International Journal of Statistics and Management System
Vol. 4 No. 2 (July-December, 2019)

Expected Overflow in a Finite Dam Model for a Class of Master Equations with Separable Kernels *

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Received: 03rd March 2018 Revised: 14th April 2018 Accepted: 10th December 2018

Abstract

We discuss the expected overflow in a finite dam model for a class of master equations with separable kernels. For this class of master equations the integral equation for the expected amount of over flow can be transformed into an ordinary differential equation. The finite dam model for overflow is considered with barriers at $X = 0$ and X = k (constant). The closed form solution for the expected amount of overflow before the dam becomes empty is arrived at. The results for expected amount of overflow with any number of emptiness in time t are also derived. In the above two cases the inputs into the dam are taken as random and the output (release rule) is taken as deterministic and linear. In another model the release is taken as proportional to the content of the dam.

1 Introduction. In research literature, several papers have appeared for storage models to study the first passage time of emptiness or overflow. For the dam of finite capacity

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Key words and phrases: Finite dam, Imbedding method, Expected loss due to overflow, Confluent Hypergeometric function, absorbing and reflecting barriers.

AMS 2000 subject classifications. Primary 60G20 ; secondary 60G40.

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results are available for first emptiness period before overflow and the first overflow time before emptiness (see Vasudevan et al [?]). Problems with one barrier absorbing and the other as reflecting were also studied by Phatarfod [?] . Keilson [?] used the compensation technique to study the first emptiness period of an infinite dam with linear release policy. The solution for the probability density function (pdf) of emptiness time is in terms of Green's function of the unbounded process. Phatarfod [?] used Wald identity to study the emptiness period for without overflow and with atleast one overflow separately. In his model, the input into the dam is random and the release rule is taken as a linear drift. Puri and Senthuria [?] considered a dam of infinite depth with random inputs and random outputs. Tsurui and Osaki [?] considered, in a different context a cumulative process with exponential decay and random inputs. They arrived at an integral equation for the first passage time density for crossing the upper barrier in the context of neuronal spike trains. Vasudevan et al [?] studied a class of two barrier problems by taking both the barriers as absorbing and in another model one barrier reflecting, with linear drift and exponential inputs. Vasudevan and Vittal [?] considered a finite dam with random inputs and random outputs together with exponential release policy. Using imbedding method, they derived closed form solutions for the first passage times of emptiness and overflow where the barriers are absorbing or when one of them is absorbing and the other reflecting. In a different context, Perry et al studied a shot noise process $X(t)$ with Poisson arrival times and exponentially diminishing random shots. They derived integral equation for the joint density of the first passage time T and $X(T)$. The solution is obtained in terms of Laplace transform functions. Perry et al [?] also studied a queueing system to calculate the total expected loss of discarded service and the total expected discounted cost of the switching and maintaining extra capacity and also the total expected loss of discarded service. They used Wald martingale for compound Poisson process. Also they studied the results between barriers ϑ^* and ϑ^{**} where the work load process $\vartheta(t)$ up crosses ϑ_u and drops down to ϑ_1.Avanzi et al [?] considered optimal dividend problem to determine dividend payment strategy that maximizes the expected discounted value of dividend paid until the company is ruined. In this model, it is assumed that the surplus process is a Levy process which is skip-free downwards. In this paper, we used the imbedding method which works out well to determine the expected loss due to overflow for a finite dam in a given time, by considering a stochastic model wherein the output is skip-free. In another model, we consider an exponentially decaying output. In section 2, we consider the model with linear drift and exponential inputs to derive a closed form solution for the expected loss before the dam becomes empty. In section 3, for the same model we take $X = 0$ as a reflecting barrier. This means the expected loss in a finite time is derived with any number of emptiness within this time. In section 4, we take the inputs into the dam as random (exponential) and the release policy is taken as proportional to the content of the dam. In this model $X = 0$ is a natural barrier.

