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# ON THE CROWN GRAPH DECOMPOSITIONS CONTAINING ODD CYCLE

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**Abstract:** We give some construction methods of graph decomposition and recursive construction of crown graph decomposition, and the existence spectrums for the $GD(v, Q_3, \lambda)$ and $GD(v, Q_5, \lambda)$ are determined when $\lambda \geq 1$. Let $n \geq 5$ be an odd number. When $v \equiv 0, 1 \pmod{4n}$, there exists a $GD(v, Q_n, \lambda)$ for any positive integer $\lambda$, and a $GD(2n, Q_n, \lambda)$ exists if and only if $\lambda \equiv 0 \pmod 4$.

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**Keywords:** Graph Decomposition; Crown Graph; *HGD*; *IGD*.

## 1. INTRODUCTION

Throughout this paper, we consider the case where $G$ is a finite simple and none of vertices is isolated. A graph $H$ is said to be $G$-decomposable if the edges of $H$ can be partitioned into subgraphs isomorphic to $G$. Let $v$ and $\lambda$ be two positive integers. We denote by $\lambda K_v$ the complete multigraph of order $v$ and index $\lambda$. A $G$-decomposition of $\lambda K_v$ is denoted by $GD(v, G, \lambda)$. Let $V$ be point set of $\lambda K_v$, $\mathcal{B}$ be the collection (called *blocks*) of subgraphs of the $GD(v, G, \lambda)$, we also denote the $GD(v, G, \lambda)$ by $(V, \mathcal{B})$. It is easy to see the following:

**Theorem 1.1:** Let $G$ be a $(p, q)$ graph. If there exists a $GD(v, G, \lambda)$, then (i). $v \geq p$; (ii). $\lambda V(V-1) \equiv 0 \pmod{2q}$; (iii). $\lambda(v-1) \equiv 0 \pmod d$.

Where $d$ is the greatest common divisor of the degree of the vertices of $G$.

One problem in design theory is the spectrum problem for $G$, i.e. for what values of $v$ and $\lambda$ is there a $GD(v, G, \lambda)$? The spectrum problem has been solved for complete simple graphs on less than six vertices for all $\lambda$, other complete simple graphs on less than nine vertices see [11, 15-17], stars for all $\lambda$ [2], and path [1], various other small graphs for at most 4 points [3]. In [5], Bermond *et al.*, dealt with necessary and sufficient conditions for the existence of $G$-decompositions, where $G$ are some graphs with five vertices. The existence of $GD(v, G, \lambda)$ for any positive integer $\lambda$ has been discussed when $G$ have six vertices and at most six edges in the literatures [1, 7, 8]. When $G$ is an $m$-cycle, Alspach *et al.*, [4] and *M*. Šajna[9] obtained the following theorem:

**Theorem 1.2:** Let $I$ be an 1-factor, and $n$ and $m$ be two positive integers with $3 \le m \le n$. Then (1). for positive even integer $n$, the graph $K_n - I$ can be decomposed into cycles of length $m$ if and only if the number of edges in $K_n - I$ is a multiple of $m$. (2). for positive odd integer $n$, the graph $K_n$ can be decomposed into cycles of length $m$ if and only if the number of edges in $K_n$ is a multiple of $m$.

$C_n$ denote the cycle of length $n$. The crown graph $Q_n$ obtained by joining single pendant edge to each vertex of $C_n$. If vertex set of $C_n$ on the $Q_n$ is $\{u_i | i = 1, 2, ..., n\}$, pendant vertex set of $Q_n$ is $\{u_i | i = n + 1, n + 2, ..., 2n\}$ and pendant edge set of $Q_n$ is $\{u_i u_i + n | i = 1, 2, ..., n\}$, then $Q_n$ is denoted by $(u_1, u_2, ..., u_n; u_{n+1}, u_{n+2}, ..., u_{2n})$. Crown graph is an important unicyclic graph with extensive application. The conception of crown graph was defined by Frucht and Harary [13]. Frucht [13] studied the gracefulness of crown graph. Grace [14] discussed the harmoniousness of crown graph.

**Theorem 1.3:** If there exists a $GD(v, Q_n, \lambda)$, then

(i) $v \ge 2n$; (ii). $\lambda V(V - 1) \equiv 0 \pmod{4n}$.

