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ON BEST SIMULTANEOUS APPROXIMATION IN METRIC SPACES

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ABSTRACT: In this paper, the existence of invariant best simultaneous approximation in metric space is proved. In doing so, we have used a recent result of Xu regarding the fixed points.

1. INTRODUCTION

In the realm of best approximation theory, it is vaible, meaningful and potentially productive to know whether some useful properties of the function being approximated is inherited by the approximating function. In this perspective, Meinardus [8] observed the general principle that could be applied, while doing so the author has employed a fixed point theorem as a tool to establish it. The result of Meinardus was further generalized by Habiniak [5], Smoluk [15] and Subrahmanyam [16].

On the other hand, Beg and Sahazad [2], Fan [4], Hicks and Humphries [6], Reich [10], Singh [13],[14] and many others have used fixed point theorems in approximation theory, to prove existence of best approximation. Various types of applications of fixed point theorems may be seen in Klee [7], Meinardus [8] and Vlasov [18]. Some applications of the fixed point theorems to best simultaneous approximation is given by Sahney and Singh [11]. For the detail survey of the subject we refer the reader to Cheney [3].

In this paper, we prove the existence of invariant best simultaneous approximation in metric space, while doing so, we use the recent result of Xu [19] on the fixed points.

2. PRELIMINARIES AND DEFINITIONS

Let (X, d) be a metric space and let I denotes the unit interval $[0, 1]$. A continuous mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if the inequality

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

holds for all $x, y, u \in X$ and $\lambda \in I$. The metric space X together with a convex structure W is called a convex metric space.

Notice that Banach space and each of its convex subsets are simple examples of convex metric spaces. However, there are many convex metric spaces which can not be imbedded in any Banach space. For examples and other details we refer to Takahashi [7]. A subset K of a convex metric space X is said to be convex if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$. The set K is said to be starshaped if there exists $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in I$. Evidently starshaped subsets of X contain all convex subsets of X as a proper subclass. Let Φ denote the collection of all function φ from $R^+ := [0, \infty) \mapsto R^+$ satisfying the properties (i) φ is continuous (ii) $\varphi(t) < t$ for all $t > 0$. A continuous mapping T from a complete metric space (X, d) into itself is said to be weakly asymptotic contraction [19] if for an arbitrary $\epsilon > 0$, there is an integer $n_\epsilon \geq 1$ such that as $\epsilon \to 0$, $n_\epsilon \to \infty$ and $(T^{n_\epsilon}x, T^{n_\epsilon}y) \leq \varphi(d(x, y)) + \epsilon$. A convex metric space X is said to satisfy condition (I) at $p \in K$ (where K is starshaped and p is star centre) if for any $x, y \in X$, $\lambda \in I$

$$\text{(I)} \qquad d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda\varphi(d(x, y)) + \epsilon.$$

It may be observed that any normed space X always satisfies condition (I) (see, for instance, Beg and Shahzad [2]).

We now introduce the following definition.

Definition 1: Let (X, d) be a metric space, W a convex structure on X and K a starshaped subset (with respect to star centre $p \in K$) of X. A mapping $T : X \to X$ is said to satisfy property (W) in K if

(i) $W(T^{n+1} x, p, k) = W\left(Tx, p, k.\left(\dfrac{n}{n+1}\right)\right)$ for each $n \in \mathbb{N}$ and $\forall x \in$ K,

and

(ii) $W(x, p, 1) = x$.

To prove our main result (Theorem 3.1 below), we shall make use of the following result due to Xu ([19], Theorem 3).

Theorem A[19]: Suppose (X, d) be a complete metric space and suppose $T : X \to X$ is a weakly asymptotic contraction. Assume that T has a bounded orbit at some $x \in X$. Then T has a unique fixed point z.

Let (X, d) be a metric space and G a nonempty subset of X. Suppose $A \in B(X)$, the set of nonempty bounded subsets of X, then we write

$$r_G(A) = \inf_{g\in G}\sup_{a\in A} d(a, g)$$

$$cent_G(A) = \{g_0 \in G : \sup_{a\in A} d(a, g_0) = r_G(A)\}.$$

The number $r_G(A)$ is called the Chebyshev radius of A w.r.t. G and an element $y_0 \in cent_G(A)$ is called a best simultaneous approximation of A w.r.t. G. If $A = \{x\}$, then $r_G(A) = d(x, G)$ and $cent_G(A)$ is the set of all best approximations of x out of G. We also refer the reader to Milman [9] for further details.