2 Expected loss due to overflow in a finite time for a finite dam model Consider a dam of finite capacity k with Poisson inputs and deterministic release. The model representing this stochastic process is

$$X(t) \;=\; x - \alpha t + \sum_{n=1}^{N(t)} Z_n \tag{2.1}$$

Here$X = x \in (0, k]$ is the initial content of the dam, α is the linear drift per unit time and Z_n are independent and identically distributed random variables each having a probability density $a(z)$ or the transition density $a(z, z^{'})$. $N(t)$ is the number of inputs in time t constituting a Poisson process with intensity μ. Thus $\sum_{n=1}^{N(t)} Z_n$ follows a compound Poisson distribution. $X(t)$ represents the content of the dam at time t. Define $S(x, k, t)$ as the expected amount of overflow in time t given that $X = x$ is the initial content of the dam ($X(0) = x$) and $X = k$ is the finite capacity of the dam.

Consider the dynamics of the process for $S(x, k, t)$ in the initial interval of time dt. The following mutually pairwise exclusive events may occur.

1 There is no random input into the dam in the initial interval of time dt.

2 There is a random input of size z in the initial interval time dt so that the content of the dam is below the level $X = k$.

3 There is a random input of size $z (z > k - x)$ in the initial interval of time dt so that overflow takes place in this interval of time.

Considering the different possible events in the initial interval of time dt we arrive at the equation

$$\begin{aligned} S(x,k,t+\mathrm{d}t) &= (1-\mu\mathrm{d}t)S(x-\alpha\,\mathrm{d}t,k,t)+\mu\mathrm{d}t\int\limits_0^{k-x} S(x+z,k,t)a(z)\mathrm{d}z \\ &+\delta(t)\mu\mathrm{d}t\int_{k-x}^{\infty} a(z)[z-(k-x)]\mathrm{d}z+\mu\mathrm{d}t\int_{k-x}^{\infty} a(z)\mathrm{d}zS(k,k,t) \quad (2.2)\end{aligned}$$

Here the third term on the RHS represents the loss due to the inputs in the initial interval of time dt and the fourth term measures the expected loss $S(k,k,t)$ after the initial input crossing the level $X=k$ of the finite dam. $\delta(t)$ is the Dirac delta function which takes the value 1 at t=0 and 0 otherwise. The equation (2.2) can be expressed as the integro-differential equation

$$\begin{aligned} \frac{\partial S}{\partial t}+\alpha\frac{\partial S}{\partial x}+\mu S &= \mu\int\limits_0^{k-x} S(x+z,k,t)a(z)\mathrm{d}z+\mu\delta(t)\int_{k-x}^{\infty} a(z)[z-(k-x)]\mathrm{d}z \\ &+\mu\mathrm{d}t\int_{k-x}^{\infty} a(z)\mathrm{d}zS(k,k,t) \quad (2.3)\end{aligned}$$

Define the Laplace transform of the function $S(x,k,t)$ as

$$\bar{S}(x,k,\ell)=\int\limits_0^{\infty} e^{-\ell t}S(x,k,t)\mathrm{d}t \quad (2.4)$$

Taking Laplace transform with respect to t, equation (2.3) becomes

$$\begin{aligned} \alpha\frac{\partial \bar{S}}{\partial x}+(\ell+\mu)\bar{S} &= \mu\int\limits_0^{k-x} \bar{S}(x+z,k,\ell)a(z)\mathrm{d}z+\mu\int_{k-x}^{\infty} a(z)[z-(k-x)]\mathrm{d}z \\ &+\mu\int_{k-x}^{\infty} a(z)\;\mathrm{d}z\;\bar{S}(k,k,\ell) \quad (2.5)\end{aligned}$$

In order to arrive at a tractable solution for $\bar{S}(x,k,\ell)$, we take the density of input of size z as exponential

$$a(z)=\lambda e^{-\lambda z}H(z)$$

where $H(z)$ is the Heaviside function which means

$$H(z)=\begin{cases} 1 & \text{if} \quad z>0 \\ 0 & \text{otherwise}\end{cases} \quad (2.6)$$