**Theorem 1.4:** (see [8]) When integer $v > 6$ and $\lambda V(V - 1) \equiv 0 \pmod{12}$, there exists a $GD(v, Q_3, \lambda)$.

In second section, we give the construction methods of graph decomposition and fundamental theorem. In third section, we obtain some recursive construction of $GD(v, Q_n, \lambda)$. We improve the result in [8], and give the spectrum of $GD(v, Q_3, \lambda)$ for any positive integer $\lambda$. We obtain also the following results: when $n \ge 5$ is an odd number and $v \equiv 0, 1 \pmod{4n}$, there exists a $GD(v, Q_n, \lambda)$ for any positive integer $\lambda$, and a $GD(2n, Q_n, \lambda)$ exists if and only if $\lambda \equiv 0 \pmod 4$. In fourth section, we give the spectrum of $GD(v, Q_5, \lambda)$ for any positive integer $\lambda$.

Let $Z$ be a ring of integers and $Z_m$ the residue class group modulo $m$ with residue classes $\{0, 1, ..., m - 1\}$. In what follows, the notations $(a, b \in Z)$: $[a, b] = \{x \in Z | a \le x \le b\}$, $[a, b]_k = \{x \in Z | a \le x \le b, x \equiv a \pmod k\}$, $(a, b, ..., c) + i = (a + i, b + i, ..., c + i)$ and $(Z_n)_m = \{i_m | i \in Z_n\}$ are used frequently.

## 2. RECURSIVE CONSTRUCTIONS

Let $h$, $\lambda$ and $v$ be positive integers. By $\lambda K_{n_1, n_2, ..., n_h}$ we mean the complete multipartite graph with $h$ parts of sizes $n_1, n_2, ..., n_h$ and index $\lambda$. Let $X = \cup_{1 \le i \le h} X_i$ be the vertex set of $K_{n_1, n_2, ..., n_h}$ where $X_i (1 \le i \le h)$ are disjoint sets with $|X_i| = n_i$ and $v = \Sigma_{1 \le i \le h} n_i$. A $G$-decomposition of $\lambda K_{n_1, n_2, ..., n_h}$ is denoted by $(G, \lambda)$-$HGD(T)$ or $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G} = \{X_i | 1 \le i \le h\}$ and $\mathcal{B}$ is the block set of the $(G, \lambda)$-$HGD(T)$. The multiset $T = \{|X_i| \| X_i \in \mathcal{G}\}$ is called type of $G$-decomposition, and also write $T = \Pi_{i=1}^{s} m_i^{ui}$ if

$G$ contains exactly $u_i$ groups with size $m_i$, $1 \le i \le s$. $(G, \lambda)$-*HGD* $(T)$ is also said to $(G, \lambda)$-*HGD* with type $T$. A $(G, \lambda)$-*HGD* $(1^{v-w} w^1)$ is called an incomplete $G$-decomposition, denoted by *IGD* $(v, w, G, \lambda)$. Obviously, a *GD* $(v, G, \lambda)$ is a $(G, \lambda)$-*HGD* (ì$^v$), which can be thought of as a *IGD* $(v, w, G, \lambda)$ with $w = 0$ or 1. The symbols $(K_k, 1)$-*HGD* $(T)$, $(G, 1)$-*HGD* $(T)$ and *IGD* $(v, w, G, 1)$ can be briefly written by $k$-*HGD* $(T)$, $G$-*HGD* $(T)$ and *IGD* $(v, w, G)$, respectively. A transversal design *TD* $(k, n)$ is a $k$-*HGD* $(n^k)$.

**Theorem 2.1:** If there exist a $(G, \lambda)$-*HGD* $(u \Pi_{i=1}^{t} n_i^{ui})$ and a *GD* $(n_i, G, \lambda)$ for $i \in [1, t]$, then there exists an *IGD* $(u + \Sigma_{1 \le i \le t}\, u_i n_i, u, G, \lambda)$.

