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Congestion in stochastic data envelopment analysis: An input relaxation approach*

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Abstract

The input relaxation model and its stochastic version, recently introduced in data envelopment analysis (DEA) literature, uses more flexibility in changes of the used input combination to find the maximum possible output and can be useful to resource management. We study congestion issues in this setting. Deterministic equivalent to the stochastic congestion model is obtained. The deterministic equivalent is typically non-linear. It is, however, shown that under fairly general conditions this non-linear model can be replaced by an ordinary deterministic DEA model. When allowable limits of data variations for evaluating decision making unit is permitted, sensitivity analysis is studied.

1 Introduction. Data envelopment analysis originated by Charnes, Cooper and Rhodes [3], provides a simple, yet powerful approach in measuring efficiency of decision making unites (DMU) with multiple inputs and outputs. Since 1978 there has been a surge of research on DEA and many further models were introduced in the literature. Banker, Charnes and Cooper [2] developed a variable return to scale version of the CCR model in

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[3], so-called BCC model. Later Charnes and Zlobec [4] and Charnes et al. [5] studied stability of efficiency tests used in DEA. A thorough discussion on new development in DEA up to 1996 can be found in Cooper, Thompson and Thrall [11]. More recently stochastic formulation of the original models were introduced to incorporate possible uncertainty in the inputs and/or outputs (e.g. [12], [13]). Morita and Seiford [23] studied robustness of the efficiency results when input and output data are subject to stochastic measurement error, while Jess, Jongen, Neralic and Stein [17] introduced a semi-infinite programming model in DEA to study an interesting chemical engineering problem. The original models, CCR and BCC, in DEA only allow changes in the input combination of decision making units that are limited to the observed inputs of evaluating decision making units. Cooper et al. [7], with a proper initiative on the data of textile industry of China for improving congestion management, increased labor input and reduced capital input and showed the new combination could have constructive results. Their method and results indicate that suitable changes to determining an input combination, which incorporates potentials and consider constraints of a society, is sometimes necessary toward increasing the output. It therefore seems plausible that using more flexibility in the input combination, when it is possible, might result in better outputs.

The idea initiated by Cooper et al. [7] motivated further work and new models in DEA, e.g., Jahanshahloo and Khodabakhshi [15], Jahanshahloo and Khodabakhshi [16] and Khodabakhshi [18], Khodabakhshi [19]. In may practical situation, one desires a model that is flexible enough to account for possible uncertainties in the outputs and/or inputs which may lead to more robust results. Input congestion obtained by the classic models such as Cooper et al. [9] or those studied by Jahanshahloo and Khodabakhshi [15] only consider deterministic DMU's, i.e. DMU's with deterministic (precise) data. Such assumption may not, however, be tenable because data in many real applications can not be precisely measured. A successful method to address uncertainty in data is replacing deterministic data by random variables, leading to stochastic DEA.

An important issue that pertains to any output oriented DEA model is congestion which essentially studies redundancy in resource allocations. The main thrust of this manuscript is congestion in stochastic DEA while we allow input relaxation. The model we study here has three components: it accommodates input relaxation, it takes into account possible uncertainties in the data, and it conveniently allows for studying congestion. Input relaxation can essentially be considered as allowing a shift in the average of the input. To study robustness with respect to the changes in the other parameter of the Normal distribution, i.e. variance-covariance matrix, we study sensitivity analysis with respect to changes in the entries of that matrix. We obtain deterministic equivalents to our stochastic models. We show that the deterministic equivalents can be transformed to quadratic programming models which can be readily solved.

The layout of the paper is as follows: The proposed model for improving outputs, input relaxation model, is described in Section 2. In Section 3, we introduce the congestion models that we use to identify congestion in deterministic forms. Stochastic form of the models are developed in Section 4. The deterministic equivalents of these stochastic models are derived in Section 5. Section 6 discusses sensitivity analysis. We used a numerical example to illustrate the proposed approach. Section 8 contains some concluding remarks.

2 Proposed underlying model Suppose that all input and output elements are non negative deterministic variables. Let DMU_j, $(j = 1, 2, \cdots, n)$ be n decision making units (DMU) that convert m inputs x_{ij} (i=1,...,m) into s outputs y_{rj} (r=1,...,s). The model for improving output, input relaxation model, recently introduced in Jahanshahloo and Khodabakhshi (2004a), is

$$\begin{aligned} \text{Maximize} \quad & \phi_o + \varepsilon\Big(\sum_{i=1}^{m} s_{i1}^- + \sum_{r=1}^{s} s_r^+ - \sum_{i=1}^{m} s_{i2}^+\Big) \\ \text{subject to} \quad & x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_{i1}^- - s_{i2}^+, i = 1, \dots, m \\ & 0 = \sum_{j=1}^{n} \lambda_j y_{rj} - \phi_o y_{ro} - s_r^+, r = 1, \dots, s \qquad (1) \\ & 1 = \sum_{j=1}^{n} \lambda_j \\ & s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+ \geq 0 \quad , \end{aligned}$$

where the first and second slacks in the input constraints are slacks for decrement and increment of the i-th input. ε is a non-Archimedean element in order to optimize slacks and ensure that all of the slacks are considered in solution. The model allows evaluating DMU to overuse the available sources. It often happens in real application that some DMU's can produce far more outputs, should we allow some input relaxation by loosening some of the existing constraints on inputs. While loosening one or more constraints may not always be possible, when it is, model (1) can result in DMU's with considerable increase in the output that is mostly due to some slight changes in one or more inputs (e.g. Jahanshahloo and Khodabakhshi [15], Khodabakhshi [18], Khodabakhshi [19] and Khodabakhshi [20].

The columns correspond to s_{i1}^{-} and s_{i2}^{+} are linearly dependent, so that at the optimal solution at most one of these variables is positive. It is obvious that s_{i1}^{-} and s_{i2}^{+} are, respectively, maximized and minimized at the optimal solution. Let $*$ shows an optimal value. The conditions of efficiency for evaluating DMU_o can therefore be stated as follows:

Definition 1. (*Efficiency according to model (1))*: DMU_o is efficient if the following two conditions are satisfied :

i) $\phi_o^* = 1$

ii) $s_{i1}^{-*} = s_{i2}^{+*} = s_r^{+*} = 0 \ \forall i \ \& \ \forall r$.