This means the transition density from x to y is given by

$$a(x,y) = \lambda e^{-\lambda(x-y)},\ x-y>0 \tag{2.7}$$

Now the integro differential equation(2.5) transforms into

$$\begin{aligned} e^{-\lambda x}\left[\alpha\frac{\partial \bar{S}}{\partial x} + (\ell+\mu)\bar{S}\right] &= \mu\lambda\int_x^k e^{-\lambda y}\bar{S}(y,k,\ell)\mathrm{d}y + \mu\,\lambda\int_k^\infty e^{-\lambda y}(y-k)\mathrm{d}y \\ &\quad +\mu\lambda\int_k^\infty e^{-\lambda y}\mathrm{d}y\,\bar{S}(k,k,\ell) \\ &= \mu\lambda\int_x^k e^{-\lambda y}\bar{S}(y,k,\ell)\mathrm{d}y + \frac{\mu}{\lambda}e^{-\lambda k} + \mu e^{-\lambda k}\bar{S}(k,k,\ell) \end{aligned} \tag{2.8}$$

Differentiating equation(2.8) with respect to x, we eliminate the integral on R.H.S and convert equation(2.8) into the differential equation,

$$\alpha\frac{\partial^2 \bar{S}}{\partial x^2} + (\ell+\mu-\lambda\alpha)\frac{\partial \bar{S}}{\partial x} - \lambda\ell\bar{S} = 0 \tag{2.9}$$

Here the coefficients in this differential equation being constants, the general solution of the differential equation is

$$\bar{S}(x,k,\ell) = A(k,\ell)\,e^{m_1\,x} + B(k,\ell)\,e^{m_2\,x} \tag{2.10}$$

Where $A(k,\ell)$ and $B(k,\ell)$ are arbitary functions independent of x. One boundary condition is $\bar{S}(0,k,\ell)=0$. Therefore from equation(2.10) we get

$$A+B=0 \tag{2.11}$$

The imbedding equation(2.7) takes care of the boundary and serves as another boundary condition. Feeding the solution (2.10)into the equation(2.8), we get,

$$
\begin{aligned}
e^{-\lambda x}\left[\alpha\frac{\partial}{\partial x}(Ae^{m_1x}+Be^{m_2x})+(\ell+\mu)(Ae^{m_1x}+Be^{m_2x})\right] &= \mu\lambda\int_x^k e^{-\lambda y}(Ae^{m_1y}+Be^{m_2y})\mathrm{d}y \\
&\quad +\frac{\mu}{\lambda}e^{-\lambda k}+\mu e^{-\lambda k}(Ae^{m_1k}+Be^{m_2k}) \\
&= \mu\lambda\left[\frac{Ae^{(m_1-\lambda)y}}{m_1-\lambda}+\frac{Be^{(m_2-\lambda)y}}{m_2-\lambda}\right]_x^k \\
&\quad +\frac{\mu}{\lambda}e^{-\lambda k}+\mu e^{-\lambda k}(Ae^{m_1k}+Be^{m_2k}) \qquad (2.12)
\end{aligned}
$$

The coefficients of Ae^{m_1x} and Be^{m_2x} vanish separately(see Appendix) and we are left with

$$
0=\mu\lambda\frac{Ae^{(m_1-\lambda)k}}{m_1-\lambda}+\mu\lambda\frac{Be^{(m_2-\lambda)k}}{m_2-\lambda}+\frac{\mu}{\lambda}e^{-\lambda k}+\mu e^{-\lambda k}(Ae^{m_1k}+Be^{m_2k})
$$

$$
\begin{aligned}
\text{i.e.,}\quad \mu\left[\lambda\, A\frac{Ae^{m_1k}}{m_1-\lambda}+Ae^{m_1k}\right]+\mu\left[\lambda B\frac{Ae^{m_2k}}{m_2-\lambda}+Be^{m_2k}\right]+\frac{\mu}{\lambda} &= 0 \\
\text{or}\quad \left(A\frac{m_1}{m_1-\lambda}e^{m_1k}+B\frac{m_2}{m_2-\lambda}e^{m_2k}\right)+\frac{1}{\lambda} &= 0 \qquad (2.13)
\end{aligned}
$$

