+5n(d_5-d_2)+\delta_5+\delta\,(v_1,v_x)$ for $x=1,2,...,5n+1$. First, we will show that $\text{expin}(v_1,D^{(2)}) = 20n^2-n+5n(d_5-d_2)+\delta_5$ and then utilize Lemma 3.2 to ensure that $\text{expin}(v_x,D^{(2)}) \leq 20n^2-n+5n(d_5-d_2)+\delta_5+\delta\,(v_1,v_x)$ for $x=1,2,...,5n+1$.

From (4.22), we have $\text{expin}(v_1,D^{(2)}) \geq 20n^2-n+5n(d_5-d_2)+\delta_5$. Next, it is necessary to prove that $\text{expin}(v_1,D^{(2)}) \leq 20n^2-n+5n(d_5-d_2)+\delta_5$ for every v_y and that for $y=1,2,...,5n+1$, the system

$$\begin{aligned}&M\mathbf{u}+\begin{bmatrix} s(P_{v_y,v_1}) \\ t(P_{v_y,v_1})\end{bmatrix}\\ &=\begin{bmatrix} 20n-1-5(d_2-d_5) \\ 20n^2-21n+1+5n(d_5-d_2)+5d_2-4\delta_5\end{bmatrix}\end{aligned} \tag{4.28}$$

has a nonnegative integer solution for the path P_{v_y,v_1}.

The solution of system (4.28) is $u_1=20n-1-5\delta_2-(5n-4)s(P_{v_y,v_1})+5t(P_{v_y,v_1})$ and $u_2=\delta_5-(1-n)s(P_{v_y,d_1})-t(P_{v_y,v_1})$. If v_y is positioned on the $v_1\rightarrow v_a$ path, then there is a $(5,5n-4-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=20n-1-5(\delta_2+\delta(v_1,v_y))\geq 4$ since $\delta_2+\delta(v_1,v_y)\leq 4n-1$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-1\geq 1$ since $\delta_5+\delta(v_1,v_y)\geq n-1$ for $n\geq 3$. If v_y is positioned on the $v_{a+1}\rightarrow v_b$ path, then there is a $(4,5n-3-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=25n-5(\delta_2+\delta(v_1,v_y))\geq 0$ since $\delta_2+\delta(v_1,v_y)\leq 5n$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-n-1\geq 0$ since $\delta_5+\delta(v_1,v_y)\geq n+1$ for $n\geq 3$. If v_y is positioned on the $v_{b+1}\rightarrow v_c$ path, then there is a $(3,5n-2-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=30n+1-5(\delta_2+\delta(v_1,v_y))\geq 6$ since $\delta_2+\delta(v_1,v_y)\leq 6n-1$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-2n-1\geq 3$ since $\delta_5+\delta(v_1,v_y)\geq 3n+1$ for $n\geq 3$. If v_y is positioned on the $v_{c+1}\rightarrow v_d$ path, then there is a $(2,5n-1-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=35n+2-5(\delta_2+\delta(v_1,v_y))\geq 12$ since $\delta_2+\delta(v_1,v_y)\leq 6n+1$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-3n-1\geq 2$ since $\delta_5+\delta(v_1,v_y)\geq 4n$ for $n\geq 3$. If v_y is positioned on the $v_{d+1}\rightarrow v_e$ path, then there is a $(1,5n-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=40n+3-5(\delta_2+\delta(v_1,v_y))\geq 18$ since $\delta_2+\delta(v_1,v_y)\leq 7n$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-4n-1\geq 2$ since $\delta_5+\delta(v_1,v_y)\geq 5n$ for $n\geq 3$. If v_y is positioned on the $v_{e+1}\rightarrow v_{5n+1}$ path, then there is a $(0,5n+1-\delta(v_1,v_y))$-path from v_y to v_1. Using this path, we determine that $u_1=45n+4-5(\delta_2+\delta(v_1,v_y))\geq 9$ since $\delta_2+\delta(v_1,v_y)\leq 9n-1$ for $n\geq 3$ and $u_2=\delta_5+\delta(v_1,v_y)-5n-1\geq 0$ since $\delta_5+\delta(v_1,v_y)\geq 5n+1$ for $n\geq 3$.

