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PARTIAL FRACTION DECOMPOSITIONS AND SOME TRIGONOMETRIC IDENTITIES

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Abstract. By means of partial fraction decomposition method, this paper investigates the problems on combinatorial computations of trigonometric identities with double free parameters, which yields a series of trigonometric sum formulae.

1. Introduction

The partial fraction decomposition of rational function is very useful in mathematics. For example, in order to integrate a rational function, it is crucial to obtain the partial fraction decomposition. Through partial fraction decompositions Chu and Marini [2] derived numerous important formulae for evaluating trigonometric sums. As continuation and extension of this approach, in this paper, we shall develop parametric decompositions of partial fractions, establish trigonometric sum identities with an extra free parameter y, which generalize naturally the corresponding results obtained by Chu and Marini.

2. Partial Fraction Decomposition

In this section, we shall establish two trigonometric identities involving double free parameters through partial fraction decompositions.

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Theorem 2.1. *If $P(\theta)$ is a Laurent polynomial of degree $\leq n$ in "$\cos\theta$", y is a real free parameter, then holds:*

$$\frac{4n\,\cos ny\,\sin n\theta P(\theta)}{\sin\theta\,(\cos 2ny-\cos 2n\theta)} \;=\; \sum_{k=0}^{2n-1}\frac{(-1)^{k+1}P(y+\frac{k\pi}{n})}{\cos\theta-\cos(y+\frac{k\pi}{n})}.$$

Proof. First, we suppose $0<y<\pi/n, 0<\theta<\pi$. The trigonometric function $\frac{\sin n\theta}{\sin\theta}P(\theta)$ may be considered as a polynomial of degree $\leq 2n-1$ in "$\cos\theta$", and $\cos 2ny-\cos 2n\theta$ as a polynomial of degree $2n$ in "$\cos\theta$", whose $2n$ distinct zeros are $\{\theta_k\}_{k=0}^{2n-1}$ with $\theta_k=y+\frac{k\pi}{n}$. We have an expansion in partial fractions

$$\frac{\sin n\theta P(\theta)}{\sin\theta\,(\cos 2ny-\cos 2n\theta)} \;=\; \sum_{k=0}^{2n-1}\frac{\lambda_k}{\cos\theta-\cos\theta_k}. \tag{2.1}$$

Multiplying across (2.1) by $\cos\theta-\cos\theta_k$ and setting $\theta\to\theta_k$, we can determined

$$\lambda_k=\lim_{\theta\to\theta_k}\frac{\sin n\theta(\cos\theta-\cos\theta_k)P(\theta)}{\sin\theta(\cos 2ny-\cos 2n\theta)}=\frac{\sin nyP(\theta_k)}{(-1)^k\sin\theta_k}\lim_{\theta\to\theta_k}\frac{\cos\theta-\cos\theta_k}{\cos 2ny-\cos 2n\theta}.$$

Therefore we derive through L'Hôpital's rule that

$$\lambda_k=\frac{(-1)^{k+1}P(y+\frac{k\pi}{n})}{4n\,\cos ny}. \tag{2.2}$$

Putting (2.1) and (2.2) together, we confirm the trigonometric sum identity displayed in Theorem 2.1.

Although we have supposed $0<y<\pi/n$, $0<\theta<\pi$ in previous proof, but we can check without difficulty that Theorem 2.1 is available for any possible y,θ. □

In terms of similarly process other trigonometric sum identities may be expressed:

Theorem 2.2. *If $P(\theta)$ is a Laurent polynomial of degree $\leq n$ in "$\cos\theta$", y is a real free parameter, then holds:*

$$\frac{4n\,\sin ny\,\sin n\theta P(\theta)}{\sin\theta\,(\cos 2ny+\cos 2n\theta)} \;=\; \sum_{k=0}^{2n-1}\frac{(-1)^{k+1}P(y+\frac{1+2k\pi}{2n})}{\cos\theta-\cos(y+\frac{1+2k\pi}{2n})}.$$

Theorem 2.3. *If $P(\theta)$ is a Laurent polynomial of degree $<n$ in "$\cos\theta$", y is a real free parameter, then holds:*

$$\frac{4n\,\cos ny\,\cos n\theta P(\theta)}{\sin\theta\,(\cos 2ny+\cos 2n\theta)} \;=\; \sum_{k=0}^{2n-1}\frac{(-1)^{k}\sin(y+\frac{1+2k\pi}{2n})P(y+\frac{1+2k\pi}{2n})}{\cos\theta-\cos(y+\frac{1+2k\pi}{2n})}.$$

The general results displayed in the previous three theorems imply numerous identities on trigonometric sums, which will be exhibited as followers:

Example 2.4. *Let* $P(\theta) = 1$ *in Theorem2.1, we derive*

$$\sum_{k=0}^{n-1} \frac{(-1)^k \cos(y + k\pi/n)}{\cos^2(y + k\pi/n) - \cos^2\theta} = \frac{2n \sin n\theta \cos ny}{\sin\theta(\cos 2ny - \cos 2n\theta)} \qquad n \text{ odd;}$$

$$\sum_{k=0}^{n-1} \frac{(-1)^k \cos\theta}{\cos^2(y + k\pi/n) - \cos^2\theta} = \frac{2n \sin n\theta \cos ny}{\sin\theta(\cos 2ny - \cos 2n\theta)} \qquad n \text{ even.}$$

Example 2.5. *Let* $P(\theta) = 2\cos n\theta$ *in Theorem2.1, we derive*

$$\sum_{k=0}^{n-1} \frac{\cos\alpha}{\cos^2(y + k\pi/n) - \cos^2\alpha} = \frac{n \sin 2n\alpha}{\sin\alpha(\cos 2ny - \cos 2n\alpha)}.$$

Example 2.6. *Let* $P(\theta) = 2\cos n\theta$ *in Theorem2.2, we derive*

$$\sum_{k=0}^{n-1} \frac{\cos\theta}{\cos^2\theta - \cos^2(y + \frac{1+2k}{2n}\pi)} = \frac{n \sin 2n\theta}{\sin\theta(\cos 2ny + \cos 2n\theta)} \qquad n \text{ even.}$$

Example 2.7. *Let* $P(\theta) = 1$ *in Theorem2.3, we derive*

$$\sum_{k=0}^{n-1} \frac{(-1)^k \sin(y + \frac{1+2k}{2n}\pi)\cos(y + \frac{1+2k}{2n}\pi)}{\cos^2\alpha - \cos^2(y + \frac{1+2k}{2n}\pi)} = \frac{2n\cos n\alpha\cos ny}{\cos 2ny + \cos 2n\alpha}.$$

When $y = 0$, the corresponding sums in Example2.4-2.7 to the moment have been studied by Chu and Marini [2].

The trigonometric identities in Example2.4-2.7 involve double free parameters y, θ, we consequently can establish a series of closed formulae of finite trigonometric sums (See [5]).

There are other interesting trigonometric sum identities, for example, those appeared in Berndt[1], Chu[3], and Gessel[4]. The reader is encouraged to try further.

References

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