Combining equation(2.11) and equation(2.13)we get,

$$
A\left(\frac{m_1}{m_1-\lambda}e^{m_1k}-\frac{m_2}{m_2-\lambda}e^{m_2k}\right)+\frac{1}{\lambda}=0 \qquad (2.14)
$$

$$
A\lambda\left[m_1(m_2-\lambda)e^{m_1k}-m_2(m_1-\lambda)e^{m_2k}\right]+(m_1-\lambda)(m_2-\lambda)=0 \qquad (2.15)
$$

The complete solution for $\bar{S}(x,k,\ell)$ is

$$
\bar{S}(x,k,\ell)=A(e^{m_1x}-e^{m_2x}) \qquad (2.16)
$$

where m_1 and m_2 are given by (from equation(2.9))

$$m_1, m_2 = \frac{-(\ell + \mu - \lambda\alpha) \pm \sqrt{(\ell + \mu - \lambda\alpha)^2 + 4\alpha\lambda\ell}}{2\alpha} \tag{2.17}$$

and A is given by equation(2.15).
When $\ell = 0, m_1 = 0, m_2 = \frac{-(\mu-\lambda\alpha)}{\alpha}$ and from equation(2.15)

$$A\lambda \left[\lambda\ m_2\ e^{m_2 k}\right] - \lambda\ (m_2 - \lambda) = 0$$

$$A = \frac{\mu e^{-m_2 k}}{\lambda(\mu - \lambda\alpha)}$$

Therefore

$$\bar{S}(x, k, 0) = S(x, k, \infty) = \frac{\mu}{\lambda(\mu - \lambda\alpha)e^{m_2 k}} \left[1 - e^{-\dfrac{(\mu - \lambda\alpha)}{\alpha}x}\right]$$

Allowing k tending to infinity. (i.e.,)$\bar{S}(x, \infty, 0) = S(x, \infty, \infty) = \infty$
where $S(x, \infty, \infty)$ is expected apparently to be zero. This is because of the loss of gain where k is infinitely far away from the original value of k.

3 Expected loss in a finite time with any number of emptiness Here we consider the expected loss in time t due to overflow allowing any number of emptiness before time t. Defining $S_1(x, k, t)$ as the expected loss in time t with any number of emptiness in time t, we have the same integro-differential equation(2.5) for $\bar{S}_1(x, k, \ell)$ where $\bar{S}_1(x, k, \ell)$ is the Laplace transform of $S_1(x, k, t)$ Here, there is only a change in the boundary condition. The boundary $X = 0$ can be reached only because of linear drift towards $X = 0$.
The boundary condition (Refer Vasudevan and Vittal [**?**]) corresponding to this is

$$\frac{\mathrm{d}\bar{S}_1}{\mathrm{d}x} = 0 \quad \text{at } X = 0 \tag{3.1}$$

In this model, both $X = 0$ and $X = k$ are reflecting barriers and we are interested in the loss due to cut off by the barrier $X = k$. Proceeding as in Section 2,

$$\bar{S}(x, k, \ell) = A_1(k, \ell)\ e^{m_1\ x} + B_1(k, \ell)\ e^{m_2\ x} \tag{3.2}$$

$$\frac{\mathrm{d}\bar{S}}{\mathrm{d}x} = 0 \text{ at } x = 0 \text{ implies } A_1 m_1 + B_1 m_2 = 0 \tag{3.3}$$