Let E be the set of efficient DMU's, F be the set of points on the frontier which are not efficient because they satisfy (i) but not (ii) in the Definition 1 and N be the set of all points which are not on a frontier and hence are inefficient.

3 Congestion Input congestion is said to exist if increasing some inputs can reduce the output. For instance, an excess of miners bumping into each other in an underground mine is an example, where further increase in the number of miners can result in reduction in the amount mined. In what follows, we provide the exact definition of congestion in general case.

Definition 2: (*Input congestion*) Input congestion occurs whenever increasing one or more inputs decreases some outputs without improving other inputs or outputs. Conversely, congestion occurs when decreasing some inputs increase some outputs without worsening other inputs or outputs.

Jahanshahloo and Khodabakhshi [15] determined input congestion based on the proposed model (1) with a two model approach. However, one can replace the two model approach introduced in Jahanshahloo and Khodabakhshi [15] for determining input congestion by a single model, a one model approach, as follows.

$$\begin{aligned}
\text{Maximize} \quad & \phi_o + \varepsilon(\sum_{i=1}^{m} -s_i^{-c} + \sum_{r=1}^{s} s_r^{+} - \sum_{i=1}^{m} s_{i2}^{+}) \\
\text{subject to} \quad & x_{io} = \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^{-c} - s_{i2}^{+}, i = 1, \dots, m \\
& 0 = \sum_{j=1}^{n} \lambda_j y_{rj} - \phi_o y_{ro} - s_r^{+}, r = 1, \dots, s \qquad (2) \\
& 1 = \sum_{j=1}^{n} \lambda_j \\
& s_i^{-c}, s_{i2}^{+}, \lambda_j, s_r^{+} \geq 0
\end{aligned}$$

It is noticeable that the way in which we could normally solve problem (1) would be via a two-stage approach. That is, at the first stage we maximize ϕ_o, and at the second stage we maximize the sum of the slacks. Therefore, we have reduced solving three problems, with two-model approach, to two problems with one-model approach. This is certainly important from a computational point of view, see Khodabakhshi [20] for a detailed discussion on this approach. See, also, Cooper et al. [9] which provided an alternative one model approach for determining input congestion based on output oriented BCC model introduced in Banker, Charnes and Cooper [2]. Note that the differences between the above model and the model introduced by Cooper et al. [2] is related to the underlying models. Cooper et al. [2] used output oriented BCC model introduced in Banker, Charnes and Cooper [2], while the above model used model (1) introduced in section 2. Both underlying models are output oriented. One may refer to Jahanshahloo and Khodabakhshi [15] which discusses differences and similarities of the two models. See, also, Jahanshahloo and Khodabakhshi [16], Khodabakhshi [18] . The variable s_i^{-c} in model (2) represents the congesting amount of i-th input, see Khodabakhshi [19] and Appendix.

4 Congestion in stochastic DEA Although DEA methodology has many advantages, such as no requirement for a priori weights or explicit specification of functional relations among the multiple inputs and outputs, there is a weakness in conventional DEA models. In fact, DEA doesn't allow stochastic variations in input and output such as data entry errors. As a result, DEA efficiency measurement may be sensitive to such variations.

A DMU which is measured as efficient relative to other DMUs, may turn inefficient if such random variations are considered. Stochastic input and output variations into DEA has been recently studied by, for example, Huang and Li [14] and Cooper et al. [10], Khodabakhshi and Asgharian [?], Khodabakhshi [20] and Khodabakhshi [21] in the literature. In what follows, we introduce stochastic version of the model (1) and congestion model (2) which allows stochastic variations in input-output data.

Following Cooper et al. [10], let $\tilde{x}_j = (\tilde{x}_{1j}, \ldots, \tilde{x}_{mj})^t$, $\tilde{y}_j = (\tilde{y}_{1j}, \ldots, \tilde{y}_{sj})^t$ be random input and output related to DMU_j $(j = 1, \ldots, n)$. Let also $x_j = (x_{1j}, \ldots, x_{mj})^t$, $y_j = (y_{1j}, \ldots, y_{sj})^t$ show the corresponding vectors of expected values of inputs and outputs for DMU_j which are used as the observed values in model (1). Suppose that all input and output components are jointly Normally distributed, therefore, it can be shown that the corresponding stochastic version of the model (1) is:

$$\begin{aligned}
\text{Maximize} \quad & \phi_o + \varepsilon(\sum_{i=1}^{m} s_{i1}^- + \sum_{r=1}^{s} s_r^+ - \sum_{i=1}^{m} s_{i2}^+) \\
\text{subject to} \quad & P\{\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} + s_{i1}^- \leq \tilde{x}_{io} + s_{i2}^+\} = 1 - \alpha, i = 1, \ldots, m \\
& P\{\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \phi_o \tilde{y}_{ro} \geq s_r^+\} = 1 - \alpha, r = 1, \ldots, s \qquad (3) \\
& 1 = \sum_{j=1}^{n} \lambda_j \\
& s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+ \geq 0 \quad .
\end{aligned}$$

In model (3), α is a predetermined value between 0 and 1 which specifies the significance level. Since a solution with $\phi_o = 1$, $\lambda_o = 1$, $\lambda_j = 0, (j \neq o)$ always exists, the optimal value of objective function is greater than or equal to 1. Stochastic efficiency by the model (3) can therefore be defined as below.

Definition 3: DMU_o is called stochastically efficient at significance level α if the following conditions are fulfilled.

i) $\phi_o^* = 1$

ii) $s_{i1}^{-*} = s_{i2}^{+*} = s_r^{+*} = 0$ $\forall i$ & $\forall r$.

DMU_o is called stochastically inefficient if one of the above conditions in Definition 3 fails to hold. In other words, if for an optimal solution $\phi_o^* > 1$, or some of slacks are non zero, then DMU_o is stochastically inefficient. In fact, if $\phi_o^* > 1$, then all outputs for evaluating DMU_o can be increased to $\phi_o^* y_{ro}$, (r=1,...,s) by using a convex combination of the other DMUs at the significance level α.

