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Reviewing Independent Set Polynomials and Independence Polynomials

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Abstract. In graph theory, independent set polynomials and independence polynomials are researched by a great of mathematicians.They are NP-hard problems(see [1] and [2]).In this paper, we will review the two problems from two aspects that they are definitions and results.

1. Reviewing main contents

In the article, two main problems on independent set polynomials and independence polynomials will be reviewed by the author .

Definition 1. Independent set polynomials $I(G;x)$ are defined as

$$I(G;x) = \sum_{k=1}^{n} b_k(G)x^k = \sum_{I\subset v(G)} \prod_{v\in I} x,$$

where let $b_k(G)$ be exactly k-independent sets of G .

Complexity: It is easy to see that $I(G;x)$ is NP-hard to compute.(see [1] Makowsky, J. A., Mariño, J. P.,2003)

Definition 2. If s_k denotes the number of stable sets of cardinality k in graph G, and $\alpha(G)$ is the size of a maximum stable set, then

$$I_\alpha(G;x) = \sum_{k=1}^{\alpha(G)} s_k x^k,$$

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is called the independence polynomial of G.(see[3] LiMin Yang,2006)

In [2], Vadim E.Levit and Eugen Mandrescu,Independence polynomials of well-covered graphs: Generic counterexamples for the unimodality conjecture ,European Journal of Combinatorics, 27(6)(2006)931-939,where independence polynomial is denoted by $I(G;x)$. In this paper[2], the authors show that for any integer $\alpha(G) \geq 8$, there exists a connected well-covered graph G with $\alpha = \alpha(G)$, whose independence polynomial is not unimodal. In addition, they present a number of sufficient conditions for a graph G with $\alpha \leq 6$ to have the unimodal independence polynomial. We think independence polynomial is denoted by$I_\alpha(G;x)$ that will distinguish the two signs in articles [1] and [2]. (also see [3])

Definition 3. *For $S^{(n)} = \{K_i : 1 \leq i \leq n\}$, $n \geq 1$, K_i is a complete graph with i vertices, if M is a subgraph of any graph G, and each component of M is all isomorphic to some element of $S^{(n)} = \{K_i : 1 \leq i \leq n\}$, then M is called one $S^{(n)}$-subgraph, if M is a spanning subgraph of G, then M is called one $S^{(n)}$-factor of G.*

In this paper[3](LiMin Yang),in the use of counting theory of $S^{(n)}$-factors with exactly k components, the author gains the representing formulas of independent set polynomials $I(G;x)$ and independence polynomials $I_\alpha(G;x)$,and presents the explicit formulas of independent set polynomials $I(G;x)$ and independence polynomials $I_\alpha(G;x)$ for a great deal of graphs.The main problems solved by the author are to compute independent set polynomials $I(G;x)$ and independence polynomials $I_\alpha(G;x)$ in [3], where independence polynomials are classes of graphs explicit formulas, here $\alpha(G)$ is the size of a maximum stable set. $\alpha(G)$ is NP-hard to compute. Because it is NP-hard that $\alpha(G)$ is the size of a maximum stable set (the size of maximum independent set),so far there does not exist the explicit formula, a number of mathematicians have studied to find new methods to solve the solutions (see[3]). In[4](J. Graph Theory, 15 (1991), pp. 99–107),Caro and Tuza have shown a certain lower bound $\alpha_k(H)$ on the size of a maximum independent set in a given k-uniform hypergraph H and have also presented an efficient sequential algorithm to find an independent set of size $\alpha_k(H)$. In [5](Hadas Shachnai and Aravind Srinivasan,SIAM Journal on Discrete Mathematics,Volume 18 Issue 3 ,Pages 488-500), the authors show that an RNC algorithm due to Beame and Luby finds an independent set of expected size $\alpha_k(H)$ and also derandomizes it for certain special cases. They also present lower bounds on independent set size for nonuniform hypergraphs using this algorithm. For graphs, they get an NC algorithm to find independent sets of size essentially that guaranteed by the general (degree-sequence based) version of Turan's theorem.

For any $i \in N$, let$c_i = c_i(G)$ denote the number of complete subgraphs of order (or dependence of cardinality) i in G. The dependence polynomial $P_G = P_G(z)$ of G is defined by

$P_G(z) := 1 + \sum_{i \in N} (-1)^i c_i(G) z^i$ (see [6])

Obviously, P_G is a polynomial of degree at most n. The dependence polynomial is also called the clique polynomial . In [6](European Journal of Combinatorics, Volume 28Issue 1 , January 2007, Pages 337-346) ,using Möbius inversion, this yields various identities involving dependence polynomials implying in particular that the dependence polynomial of the line graph L(G) of G is determined uniquely by the (multiset of) vertex degrees of G and the number of triangles in G. There exists difference between two definitions of articles [2]and [6],where s_k denotes the number of stable sets of cardinality k in graph G, and $\alpha(G)$ is the size of a maximum stable set in [2], $c_i = c_i(G)$ denotes the number of complete subgraphs of order i in G in [6]. But in this paper[3], $\alpha(G)$ is the size of a maximum stable set,in [7] $\alpha(G)$ is the number of all partitions of $V(G)$ into exactly k non-empty independent sets of any graph G,the two concepts is not the same. I think the latter is denoted by $\alpha_I(G)$,we will distinguish the two signs between the size of a maximum stable set and the number of all partitions of $V(G)$ into exactly k non-empty independent sets of any graph G(see[11] and [12]).

In this paper[7], enumeration of Independent Sets of Graphs is NP-hard, our ways are combinatorial counting methods. In the use of $S^{(n)}$-factors with exactly k components, the authors gain the representing formula of the number of all k independent sets of graphs and present the explicit formulas of enumeration of independent sets of graphs for a great deal of graphs.Finally, we solve the problem of NP-hard on enumeration of Independent Sets of Graphs.

2. Conclusion

In the reviewing paper,related to independent set polynomials ,independence polynomials, $\alpha(G)$ is the size of a maximum stable set and counting theory of $S^{(n)}$-factors,we have reviewed these aspects in recent years from Adv.in Appl.Math.,European Journal of Combinatorics,J. Graph Theory,SIAM Journal on Discrete Mathematics and International Journal of Combinatorial Graph Theory and Applications.

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