As in the last section, the solutions for m_1 and m_2 are given by

$$\alpha m^2 + (\ell + \mu - \lambda\alpha)m - \lambda\ell = 0 \tag{3.4}$$

$$m_1, m_2 = \frac{-(\ell + \mu - \lambda\alpha) \pm \sqrt{(\ell + \mu - \lambda\alpha)^2 + 4\alpha\lambda\ell}}{2\alpha} \tag{3.5}$$

and the boundary condition of crossing the barrier $X = k$ is taken care of by the equation(2.8) and resulting in the second relation connecting A and B, the same as in section 2 with A and B replaced by A_1 and B_1.
Therefore,

$$\frac{A_1 m_1}{m_1 - \lambda} e^{m_1 k} + \frac{B_1 m_2}{m_2 - \lambda} e^{m_2 k} + \frac{1}{\lambda} = 0 \tag{3.6}$$

The equation(3.2),equation(3.3)and equation(3.6) completely determine the closed form solution for $\bar{S}(x, k, \ell)$.
As in the lastt case (Section 2) when $\ell = 0$,

$$m_1 = 0 \qquad \text{and} \qquad m_2 = \frac{-(\mu - \lambda\alpha)}{\alpha}$$

From equation(3.2) we get $B_1 m_2 = 0$ implying $B_1 = 0$ as $m_2 \neq 0$.
The general solution reduces to

$$\bar{S}(x, k, \ell) = S_1(x, k, \infty) = A_1 e^{m_1 x} \tag{3.7}$$

From equation(2.15)

$$A_1 m_1 (m_2 - \lambda) e^{m_1 k} + B_1 m_2 (m_1 - \lambda) e^{m_2 k} + (m_1 - \lambda)(m_2 - \lambda) = 0 \tag{3.8}$$

When $m_1 = 0,\ m_2 = \dfrac{-(\mu - \lambda\alpha)}{\alpha},\ A = \dfrac{\lambda(m_2 - \lambda)}{m_1(m_2 - \lambda)} = \dfrac{\lambda}{m_1} = \infty, \quad \lambda > 0$
Therefore,

$$S_1(x, k, \infty) = \infty \quad \text{and so also} \quad S_1(x, \infty, \infty) = \infty$$

Here also we note that $S_1(x, \infty, \infty) \neq 0$ as could be expected. This is because of the loss of gain when k is infinitely far away from the beginning k.

4 Expected loss when the release is proportional to the content of the dam

In this section we consider a dam model of the type

$$X(t) = x - \alpha \int_0^t X(t^{'})\mathrm{d}t^{'} + \sum_{n=1}^{N(t)} Z_n \tag{4.1}$$

This only means that the release from the dam is proportional to the content(exponential release)

Define$S_2(x,k,t)$ as the expected quantity of overflow in time t.

Proceeding as in section 2, the integro differential equation for $S_2(x,k,t)$ is

$$\begin{aligned}\frac{\partial S_2}{\partial t} + \alpha x \frac{\partial S_2}{\partial x} + \mu S_2 &= \mu \int_0^{k-x} a(z) S_2(x+z,k,t)\mathrm{d}z + \delta(t)\mu \int_{k-x}^{\infty} a(z)[z-(k-x)]\mathrm{d}z \\ &+ \mu \int_{k-x}^{\infty} a(z)\mathrm{d}z S_2(k,k,t) \end{aligned}\tag{4.2}$$

Define

$$\bar{S}_2(x,k,\ell) = \int_0^{\infty} e^{-\ell t} S_2(x,k,t)\mathrm{d}t \tag{4.3}$$

Taking the Laplace transform with respect to t, the equation(4.2) becomes,

$$\begin{aligned}\alpha x \frac{\partial \bar{S}_2}{\partial x} + (\ell+\mu)\bar{S}_2 &= \mu \int_0^{k-x} a(z)\bar{S}_2(x+z,k,\ell)\mathrm{d}z + \mu \int_{k-x}^{\infty} a(z)[z-(k-x)]\mathrm{d}z \\ &+ \mu \int_{k-x}^{\infty} a(z)\mathrm{d}z \bar{S}_2(k,k,\ell)\end{aligned}\tag{4.4}$$