It is worth emphasizing that optimal values of slack variables may not be attainable in an ε-free model. A non-Archimedian ε is therefore needed in model (3) to avoid such undesirable feature. The stochastic form of the model (2) also can be obtained as follows:

$$\begin{aligned}
\text{Maximize} \quad & \phi_o + \varepsilon\Big(\sum_{i=1}^{m} -s_i^{-c} + \sum_{r=1}^{s} s_r^{+} - \sum_{i=1}^{m} s_{i2}^{+}\Big) \\
\text{subject to} \quad & P\Big\{\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} + s_i^{-c} \leq \tilde{x}_{io} + s_{i2}^{+}\Big\} = 1-\alpha, i = 1,\dots,m \\
& P\Big\{\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} - \phi_o \tilde{y}_{ro} \geq s_r^{+}\Big\} = 1-\alpha, r = 1,\dots,s \qquad (4) \\
& 1 = \sum_{j=1}^{n} \lambda_j \\
& s_i^{-c}, s_{i2}^{+}, \lambda_j, s_r^{+} \geq 0 \quad .
\end{aligned}$$

Definition 4: *Congestion is present for DMU_o to the predetermined level of probability in model (4) if and only if for any optimal solution $(\phi_o^*, \lambda^*, S^{+*}, S^{-c*})$, there exists at least one $s_i^{-c*} > 0$ $(1 \leq i \leq m)$.*

The next theorem follows immediately.

Theorem 1: *For any optimal solution of model (4), we have the following:*

i) *If $\phi_o^* > 1$, then DMU_o is stochastically inefficient.*

ii) *If there exists at least one r $(1 \leq r \leq s)$ for which $s_r^{+*} > 0$ then DMU_o is stochastically inefficient.*

iii) *If there exists at least one $s_i^{-c*} > 0$ $(1 \leq i \leq m)$, then DMU_o is stochastically inefficient.*

iv) *If $\phi_o^* = 1$, $S^{+*} = 0$, and $S^{-c*} = 0$, then DMU_o is on a segment of the stochastic frontier.*

5 Deterministic equivalents In this section we exploit the normality assumption to introduce a deterministic equivalent to model (3). We need first recall a well-known fact about normally distributed random vectors that is used below. Suppose that $\vec{X}_k \sim N(\vec{\mu}_{k\times 1}, \Sigma_{k\times k})$, where $\mu_{k\times 1}$ and $\Sigma_{k\times k}$ are, respectively, the mean value vector and the variance-covariance matrix. Then for any matrix $A_{m\times k} \neq 0$ we have $A\vec{X} \sim N(A\vec{\mu}, A\Sigma_{k\times k}A^T)$, where A^T is the transpose of A. Using this result, one can obtain the following deterministic equivalent to model (3).

$$\begin{aligned}
\text{Maximize} \quad & \phi_o + \varepsilon(\sum_{i=1}^{m} s_{i1}^{-} + \sum_{r=1}^{s} s_r^{+} - \sum_{i=1}^{m} s_{i2}^{+}) \\
\text{subject to} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_{i1}^{-} - s_{i2}^{+} - \Phi^{-1}(\alpha)\sigma_i^I(\lambda) = x_{io}, i = 1, \ldots, m \\
& \phi_o y_{ro} - \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^{+} - \Phi^{-1}(\alpha)\sigma_r^o(\phi_o, \lambda) = 0, r = 1, \ldots, s \qquad (5) \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& s_{i1}^{-}, s_{i2}^{+}, \lambda_j, s_r^{+} \geq 0 \quad ,
\end{aligned}$$

where Φ is the cumulative distribution function (cdf) of a standard Normal random variable and Φ^{-1} is its inverse. We assume that x_{ij} and y_{rj} also are the means of the inputs and output variables which are known. Similarly, the deterministic equivalent of model (4) can be represented by

$$\begin{aligned}
\text{Maximize} \quad & \phi_o + \varepsilon(\sum_{i=1}^{m} -s_i^{-c} + \sum_{r=1}^{s} s_r^{+} - \sum_{i=1}^{m} s_{i2}^{+}) \\
\text{subject to} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^{-c} - \Phi^{-1}(\alpha)\sigma_i^I(\lambda) = x_{io}, i = 1, \ldots, m \\
& \phi_o y_{ro} - \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^{+} - \Phi^{-1}(\alpha)\sigma_r^o(\phi_o, \lambda) = 0, r = 1, \ldots, s \qquad (6) \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& s_i^{-c}, \lambda_j, s_r^{+} \geq 0 \quad ,
\end{aligned}$$

It is obvious, from the forms of $\sigma_i^I(\lambda)$ and $\sigma_r^o(\phi_o, \lambda)$, that model (5) is a non-linear program. Following Cooper et al. [10], we show that this non-linear program can be transformed to a quadratic program. Suppose that w_i^I and w_r^o are nonnegative variables. Replacing w_i^I and w_r^o, respectively, by $\sigma_i^I(\lambda)$ and $\sigma_r^o(\phi_o, \lambda)$ and adding the following quadratic equality constraints

$$(w_i^I)^2 = (\sigma_i^I(\lambda))^2 \qquad\qquad (w_r^o)^2 = (\sigma_r^o(\phi_o, \lambda))^2,$$

model (5) is transformed to a quadratic programming problem. One can therefore obtain the optimal values ϕ_o^* s_{i1}^{-*}, s_{i2}^{+*} and s_r^{+*} by solving the quadratic program. If $\alpha = 0.5$, then $\Phi^{-1}(\alpha) = 0$ and optimal values ϕ_o^*, s_{i1}^{-*}, s_{i2}^{+*} and s_r^{+*} in the stochastic form can be obtained by solving the model (1) in which the mean values of inputs and outputs are used.