Therefore, for every $y = 1, 2, ..., 5n + 1$, the system (4.28) has a nonnegative integer solution. Proposition 3.1 ensures that for every $y = 1, 2, ..., 5n + 1$, there is a $v_y \overset{(m,h)}{\longrightarrow} v_1$ walk with $m = 20n - 1 - 5(d_2 - d_5)$ and $h = 20n^2 - 21n + 1 + 5n(d_5 - d_2) + 5d_2 - 4\delta_5$. Consequently, $\text{expin}(v_1, D^{(2)}) \leq 20n^2 - n + 5n(d_5 - d_2) + \delta_5$. So, $\text{expin}(v_1, D^{(2)}) = 20n^2 - n + 5n(d_5 - d_2) + \delta_5$. By Lemma 3.2, we conclude that expin $(v_x, D^{(2)}) \leq 20n^2 - n + 5n(d_5 - d_2) + \delta_5 + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n+1$.

Case 3. (for $\delta_{1,2} - \delta_2 \geq 3n$)

First, we need to show that $\text{expin}(v_x, D^{(2)}) \geq 20n^2 - n + 5n(\delta_{1,1} - \delta_2) + \delta_{1,1} + \delta(v_1, v_x)$. Choose paths P_{v_b,v_x} and P_{v_{a+1},v_x} and define $k_1 = t(Q_2)s(P_{v_b,v_x}) - s(Q_2)t(P_{v_b,v_x})$ and $k_2 = s(Q_1)t(P_{v_{a+1},v_x}) - t(Q_1)s(P_{v_{a+1},v_x})$. We consider six subcases.

The vertex v_x is positioned on path $v_1 \to v_a$. Using path P_{v_b,v_x}, we get path $(4, \delta_2 - 3 + \delta(v_1, v_x))$, arriving at $k_1 = 20n - 1 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} + \delta(v_1, v_x))$, leading to $k_2 = \delta_{1,1} + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$
$$= \begin{bmatrix} 20n - 1 - 5(d_2 - d_{11}) \\ 20n^2 - 21n + 1 + 5n(d_{11} - d_2) + 5d_2 - 4\delta_{1,1} + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\text{expin}(v_x, D^{(2)}) \geq 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1} + \delta(v_1, v_x) \tag{4.29}$$

for every vertex v_x positioned on the path $v_1 \to v_a$.

The vertex v_x is positioned on path $v_{a+1} \to v_b$. Using path P_{v_b,v_x}, we get path $(5, \delta_2 - 4 + \delta(v_1, v_x))$, ending up with $k_1 = 25n - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} - n + \delta(v_1, v_x))$, leading to $k_2 = \delta_{1,1} - n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$
$$= \begin{bmatrix} 20n - 5(d_2 - d_{11}) \\ 20n^2 - 21n + 5n(d_{11} - d_2) + 5d_2 - 4\delta_{1,1} + \delta(v_1, v_x) \end{bmatrix}.$$

Hence,

$$\text{expin}(v_x, D^{(2)}) \geq 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1} + \delta(v_1, v_x) \tag{4.30}$$

for every vertex v_x positioned on the path $v_{a+1} \to v_b$.

The vertex v_x is positioned on path $v_{b+1} \to v_c$. Using path P_{v_b,v_x}, we get path $(1, \delta_2 - 5n - 1 + \delta(v_1, v_x))$, arriving at $k_1 = 30n + 1 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} - 2n - 3 + \delta(v_1, v_x))$, resulting in $k_2 = \delta_{1,1} - 2n - 3 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$
$$\begin{bmatrix} 20n - 14 - 5(d_2 - d_{11}) \\ 20n^2 + 5n(d_{11} - d_2) - 36n + 11 + 5\delta_2 - 4d_{11} + \delta(v_1, v_x) \end{bmatrix}.$$

Let $p_1 = 20n - 14 - 5(d_2 - d_{11})$ and $p_2 = 20n^2 + 5n(d_{11} - d_2) - 36n + 11 + 5\delta_2 - 4d_{11} + \delta(v_1, v_x)$. We consider the walk (p_1, p_2) from v_{a+1} to v_x. Note that path P_{v_{a+1},v_x} is $(0, \delta_{1,1} - 2n - 3 + \delta(v_1, v_x))$ and that solving the system $M\mathbf{u} + \begin{bmatrix} s(P_{v_{a+1},v_x}) \\ t(P_{v_{a+1},v_x}) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ leads to $u_1 = 20n - 14 - 5(d_2 - d_{11})$ and $u_2 = 0$. Because the path P_{v_{a+1},v_x} lies totally on cycle Q_2, there is no (p_1, p_2)-walk from v_{a+1} to v_x. Therefore, $\text{expin}(v_x, D^{(2)}) > p_1 + p_2$. The shortest walk from v_{a+1} to v_x with minimal p_1 red arcs and minimal p_2 red arcs is a $(p_1 + s(Q_2), p_2 + t(Q_2))$-walk. Since $s(Q_2) + t(Q_2) = 5n + 1$, we get