Take the transition density as

$$a(x,x^{'}) = e^{-\lambda(x-x^{'})} \tag{4.5}$$

Then equation(4.4) becomes

$$\begin{aligned} e^{-\lambda x}\left[\alpha x \frac{\partial \bar{S}_2}{\partial x} + (\ell+\mu)\bar{S}_2\right] &= \mu\lambda \int_0^{k} e^{-\lambda y}\bar{S}_2(y,k,\ell)\mathrm{d}y + \mu\lambda \int_k^{\infty} e^{-\lambda y}(y-k)\mathrm{d}y \\ &+ \mu\lambda \int_k^{\infty} e^{-\lambda y}\mathrm{d}y\, \bar{S}_2(k,k,\ell) \\ &= \mu\lambda \int_x^{k} e^{-\lambda y}\bar{S}_2(y,k,\ell)\mathrm{d}y + \frac{\mu}{\lambda}e^{-\lambda k} + \mu e^{-\lambda k}\bar{S}_2(k,k,\ell) \end{aligned}\tag{4.6}$$

Differentiating equation(4.6) with respect to x we get,

$$\alpha x\frac{\partial^2 \bar{S}_2}{\partial x^2} + (\ell + \mu + \alpha - \lambda\alpha x)\frac{\partial \bar{S}_2}{\partial x} - \lambda\ell\bar{S}_2 = 0 \tag{4.7}$$

Taking $\lambda x = v$, the equation(**??**) transforms to

$$v\frac{\partial^2 \bar{S}_2}{\partial x^2} + \left(\frac{\ell+\mu}{\alpha} + 1 - v\right)\frac{\partial \bar{S}_2}{\partial x} - \frac{\ell}{\alpha}\bar{S}_2 = 0 \tag{4.8}$$

The general solution of this equation is

$$\bar{S}_2(x,k,\ell) = A(k,\ell)\ {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\mu}{\alpha} + 1, \lambda x\right) + B(k,\ell)(\lambda x)^{-\dfrac{(\ell+\mu)}{\alpha}}\ {}_1F_1\left(\frac{-\mu}{\alpha}, 1 - \frac{\ell+\mu}{\alpha}, \lambda x\right) \tag{4.9}$$

Where $A(k,\ell)$ and $B(k,\ell)$ are functions of k and l and independent of x. ${}_1F_1$ which occurs in the above equation is given by

$${}_1F_1(a,c,z) = 1 + \sum \frac{(a)_n}{(c)_n}\frac{z^n}{n!} \tag{4.10}$$

where $(a)_n = a(a+1)(a+2)\dots(a+n-1)$
The above solution is valid for all initial values of x where $0 \le x \le k$.
The second term on the right hand side of equation(**??**) is infinite for $\left(\frac{\ell+\mu}{\alpha}\right) > 0$ as α and μ are positive and real $\ell > 0$. Hence we choose $B(k,\ell) = 0$ in order that the solution exists. Therefore, the general solution reduces to

$$\bar{S}_2(x,k,\ell) = A(k,\ell)\ {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\mu}{\alpha} + 1, \lambda x\right) \quad \text{for} \quad 0 \le x \le k \tag{4.11}$$

The first order integro-differential equation(4.6) serves as a boundary condition and so substituting the solution(**??**) in equation(4.6) we get,

$$Ae^{-\lambda x}\left[\alpha x\frac{\partial}{\partial x}{}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\mu}{\alpha} + 1, \lambda x\right) + (\ell+\mu)\ {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\mu}{\alpha} + 1, \lambda x\right)\right]$$

$$= \mu\lambda A\int_x^k e^{-\lambda y}\ {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\mu}{\alpha} + 1, \lambda y\right)\mathrm{d}y + \frac{\mu}{\lambda}e^{-\lambda k} + \mu e^{-\lambda k}A\ {}_1F_1\left(\frac{\ell}{\alpha}, \frac{\ell+\mu}{\alpha} + 1, \lambda k\right) \tag{4.12}$$