One of the following three cases must naturally occur for the i-th input of evaluating DMU_o,:

i) increase, which corresponds to $s_{i2}^{+*} > 0$

ii) decrease, which corresponds to $s_{i1}^{-*} > 0$

iii) no change, which corresponds to $s_{i1}^{-*} = s_{i2}^{+*}$

To take the above point into our model, we need to impose the following constraint on s_{i1}^- and s_{i2}^+

$$s_{i1}^- \cdot s_{i2}^+ = 0 \quad .$$

It is worth noting that in model (1) at most one of the three aforementioned cases could occur. For, we use the simplex method and the corresponding columns of s_{i1}^{-*}, s_{i2}^{+*} in model (1) are linearly dependent. We therefore have the following deterministic equivalent to model (3),

$$\begin{aligned}
\text{Maximize} \quad & \phi_o + \varepsilon(\sum_{i=1}^{m} s_{i1}^- + \sum_{r=1}^{s} s_r^+ - \sum_{i=1}^{m} s_{i2}^+) \\
\text{subject to} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_{i1}^- - s_{i2}^+ - \Phi^{-1}(\alpha) w_i^I = x_{io}, i = 1, \ldots, m \\
& \phi_o y_{ro} - \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^+ - \Phi^{-1}(\alpha) w_r^o = 0, r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j = 1 \qquad\qquad (7) \\
(w_i^I)^2 = & \sum_{j \neq o} \sum_{k \neq o} \lambda_j \lambda_k Cov(\tilde{x}_{ij}, \tilde{x}_{ik}) + 2(\lambda_o - 1) \sum_{j \neq o} \lambda_j Cov(\tilde{x}_{ij}, \tilde{x}_{io}) + (\lambda_o - 1)^2 Var(\tilde{x}_{io}) \\
(w_r^o)^2 = & \sum_{k \neq o} \sum_{j \neq o} \lambda_k \lambda_j Cov(\tilde{y}_{rk}, \tilde{y}_{rj}) + 2(\lambda_o - \phi_o) \sum_{k \neq o} \lambda_k Cov(\tilde{y}_{rk}, \tilde{y}_{ro}) + (\lambda_o - \phi_o)^2 Var(\tilde{y}_{ro}) \\
& s_{i1}^- \cdot s_{i2}^+ = 0, i = 1, \ldots, m \\
& s_{i1}^-, s_{i2}^+, \lambda_j, s_r^+, w_i^I, w_r^o \geq 0;
\end{aligned}$$

and similarly to model (4),

$$\begin{aligned}
\text{Maximize} \quad & \phi_o + \varepsilon(\sum_{i=1}^{m} -s_i^{-c} + \sum_{r=1}^{s} s_r^+ - \sum_{i=1}^{m} s_{i2}^+) \\
\text{subject to} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_{io}^{-c} - s_{i2}^+ - \Phi^{-1}(\alpha) w_i^I = x_{io}, i = 1, \ldots, m \\
& \phi_o y_{ro} - \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^+ - \Phi^{-1}(\alpha) w_r^o = 0, r = 1, \ldots, s \\
& \sum_{j=1}^{n} \lambda_j = 1 \qquad\qquad (8) \\
(w_i^I)^2 = & \sum_{j \neq o} \sum_{k \neq o} \lambda_j \lambda_k Cov(\tilde{x}_{ij}, \tilde{x}_{ik}) + 2(\lambda_o - 1) \sum_{j \neq o} \lambda_j Cov(\tilde{x}_{ij}, \tilde{x}_{io}) + (\lambda_o - 1)^2 Var(\tilde{x}_{io}) \\
(w_r^o)^2 = & \sum_{k \neq o} \sum_{j \neq o} \lambda_k \lambda_j Cov(\tilde{y}_{rk}, \tilde{y}_{rj}) + 2(\lambda_o - \phi_o) \sum_{k \neq o} \lambda_k Cov(\tilde{y}_{rk}, \tilde{y}_{ro}) + (\lambda_o - \phi_o)^2 Var(\tilde{y}_{ro}) \\
& s_{io}^{-c} \cdot s_{i2}^+ = 0, i = 1, \ldots, m \\
& s_{io}^{-c}, \lambda_j, s_r^+, s_{i2}^+, w_i^I, w_r^o \geq 0.
\end{aligned}$$

To simplify our presentation, we assume in the sequel that $Cov(\tilde{x}_{ij}, \tilde{x}_{ik}) = Cov(\tilde{y}_{rj}, \tilde{y}_{rk}) = 0$ for $j \neq k$, $i = 1, \cdots, m$ and $r = 1, \cdots, s$. This simplifies model (7) and model (8) to a considerable extent. Such assumption is natural in many applications, while unrealistic in many others. See §7 for further comments.

6 Sensitivity analysis In this section we discuss sensitivity of the stochastic model and their deterministic equivalents with respect to changes in the parameter values. To simplify our presentation we confine ourselves to changes in the variance parameters and following Cooper et al. (2004), we permit allowable limits of data variations for only one DMU at a time. There are other approaches, to sensitivity analysis in DEA; for example, Charnes et al. [6] and Seiford and Zhu [24] that allow all data for all DMUs to be varied simultaneously until at least one DMU changes its status from efficient to inefficient, or vice versa. Without loss of generality, we assume that only variances of the input and outputs of DMU$_o$ are subject to changes. Let σ^I_{io}, and σ^O_{ro}, be respectively replaced by $\tilde{\sigma}^I_{io} = \sigma^I_{io} + \tau^I_{io}$ and $\tilde{\sigma}^O_{ro} = \sigma^O_{ro} + \tau^O_{ro}$, while the other parameters remain unchanged. Having incorporated these changes in model (5) and model (6) we respectively obtain

$$
\begin{aligned}
\text{Maximize} \quad & \phi_o + \varepsilon\Big(\sum_{i=1}^{m} s^-_{i1} + \sum_{r=1}^{s} s^+_r - \sum_{i=1}^{m} s^+_{i2}\Big) \\
\text{subject to} \quad & x'_{io} = \sum_{j=1}^{n} \lambda_j x'_{ij} + s^-_{i1} - s^+_{i2}, i = 1, \ldots, m \\
& 0 = \sum_{j=1}^{n} \lambda_j y'_{rj} - \phi_o y'_{ro} - s^+_r, r = 1, \ldots, s \\
& 1 = \sum_{j=1}^{n} \lambda_j \\
& s^-_{i1}, s^+_{i2}, \lambda_j, s^+_r \geq 0 \quad ,
\end{aligned}
\tag{9}
$$