3. MAIN RESULTS

Now we state and prove our main result.

Theorem 3.1: Suppose X is a complete convex metric space satisfying condition (I). Let G be a non empty subset and A be a bounded subset of X. Also let T be a weakly asymptotic self contraction of G. If the set $cent_G(A)$ of best simultaneous G-approximates to A is nonempty, compact, starshaped, T-invariant and T is continuous on $cent_G(A)$, then $cent_G(A)$ contains a T-invariant point.

Proof: Let p be the star-centre of $cent_G(A)$. Then $W(x, p, \lambda) \in cent_G(A)$ for each $x \in cent_G(A)$. Let for an arbitrary $\epsilon > 0$, there is an integer $n_\epsilon \geq 1$ such that as $\epsilon \to 0$, $n_\epsilon \to \infty$ 1 and $\left\{k_{n_\epsilon}\right\}_{n_\epsilon=1}^{\infty}$ be a real sequence with $0 \leq k_{n_\epsilon} < 1$ such that $\lim\limits_{n_\epsilon \to \infty} k_{n_\epsilon} = 1$. Define $T_{n_\epsilon} : cent_G(A) \to cent_G(A)$ by

$$T_{n_\epsilon} x = W(T^{n_\epsilon} x, p, k_{n_\epsilon})$$

for all $x \in cent_G(A)$ and $n_\epsilon \geq 1$. Since p is star-center of $cent_G(A)$ and $T(cent_G(A)) \subset cent_G(A)$ it follows that T_{n_ϵ} maps $cent_G(A)$ to itself for each n_ϵ. Now applying condition (I), we obtain

$$\begin{aligned} d(T_{n_\epsilon} x, T_{n_\epsilon} y) &= d(W(T^{n_\epsilon} x, p, k_{n_\epsilon}), W(T^{n_\epsilon} y, p, k_{n_\epsilon})) \\ &\leq k_{n_\epsilon} d(T^{n_\epsilon} x, T^{n_\epsilon} y) \\ &\leq k_{n_\epsilon}(\varphi(d(x, y)) + \epsilon) \\ &\leq \lim_{n_\epsilon \to \infty} k_{n_\epsilon} \left(\varphi\left(d\left(x, y\right)\right) + \epsilon\right) \\ &= \varphi(d(x, y)) + \epsilon \end{aligned}$$

for all $\varphi \in \Phi$ and, thereby, implying that T_{n_ϵ} is a weakly asymptotic contraction for each $n_\epsilon \geq 1$. It follows, by Theorem A, that each T_{n_ϵ} has a fixed point, say z_{n_ϵ}. Since $cent_G(A)$ is compact, $\{z_{n_\epsilon}\}$ has a convergent subsequence $\left\{z_{(n_\epsilon)_i}\right\}$ such that $z(n_\epsilon)_i \to z$ (say) as $i \to \infty$. Since T satisfies condition (W) in $cent_G(A)$, we have

$$z = \lim_{i\to\infty} z(n_\epsilon)_i = \lim_{i\to\infty} T_{(n_\epsilon)_i} z(n_\epsilon)_i = \lim_{i\to\infty} W(T^{(n_\epsilon)_i} z(n_\epsilon)_i, p, k(n_\epsilon)_i)$$

$$= \lim_{i\to\infty} W\left(Tz(n_\epsilon)_i, p, k(n_\epsilon)_i \cdot \left(\frac{(n_\epsilon)_i - 1}{(n_\epsilon)_i} \right) \right) = W(Tz, p, 1) = Tz.$$

Hence $cent_G(A)$ contains a T-invariant point. This completes the proof.

Remark 3.2: Unification of the concept of weakly asymptotic contraction with property (W) in convex metric structure improves several known results in invariant best simultaneous approximation.

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