$$\begin{aligned} \text{expin}(v_x, D^{(2)}) &\geq p_1 + p_2 + 3(s(Q_2) + t(Q_2)) \\ &= 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1} + \delta(v_1, v_x) \end{aligned} \tag{4.31}$$

for every vertex v_x positioned on the path $v_{b+1} \to v_c$.

The vertex v_x is positioned on path $v_{c+1} \to v_d$. Using path P_{v_b,v_x}, we get path $(2, \delta_2 - 5n - 2 + \delta(v_1, v_x))$, leading to $k_1 = 35n + 2 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} - 3n - 2 + \delta(v_1, v_x))$, resulting in $k_2 = \delta_{1,1} - 3n - 2 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 20n - 8 - 5(d_2 - d_{11}) \\ 20n^2 + 5n(d_{11} - d_2) - 31n + 6 + 5\delta_2 - 4d_{11} + \delta(v_1, v_x) \end{bmatrix}.$$

Let $p_1 = 20n - 8 - 5(d_2 - d_{11})$ and $p_2 = 20n^2 + 5n(d_{11} - d_2) - 31n + 6 + 5\delta_2 - 4d_{11} + \delta(v_1, v_x)$. We consider the walk (p_1, p_2) from v_{a+1} to v_x. Note that path P_{v_{a+1},v_x} is $(0, \delta_{1,1} - 3n - 2 + \delta(v_1, v_x))$ and that solving the system $M\mathbf{u} + \begin{bmatrix} s(P_{v_{a+1},v_x}) \\ t(P_{v_{a+1},v_x}) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ leads to $u_1 = 20n - 8 - 5(d_2 - d_{11})$ and $u_2 = 0$. Because the path P_{v_{a+1},v_x} lies totally on cycle Q_2, there is no (p_1, p_2)-walk from v_{a+1} to v_x. Therefore, $\text{expin}(v_x, D^{(2)}) > p_1 + p_2$. The shortest walk from v_{a+1} to v_x with minimal p_1 red arcs and minimal p_2 red arcs is a $(p_1 + s(Q_2), p_2 + t(Q_2))$-walk. Since $s(Q_2) + t(Q_2) = 5n + 1$, we get

$$\begin{aligned} \text{expin}(v_x, D^{(2)}) &\geq p_1 + p_2 + 2(s(Q_2) + t(Q_2)) \\ &= 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1} + \delta(v_1, v_x) \end{aligned} \tag{4.32}$$

for every vertex v_x positioned on the path $v_{c+1} \to v_d$.

The vertex v_x is positioned on path $v_{d+1} \to v_e$. Using path P_{v_b,v_x}, we get path $(3, \delta_2 - 5n - 3 + \delta(v_1, v_x))$, ending up with $k_1 = 40n + 3 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} - 4n - 1 + \delta(v_1, v_x))$, arriving at $k_2 = \delta_{1,1} - 4n - 1 + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{bmatrix} m_x \\ h_x \end{bmatrix} \geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} =$$

$$\begin{bmatrix} 20n - 2 - 5(d_2 - d_{11}) \\ 20n^2 + 5n(d_{11} - d_2) - 26n + 1 + 5\delta_2 - 4d_{11} + \delta(v_1, v_x) \end{bmatrix}.$$

Let $p_1 = 20n - 2 - 5(d_2 - d_{11})$ and $p_2 = 20n^2 + 5n(d_{11} - d_2) - 26n + 1 + 5\delta_2 - 4d_{11} + \delta(v_1, v_x)$. We consider the walk (p_1, p_2) from v_{a+1} to v_x. Note that path P_{v_{a+1},v_x} is $(0, \delta_{1,1} - 4n - 1 + \delta(v_1, v_x))$ and that solving the system $M\mathbf{u} +$