and

$$\begin{aligned}
\text{Maximize} \quad & \phi_o + \varepsilon(-\sum_{i=1}^{m} s_i^{-c} + \sum_{r=1}^{s} s_r^{+} - \sum_{i=1}^{m} s_{i2}^{+}) \\
\text{subject to} \quad & x'_{io} = \sum_{j=1}^{n} \lambda_j x'_{ij} + s_i^{-c} - s_{i2}^{+}, i = 1, \ldots, m \\
& 0 = \sum_{j=1}^{n} \lambda_j y'_{rj} - \phi_o y'_{ro} - s_r^{+}, r = 1, \ldots, s \\
& 1 = \sum_{j=1}^{n} \lambda_j \\
& s_i^{-c}, \lambda_j, s_r^{+} \geq 0 \quad ,
\end{aligned} \tag{10}$$

where

$$y'_{ro} = y_{ro} - \tilde{\sigma}_{ro}^{O} \Phi^{-1}(\alpha),\ r = 1, ..., s \qquad y'_{rj} = y_{rj}, j \neq o,\ r = 1, ..., s \tag{11}$$

$$x'_{io} = x_{io} + \tilde{\sigma}_{io}^{I} \Phi^{-1}(\alpha),\ i = 1, ..., m \qquad x'_{ij} = x_{ij}, j \neq o, \imath = 1, ..., m \tag{12}$$

Models (9) and (10) are the deterministic equivalents of stochastic models (3) and (4) under the above assumptions.

We divide the following theorems into three categories, $\alpha = 0.5$, $0 < \alpha < 0.5$ and $0.5 < \alpha < 1$. Note that the closer α to 0, the closer we are to an unconstraint optimization problem. Likewise, the closer α to 1, the closer we are to a constraint optimization. It is interesting to note that the most uncertain case, i.e. $\alpha = 0.5$, corresponds to deterministic model (1) as shown in the next theorem.

Theorem 2: *Let $\alpha = 0.5$. Then congestion is present for DMU_o in input-output mean model (1) if and only if congestion is present for DMU_o in stochastic model (3).*

Theorem 3: *For $0 < \alpha < 0.5$, assume that $(\bar{\phi}_o, \bar{\lambda}, \bar{S_1}^{-}, \bar{S_2}^{+}, \bar{S}^{+})$ is a feasible solution of (9), and is therefore, also a feasible solution of (3), consider the transformations defined by*

$$\phi_o = \bar{\phi_o}, \quad \lambda = \bar{\lambda}, \quad s_r^{+} = \bar{s}_r^{+} - (\bar{\phi_o} - \lambda_o)\tilde{\sigma}_{io}^{I} \Phi^{-1}(\alpha),\ r = 1, ..., s$$

$$s_{i1}^{+} = \bar{s}_{i1}^{-} - (1 - \bar{\lambda_o})\tilde{\sigma}_{io}^{I} \Phi^{-1}(\alpha), \quad s_{i2}^{+} = \bar{s_{i2}}^{+}, i = 1, ..., m$$

then $(\phi_o, \lambda, S_1^{-}, S_2^{+}, S^{+})$ is a feasible solution of model (1).

Theorem 4: *For $0 < \alpha < 0.5$,*

i) *Suppose that congestion is not present for DMU_o in input-output model (1). Then congestion is also not present for DMU_o in stochastic model (3).*

ii) *Suppose that congestion is present for DMU_o and $DMU_o \in N$ in input-output model (1). Then congestion is also present for DMU_o and $DMU_o \in N$ in stochastic model (3). Moreover, $s_i^{-c'*} = s_i^{-c*} + \tilde{\sigma}_{io}^I \Phi^{-1}(\alpha)$ is the congesting amount of the i-th input of DMU_o in stochastic model (3), if $-1 < \tau_{io}^I/\sigma_{io}^I < s_i^{-c*}/(-\Phi^{-1}(\alpha)) - 1$ and $-1 < \tau_{ro}^O/\sigma_{ro}^O < \beta_r^{+*}/(-\Phi^{-1}(\alpha)) - 1$, where s_i^{-c*} is the congesting amount of the i-th input of DMU_o in input-output mean model (1), and $\sum_{r=1}^{s} \beta_r^{+*}$ is the optimal value of*

$$\begin{aligned}
\text{Maximize} \quad & \sum_{r=1}^{s} \beta_r^+ \\
\text{subject to} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} - \beta_{i2}^+ \leq x_{io}, i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - \beta_r^+ \geq y_{ro}, r = 1, \ldots, s \\
& 1 = \sum_{j=1}^{n} \lambda_j, \\
& \beta_{i2}^+, \lambda_j, \beta_r^+ \geq 0 \quad ,
\end{aligned} \tag{13}$$

Proof:

i) Consider model (9), the deterministic equivalent of model (3). Note that $\Phi^{-1}(\alpha) < 0$ since $0 < \alpha < 0.5$, and hence $y'_{ro} \geq y_{ro}$ and $x'_{io} \leq x_{io}$. This then implies that congestion is not present for DMU_0 in model (3) when DMU_o improves from (x_o, y_o) to (x'_o, y'_o).

ii) Suppose that $(\beta_2^{+*}, \beta^{+*}, \lambda^*)$ is an optimal solution of model (13). Then as a necessary condition of being maximal we must have

$$\sum_{j=1}^{n} \lambda_j^* y_{rj} = y_{ro} + \beta_r^{+*}, \ r = 1, \ldots, s,$$

we also have

$$\sum_{j=1}^{n} \lambda_j^* x_{ij} \leq x_{io} + \beta_{i2}^{+*},\ i = 1, ..., m.$$