Without loss of generality, we choose $\lambda = 1$ and use the property of confluent hypergeometric function(Abrarnowitz and Stegun 1968)

$$\int e^{-x}{}_1F_1(a,b,x)\mathrm{d}x = \frac{e^{-x}(b-1)}{1+a-b}{}_1F_1(a,b-1,x) + \text{ constant} \qquad \text{for } b - a \neq 1$$

Then equation(4.6) becomes,

$$e^{-x}A\left\{\alpha x\frac{\partial}{\partial x}{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha}+1,x\right)+(\ell+\mu)\left[{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha}+1,x\right)-{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha},x\right)\right]\right\}$$
$$=e^{-k}\left[\mu-A(\ell+\mu)\,{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha},k\right)+\mu A\,{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha}+1,k\right)\right] \tag{4.13}$$

LHS is zero because of the identity,

$$(b-1){}_1F_1(a,b-1,z)=(b-1){}_1F_1(a,c,z)+z\frac{d}{dz}{}_1F_1(a,c,z) \tag{4.14}$$

Hence,

$$A(k,\ell)=\frac{\mu}{(\ell+\mu)\,{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha},k\right)-\mu\,{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha}+1,k\right)} \tag{4.15}$$

The complete solution for $\bar{S}_2(x,k,\ell)$ is

$$\bar{S}_2(x,k,\ell)=\frac{\mu\,{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha}+1,x\right)}{(\ell+\mu)\,{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha},k\right)-\mu\,{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha}+1,k\right)} \tag{4.16}$$

If we start with an empty dam,

$$\bar{S}_2(0,k,\ell)=\frac{\mu}{(\ell+\mu)\,{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha},k\right)-\mu\,{}_1F_1\left(\frac{\ell}{\alpha},\frac{\ell+\mu}{\alpha}+1,k\right)} \tag{4.17}$$

Also when $t=\infty$,

$$\bar{S}_2(x,k,0)=S_2(x,k,\infty)=\infty \tag{4.18}$$

This only concludes that as $t\to\infty,\ S_2(x,k)=\infty$ and as $t\to\infty$ and $k\to\infty$, $S_2(x,\infty,\infty)=\infty$. This is due to the fact that the loss of gain is infinitely far away already from the initial position of the barrier.

5 Conclusion In conclusion, we like to point out that the procedure used to derive the different types of first passage time densities with two barriers either both absorbing or one reflecting and the other absorbing also works out in obtaining the expected overflow in dam models with linear release rule or exponential release rule with the kernel in the master equations in separable form. These types of problems with random inputs, random outputs and exponential release policy is being studied which will appear in our later publication.

Acknowledgement. The authors gratefully acknowledge professor Jiming Jiang, chief Editor of IJSMS and anonymous referees for their valuable comments and suggestions for the improvement of the paper. The first author sincerely thanks the Indian statistical Institute, Bangalore for the support and encouragement.

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APPENDIX

We will show that co-efficients of Ae^{m_1x} and Be^{m_2x} vanish separately.
The co-efficient of Ae^{m_1x} is

$$e^{-\lambda x}\left[\alpha m_1+(\ell+\mu)+\frac{\mu\lambda}{m_1-\lambda}\right]=\frac{e^{-\lambda x}}{m_1-\lambda}\left[\alpha m_1^2+(\ell+\mu-\alpha\lambda)m_1-\lambda\ell\right] \qquad (A_1)$$

Since m_1 is a characteristic root of the equation(2.9)

$$\alpha m^2+(\ell+\mu-\alpha\lambda)m-\lambda\ell=0$$

From equation(A_1)we note that the co-efficient Ae^{m_1x} is zero. On the same lines the co-efficient of Be^{m_2x} is also zero.

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