$\begin{bmatrix} s(P_{v_{a+1},v_x}) \\ t(P_{v_{a+1},v_x}) \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ results in $u_1 = 20n - 2 - 5(d_2 - d_{11})$ and $u_2 = 0$. Because the path P_{v_{a+1},v_x} lies totally on cycle Q_2, there is no (p_1, p_2)-walk from v_{a+1} to v_x. Therefore, $\text{expin}(v_x, D^{(2)}) > p_1 + p_2$. The shortest walk from v_{a+1} to v_x with minimal p_1 red arcs and minimal p_2 red arcs is a $(p_1+s(Q_2), p_2+t(Q_2))$-walk. Since $s(Q_2) + t(Q_2) = 5n + 1$, we get

$$\begin{aligned} \text{expin}(v_x, D^{(2)}) &\geq p_1 + p_2 + s(Q_2) + t(Q_2) \\ &= 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1} + \delta(v_1, v_x) \end{aligned} \tag{4.33}$$

for every vertex v_x positioned on the path $v_{d+1} \to v_e$.

The vertex v_x is positioned on path $v_{e+1} \to v_{5n+1}$. Using path P_{v_b,v_x}, we get path $(4, \delta_2 - 5n - 4 + \delta(v_1, v_x))$, leading to $k_1 = 45n + 4 - 5(\delta_2 + \delta(v_1, v_x))$. Using path P_{v_{a+1},v_x}, we get path $(0, \delta_{1,1} - 5n + \delta(v_1, v_x))$, arriving at $k_2 = \delta_{1,1} - 5n + \delta(v_1, v_x)$. Utilizing Lemma 3.3, we get

$$\begin{aligned} \begin{bmatrix} m_x \\ h_x \end{bmatrix} &\geq M \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \\ &= \begin{bmatrix} 20n + 4 - 5(d_2 - d_{11}) \\ 20n^2 - 21n - 4 + 5n(d_{11} - d_2) + 5d_2 - 4\delta_{1,1} + \delta(v_1, v_x) \end{bmatrix}. \end{aligned}$$

Hence,

$$\text{expin}(v_x, D^{(2)}) \geq 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1} + \delta(v_1, v_x) \tag{4.34}$$

for every vertex v_x positioned on the path $v_{e+1} \to v_{5n+1}$.

From (4.29), (4.30), (4.31), (4.32), (4.33), and (4.34), we conclude that expin $(v_x, D^{(2)}) \geq 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1} + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n + 1$.

Next, we will show that $\text{expin}(v_x, D^{(2)}) \leq 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1} + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n + 1$. First, we will show that $\text{expin}(v_1, D^{(2)}) = 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1}$ and then utilize Lemma 3.2 to ensure that $\text{expin}(v_x, D^{(2)}) \leq 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1} + \delta(v_1, v_x)$ for $x = 1, 2, ..., 5n + 1$.

From (4.29), we have $\text{expin}(v_1, D^{(2)}) \geq 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1}$. Next, it is necessary to prove that $\text{expin}(v_1, D^{(2)}) \leq 20n^2 - n + 5n(d_{11} - d_2) + \delta_{1,1}$ for every v_y and that for $y = 1, 2, ..., 5n + 1$, the system

$$\begin{aligned} M\mathbf{u} &+ \begin{bmatrix} s(P_{v_y,v_1}) \\ t(P_{v_y,v_1}) \end{bmatrix} \\ &= \begin{bmatrix} 20n - 1 - 5(d_2 - d_{11}) \\ 20n^2 - 21n + 1 + 5n(d_{11} - d_2) + 5d_2 - 4\delta_{1,1} \end{bmatrix} \end{aligned} \tag{4.35}$$

has a nonnegative integer solution for the path P_{v_y,v_1}.