Since $\tilde{\sigma}_{io}^I < s_i^{-c*}/(-\Phi^{-1}(\alpha))$ and $\tilde{\sigma}_{ro}^O < \beta_r^{+*}/(-\Phi^{-1}(\alpha))$, we therefore obtain

$$\begin{aligned} y'_{ro} = y_{ro} - \tilde{\sigma}_{ro}^o \Phi^{-1}(\alpha) < y_{ro} + \beta_r^{+*} &= \sum_{j=1}^{n} \lambda_j^* y_{rj} \\ &\leq \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j^* y_{rj} + \lambda_o^*(y_{ro} - \tilde{\sigma}_{ro}^o \Phi^{-1}(\alpha)) \\ &= \sum_{j=1}^{n} y'_{rj} \lambda_j^*,\ r = 1, .., s \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{n} \lambda_j^* x'_{ij} = \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j^* x_{ij} + \lambda_o^*(x_{io} + \tilde{\sigma}_{io}^I \Phi^{-1}(\alpha)) &\leq \sum_{j=1}^{n} \lambda_j^* x_{ij} \\ \leq x_{io} + \beta_{i2}^{+*} = x'_{io} - \tilde{\sigma}_{io}^I \Phi^{-1}(\alpha) &+ \beta_{i2}^{+*} \\ &= x'_{io} + s_{i2}^+,\ i = 1, .., m \end{aligned}$$

where $s_{i2}^+ = -\tilde{\sigma}_{io}^I \Phi^{-1}(\alpha) + \beta_{i2}^{+*} \geq 0$.

This means that DMU$_o$ with input-output combination of $(x'_o,\ y'_o)$ in which $x'_{io} = x_{io} + \tilde{\sigma}_{io}^I \Phi^{-1}(\alpha)$ and $y'_{ro} = y_{ro} - \tilde{\sigma}_{ro}^o \Phi^{-1}(\alpha)$ is inefficient in model (9). Hence, DMU$_o \in N$ in stochastic model (3). Therefore, there are still improvements that can be made for output y$'_o$ of DMU$_o$ in deterministic equivalent (9) of model (3). Note that $x_{io} - x'_{io} = -\tilde{\sigma}_{io}^I \Phi^{-1}(\alpha) < s_i^{-c*}$. Let $s_i^{-c'*} = s_i^{-c*} + \tilde{\sigma}_{io}^I \Phi^{-1}(\alpha)$. Then $s_i^{-c'*}$ is the congesting amount of the i-th input of DMU$_o$ in stochastic model (3). Since $s_i^{-c'*} > 0$, congestion is present at DMU$_o$ in stochastic model (3).

Theorem 5: *For* $0.5 < \alpha < 1$,

i) *Suppose that congestion is present for DMU$_o$ in input-output model (1). Then congestion is also present for DMU$_o$ in stochastic model (3). Furthermore,* $s_i^{-c'*} = s_i^{-c*} + \tilde{\sigma}_{io}^I \Phi^{-1}(\alpha)$ *is the congesting amount of the i-th input of DMU$_o$ in stochastic* *model (3), where s_i^{-c*} is the input congesting value as determined from input-output model (1).*

ii) *Suppose that $DMU_o \in F$ and DMU_o is an extreme point in input-output mean model (1), then $DMU_o \in N$ and congestion is present for DMU_o in stochastic model (3). Furthermore, $s_i^{-c'*} = s_{i1}^{-*} + \tilde{\sigma}_{io}^{I}\Phi^{-1}(\alpha)$ is the congesting amount of the i-th input of DMU_o in stochastic model (3), where s_{i1}^{-*} is the optimal slack value obtained from input-output model (1).*

Proof:

i) Since $0.5 < \alpha < 1$, $\Phi^{-1}(\alpha) > 0$. We have $y'_{ro} \leq y_{ro}$ and $x'_{io} \geq x_{io}$, where y'_{ro} and x'_{io} are respectively given by (11) and (12). Therefore, worsening DMU$_o$ from (x_o, y_o) to (x'_o, y'_o) implies that congestion should be present for DMU$_o$ in stochastic model (3). The second part of (i) is obvious.

ii) Since DMU$_o \in F$ in input-output mean model (1), there exists a $\lambda \geq 0$, with $\lambda_o = 0$, $\lambda_j \geq 0, (j \neq o)$ along with non-negative slacks for which at least one of the output slacks is zero and $\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j = 1$ such that

$$s_r^+ + y_{ro} = \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} = \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y'_{rj} > y_{ro} - \tilde{\sigma}_{ro}^{O}\Phi^{-1}(\alpha) = y'_{ro}, r = 1, ..., s$$

$$x_{io} - s_{i1}^- = \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij} - s_{i2}^+ = \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x'_{ij} - s_{i2}^+ \leq x_{io} + \tilde{\sigma}_{io}^{I}\Phi^{-1}(\alpha) = x'_{io}, i = 1, ..., m$$

i.e. DMU$_o$ is inefficient in model (9). In fact, when evaluating DMU$_o$ with adjusted inputs and outputs, a solution with $\phi > 1$ exists for model (9). Therefore, DMU$_o \in$N in stochastic model (3). Since DMU$_o$ is the only DMU with random variations in inputs and outputs and it is an extreme point on F in input-output mean model (1), any reductions in its outputs will move it away from the frontier of model (1). Notice that $y'_{ro} = y_{ro} - \tilde{\sigma}_{ro}^{O}\Phi^{-1}(\alpha) < y_{ro}$ and $x'_{io} = x_{io} + \tilde{\sigma}_{io}^{I}\Phi^{-1}(\alpha) > x_{io}$, while $s_i^{-c'*} = s_{i1}^{-*} + \tilde{\sigma}_{io}^{I}\Phi^{-1}(\alpha)$ is the congesting amount of the i-th input of DMU$_o$ in stochastic model (3).

Theorem 6: *For* $0 < \alpha < 0.5$,

i) *Suppose that congestion is present for* DMU_o *in stochastic model (3). Then congestion is also present for* DMU_o *in input-output model (1). Furthermore,* $s_i^{-c*} = s_i^{-c'*} - \tilde{\sigma}_{io}^I \Phi^{-1}(\alpha)$ *is the congesting amount of the i-th input of* DMU_o *in input-output model (1), where* $s_i^{-c'*}$ *is the input congesting value as determined from stochastic model (3).*

ii) *Suppose that* $DMU_o \in F$ *and* DMU_o *is an extreme point in stochastic model (3), then* $DMU_o \in N$ *and congestion is present for* DMU_o *in input-output model (1). Furthermore,* $s_i^{-c*} = s_{i1}^{-*} - \tilde{\sigma}_{io}^I \Phi^{-1}(\alpha)$ *is the congesting amount of the i-th input of* DMU_o *in input-output model (1), where* s_{i1}^{-*} *is the optimal slack value obtained from stochastic model (3).*

Proof: It is similar to the proof of Theorem 5.

Theorem 7: *For* $0.5 < \alpha < 1$,

i) *Suppose that congestion is not present for* DMU_o *in stochastic model (3) . Then congestion is also not present for* DMU_o *in input-output model (1).*

ii) *Suppose that congestion is present for* DMU_o *and* $DMU_o \in N$ *in stochastic model (3). Then congestion is also present for* DMU_o *and* $DMU_o \in N$ *in input-output model (1). Moreover,* $s_i^{-c*} = s_i^{-c'*} - \tilde{\sigma}_{io}^I \Phi^{-1}(\alpha)$ *is the congesting amount of the i-th input of* DMU_o *in input-output model (1), if* $-1 < \tau_{io}^I / \sigma_{io}^I < s_i^{-c'*} / (\Phi^{-1}(\alpha)) - 1$ *and* $-1 < \tau_{ro}^O / \sigma_{ro}^O < \beta_r^{+*} / (\Phi^{-1}(\alpha)) - 1$*, where* $s_i^{-c'*}$ *is the congesting amount of the i-th input of* DMU_o *in stochastic model (3), and* $\sum_{r=1}^{s} \beta_r^{+*}$ *is the optimal value of*

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{r=1}^{s} \beta_r^+ \\
\text{subject to} \quad & \sum_{j=1}^{n} \lambda_j x'_{ij} - \beta_{i2}^+ \leq x'_{io}, i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y'_{rj} - \beta_r^+ \geq y'_{ro}, r = 1, \ldots, s \\
& 1 = \sum_{j=1}^{n} \lambda_j, \\
& \beta_{i2}^+, \lambda_j, \beta_r^+ \geq 0 \quad ,
\end{aligned} \tag{14}
$$

Proof: Refer to Appendix.

Illustration In Table 1, data and numerical results for five DMUs are presented. Column 4 of Table 1 shows the optimal value of the objective function of the model (1) without ϵ. In order to illustrate some of our theoretical results, we consider only two cases with

Table 1: Data for illustrative example

DMU	x_1	y_1	ϕ_o^*
A	1	0.5	4
B	2	2	1
C	3	2	1
D	5	1	2
E	4	1	2

$\alpha = 0.2$ and $\alpha = 0.8$. We also assume that $\sigma^I = \sigma^o = 0.45$ and $\tau^I = \tau^o = 0.05$, therefore, $\tilde{\sigma}^I = \tilde{\sigma}^o = 0.5$. In both cases numerical results are presented in Table 2. The first point in pranthesis shows ϕ_o^* and the second one shows $s^{-c'*}$.
Case 1. $\alpha = 0.2$.
From a cumulative normal distribution table, we have $\Phi^{-1} = -0.84$ and therefore $\tilde{\sigma}^I \Phi^{-1}(\alpha) = \tilde{\sigma}^o \Phi^{-1}(\alpha) = -0.42$ for use in the following example. Assume that only point C has random variations in its input and output. Based on (11) and (12), the adjusted input and output

for point C with $\alpha = 0.2$ is
C′: $x' = 3 + (-0.42) = 2.58$ and $y' = 2 - (-0.42) = 2.42$.
We have $\tilde{\phi_{C'}}^* = 1$, therefore, C′ is efficient. Hence, C=C′ has no congestion. This is consistent with Theorem 4(i). Adjusted input output for point E is also E′(3.58,1.42). For E′ we have $\tilde{\phi_{E'}}^* = 1.41$. Therefore, E′ is inefficient, and input congestion for point E′ is 0.58.

Using model (13) on point E with coordinates as given in Table 1, we have the following linear program.

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{r=1}^{s} \beta_r^+ \\
\text{subject to} \quad & 0.5\lambda_A + 2\lambda_B + 2\lambda_C + 1\lambda_D + \lambda_E - \beta^+ \leq 1, i = 1, \ldots, m \\
& 1\lambda_A + 2\lambda_B + 3\lambda_C + 5\lambda_D + 4\lambda_E - \beta_2^+ \geq 4, r = 1, \ldots, s \\
& 1 = \lambda_A + \lambda_B + \lambda_C + \lambda_D + \lambda_E, \\
& \beta_2^+, \lambda_j, \beta^+ \geq 0 \quad ,
\end{aligned}
$$

This gives that $\beta^{+*} = 0.58$. Therefore, $\tilde{\sigma}^o(-\Phi^{-1}(\alpha)) = +0.42 \leq \beta^+ = 0.58$ where $s^{-c*} = 1$ from Figure 1 on point E. Hence, the conditions of Theorem 4(ii) are satisfied and we have $s^{-c'*} = s^{-c*} + \tilde{\sigma}^o\Phi^{-1}(\alpha) = 0.58$. This is the value of input congestion for E in stochastic model ().

Case 2. $\alpha = 0.8$.
In this case, we again have $\Phi^{-1}(\alpha) = 0.84$ and therefore $\tilde{\sigma}^I\Phi^{-1}(\alpha) = \tilde{\sigma}^o\Phi^{-1}(\alpha) = 0.42$. Assume that only point C has random variations in its input and output. Based on (11) and (12), the adjusted input and output for point C with $\alpha = 0.8$ is $x' = 3 + (0.42) = 3.42$ and $y' = 2 + (-0.42) = 1.58$. $\tilde{\phi_{C'}}^*$ is 1.266, therefore, C′ is inefficient. We notice that the optimal value of slack for input at C is $s^{-*} = 1$. Therefore, based on Theorem 5(ii), $s^{-c'*} = s^{-*} + \tilde{\sigma}^o\Phi^{-1}(\alpha) = 1 + 0.42 = 1.42$ is the input congestion for C in stochastic model (3). Furthermore, adjusted input output for DMU E is E′(4.42,0.58) with $\tilde{\phi_{E'}}^* = 3.448$. Therefore, E′ is inefficient. Input congestion for E′ is also 1.42 which is presented in table 2. This is consistent with Theorem 5(i). Note that from Figure 1, it is obvious that the input congestion for E is s^{-c*}=1. Therefore, utilizing Theorem 5(i) we have $s^{-c'*} = s^{-c*} + \tilde{\sigma}^o\Phi^{-1}(\alpha) = 1 + 0.42 = 1.42$.