The solution of system (4.35) is $u_1 = 20n - 1 - 5\delta_2 - (5n-4)s(P_{v_y,v_1}) + 5t(P_{v_y,v_1})$ and $u_2 = \delta_{1,1} - (1 - n)s(P_{v_y,d_1}) - t(P_{v_y,v_1})$. If v_y is positioned on the $v_1 \to v_a$ path, then there is a $(5, 5n - 4 - \delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 20n - 1 - 5(\delta_2 + \delta(v_1, v_y)) \geq 24$ since $\delta_2 + \delta(v_1, v_y) \leq 2n + 1$ for $n \geq 3$ and $u_2 = \delta_{1,1} + \delta(v_1, v_y) - 1 \geq 0$ since $\delta_{1,1} + \delta(v_1, v_y) \geq n - 2$ for $n \geq 3$. If v_y is positioned on the $v_{a+1} \to v_b$ path, then there is a $(4, 5n - 3 - \delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 25n - 5(\delta_2 + \delta(v_1, v_y)) \geq 0$ since $\delta_2 + \delta(v_1, v_y) \leq 5n$ for $n \geq 3$ and $u_2 = \delta_{1,1} + \delta(v_1, v_y) - n - 1 \geq 0$ since $\delta_{1,1} + \delta(v_1, v_y) \geq n + 1$ for $n \geq 3$. If v_y is positioned on the $v_{b+1} \to v_c$ path, then there is a $(3, 5n-2-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 30n+1-5(\delta_2+\delta(v_1, v_y)) \geq 6$ since $\delta_2+\delta(v_1, v_y) \leq 6n-1$ for $n \geq 3$ and $u_2 = \delta_{1,1}+\delta(v_1, v_y)-2n-1 \geq 4$ since $\delta_{1,1}+\delta(v_1, v_y) \geq 4n-1$ for $n \geq 3$. If v_y is positioned on the $v_{c+1} \to v_d$ path, then there is a $(2, 5n-1-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 35n+2-5(\delta_2+\delta(v_1, v_y)) \geq 12$ since $\delta_2+\delta(v_1, v_y) \leq 6n+1$ for $n \geq 3$ and $u_2 = \delta_{1,1}+\delta(v_1, v_y)-3n-1 \geq 3$ since $\delta_{1,1}+\delta(v_1, v_y) \geq 4n+1$ for $n \geq 3$. If v_y is positioned on the $v_{d+1} \to v_e$ path, then there is a $(1, 5n-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 40n+3-5(\delta_2+\delta(v_1, v_y)) \geq 18$ since $\delta_2+\delta(v_1, v_y) \leq 7n$ for $n \geq 3$ and $u_2 = \delta_{1,1}+\delta(v_1, v_y)-4n-1 \geq 2$ since $\delta_{1,1}+\delta(v_1, v_y) \geq 5n$ for $n \geq 3$. If v_y is positioned on the $v_{e+1} \to v_{5n+1}$ path, then there is a $(0, 5n+1-\delta(v_1, v_y))$-path from v_y to v_1. Using this path, we determine that $u_1 = 45n+4-5(\delta_2+\delta(v_1, v_y)) \geq 29$ since $\delta_2+\delta(v_1, v_y) \leq 7n+1$ for $n \geq 3$ and $u_2 = \delta_{1,1}+\delta(v_1, v_y)-5n-1 \geq 1$ since $\delta_{1,1}+\delta(v_1, v_y) \geq 5n+2$ for $n \geq 3$.

Therefore, for every $y = 1, 2, ..., 5n+1$, the system (4.35) has a nonnegative integer solution. Proposition 3.1 ensures that for every $y = 1, 2, ..., 5n+1$, there is $v_y \stackrel{(m,h)}{\longrightarrow} v_1$ walk with $m = 20n-1-5(d_2-d_{11})$ and $h = 20n^2-21n+1+5n(d_{11}-d_2)+5d_2-4\delta_{1,1}$. Consequently, $\text{expin}(v_1, D^{(2)}) \leq 20n^2-n+5n(d_{11}-d_2)+\delta_{1,1}$. So, $\text{expin}(v_1, D^{(2)}) = 20n^2-n+5n(d_{11}-d_2)+\delta_{1,1}$. By Lemma 3.2, we conclude that expin $(v_x, D^{(2)}) \leq 20n^2-n+5n(d_{11}-d_2)+\delta_{1,1}+\delta(v_1, v_x)$ for every $x = 1, 2, ..., 5n+1$. □

5. Conclusion

Incoming local exponents in a two-cycle Hamiltonian bicolour digraph with a cycle difference of $4n+1$ can generally be obtained with the formula $\text{expin}(v_x, D^{(2)}) \leq \text{expin}(v_y, D^{(2)}) + \delta(v_y, v_x)$. Future research is expected to generalize incoming-local-exponent two-cycle Hamiltonian bicolour digraphs formulas for cycles with lengths n and $kn+1$.

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