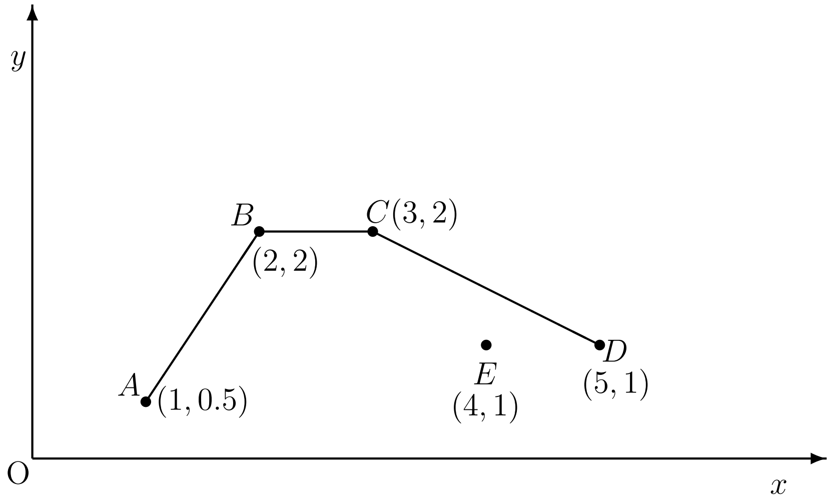

Figure 1: Congestion

Table 2: Computational results, $(\tilde{\phi}_o^{s*}, s^{-c'*})$

Possibility level, α	DMU_C	DMU_E
0.2	(1, 0)	(1.41, 0.58)
0.8	(1.266, 1.42)	(3.45, 1.42)

7 Concluding Remarks In this paper, we formulated stochastic version of the proposed model have been introduced for determining input congestion. We found deterministic equivalents to our stochastic programs. The deterministic equivalents are quadratic programs that can be solved to identify the efficiency ratio, slacks and input congestion of the evaluating DMU at a predetermined probability significance level. Considering fuzzy counterparts of the additive model correspond to the underlying model can be suggested for further research.

To simplify our presentation in §6, we assumed that all covariances are zero. Although are many examples for which this assumption holds, there are also many examples in which such assumption is realistic. Modeling covariance is very problem-specific. If the DMU's are indexed by time, sometimes time series models can help to model a suitable covariance structure. A general approach used in statistical analysis is the Bayesian approach which puts a prior on the covariance matrix. Wishart distribution is often a reasonable choice. One may further put a a flat prior on the hyper parameters involved in Wishart distribution and analyze data using a formal hierarchical Bayesian approach. Such hierarchical approach can also address sensitivity analysis since it essentially discusses changes in the parameters of the imposed distribution on the inputs and outputs.

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Appendix

Proof. Proof of Theorem 7:

i) Consider model (9), the deterministic equivalent of model (3). Since $0.5 < \alpha < 1$, $\Phi^{-1}(\alpha) > 0$. Hence $y'_{ro} \le y_{ro}$ and $x'_{io} \ge x_{io}$. This then implies that congestion is not present for DMU_0 in model (1) when DMU_o improves from (x'_o, y'_o) to (x_o, y_o).

ii) Suppose that $(\beta_2^{+*}, \beta^{+*}, \lambda^*)$ is an optimal solution of model (14). Then as a necessary condition of being maximal we must have

$$\sum_{j=1}^{n} \lambda_j^* y'_{rj} = y'_{ro} + \beta_r^{+*},\ r = 1, \ldots, s,$$

we also have

$$\sum_{j=1}^{n} \lambda_j^* x'_{ij} \le x'_{io} + \beta_{i2}^{+*},\ i = 1, ..., m.$$

Since $\tilde{\sigma}_{io}^I < s_i^{-c'*}/(\Phi^{-1}(\alpha))$ and $\tilde{\sigma}_{ro}^O < \beta_r^{+*}/(\Phi^{-1}(\alpha))$, we therefore obtain

$$
\begin{aligned}
y_{ro} = y'_{ro} + \tilde{\sigma}^{o}_{ro}\Phi^{-1}(\alpha) < y'_{ro} + \beta^{+*}_{r} &= \sum_{j=1}^{n} \lambda^*_j y'_{rj} \\
&= \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda^*_j y_{rj} + \lambda^*_o(y_{ro} - \tilde{\sigma}^{o}_{ro}\Phi^{-1}(\alpha)) \\
&\leq \sum_{j=1}^{n} y_{rj}\lambda^*_j, \; r = 1, .., s
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j=1}^{n} \lambda^*_j x_{ij} = \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda^*_j x_{ij} + \lambda^*_o(x_{io} + \tilde{\sigma}^{I}_{io}\Phi^{-1}(\alpha)) &= \sum_{j=1}^{n} \lambda^*_j x'_{ij} \\
&\leq x'_{io} + \beta^{+*}_{i2} = x_{io} + \tilde{\sigma}^{I}_{io}\Phi^{-1}(\alpha) + \beta^{+*}_{i2} \\
&= x_{io} + s^{+}_{i2}, \; i = 1, .., m
\end{aligned}
$$

where $s^{+}_{i2} = \tilde{\sigma}^{I}_{io}\Phi^{-1}(\alpha) + \beta^{+*}_{i2} \geq 0$.

This means that DMU_o with input-output combination of (x_o, y_o) is inefficient in model (1). Hence, $\text{DMU}_o \in N$ in input-output model (1). Therefore, there are still improvements that can be made for output y_o of DMU_o in model (1). Note that $x'_{io} - x_{io} = \tilde{\sigma}^{I}_{io}\Phi^{-1}(\alpha) < s^{-c'*}_{i}$. Let $s^{-c*}_{i} = s^{-c'*}_{i} - \tilde{\sigma}^{I}_{io}\Phi^{-1}(\alpha)$. Then s^{-c*}_{i} is the congesting amount of the i-th input of DMU_o in model (1). Since $s^{-c*}_{i} > 0$, congestion is present at DMU_o in model (1).

□

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