**Proof:** Let $(X, \mathcal{G}, \mathcal{B}_0)$ be a $(G, \lambda)$-*HGD* $(u \Pi_{i=1}^{t} n_i^{ui})$ and $(G, \mathcal{B}_G)$ be a *GD* $(n_i, G, \lambda)$ for each $G \in \mathcal{G}$ with $|G| = n_i$. Then $\mathcal{B}_0 \cup (\cup_{G \in \mathcal{G} \setminus \{G_0\}} \mathcal{B}_G)$ is a block set of the *IGD* $(u + \Sigma_{1 \le i \le t}\, u_i n_i, u, G, \lambda)$, where $G_0 \in \mathcal{G}$ and $|G_0| = u$. Indeed, every pair $\{x, y\}$ of distinct elements which satisfy $x, y \in \cup_{G \in \mathcal{G} \setminus \{G_0\}} G$ or $x \in G_0$ and $y \in \cup_{G \in \mathcal{G} \setminus \{G_0\}} G$ occurs in exactly $\lambda$ blocks of $\mathcal{B}_0 \cup (\cup_{G \in \mathcal{G} \setminus \{G_0\}} \mathcal{B}_G)$. This completes the proof.

**Theorem 2.2:** If there exist a $(G, \lambda)$-*HGD* $(\Pi_{i=1}^{t} n_i^{ui})$ and an *IGD* $(n_i + w, w, G, \lambda)$ for $i \in [1, t]$, then there exists an *IGD* $(w + \Sigma_{1 \le i \le t}\, u_i n_i, w, G, \lambda)$.

**Proof:** Let $(X, \mathcal{G}, \mathcal{B}_0)$ be a $(G, \lambda)$-*HGD* $(\Pi_{i=1}^{t} n_i^{ui})$. By hypothetical conditions of the theorem, there exists an *IGD* $(n_i + w, w, G, \lambda)$ for $i \in [1, t]$. For each $G \in \mathcal{G}$ with $|G| = n_i$, let $\mathcal{B}_G$ be the block set of the *IGD* $(n_i + w, w, G, \lambda)$. We can prove that $\mathcal{B}_0 \cup (\cup_{G \in \mathcal{G}} \mathcal{B}_G)$ is a block set of the *IGD* $(w + \Sigma_{1 \le i \le t}\, u_i n_i, w, G, \lambda)$.

**Theorem 2.3:** If there exist a $(G, \lambda)$-*HGD* $(\Pi_{i=1}^{t} n_i^{ui})$ and an *IGD* $(n_i + w, w, G, \lambda)$ for $i \in [2, t]$, then there exists an *IGD* $(w + \Sigma_{1 \le i \le t}\, n_i, n_1 + w, G, \lambda)$.

**Theorem 2.4:** (1). If there exist a *GD* $(v, G, \lambda_1)$ and a *GD* $(v, G, \lambda_2)$, then there exists a *GD* $(v, G, \lambda_1 + \lambda_2)$.

(2) If there exist an *IGD* $(v, w, G, \lambda_1)$ and an *IGD* $(v, w, G, \lambda_2)$, then there exists an *IGD* $(v, w, G, \lambda_1 + \lambda_2)$.

(3) If there exists a *GD* $(v, G, \lambda)$, then there exists a *GD* $(v, G, s\lambda)$ for any integer $s \ge 1$.

(4) If there exists an *IGD* $(v, w, G, \lambda)$, then there exists an *IGD* $(v, w, G, s\lambda)$ for any integer $s \ge 1$.

The technique of filling in holes will be useful in our constructions.

**Theorem 2.5:** If there exist an *IGD* $(n, w, G, \lambda)$ and a *GD* $(w, G, \lambda)$, then there exists a *GD* $(n, G, \lambda)$.

**Theorem 2.6:** If there exists a $(G, \lambda)$-$HGD$ $(\Pi^t_{i=1} n_i^{u_i})$, then

(i) there exists a $GD(\Sigma_{1 \le i \le t} u_i n_i, G, \lambda)$ when a $GD(n_i, G, \lambda)$ exists for any $i \in [1, t]$;

(ii) there exists a $GD(1 + \Sigma_{1 \le i \le t} u_i n_i, G, \lambda)$ when a $GD(n_i + 1, G, \lambda)$ exists for any $i \in [1, t]$.

**Proof:** The result (i) is obtained by the method of filling in holes. For part (ii), let $(X, \mathcal{G}, \mathcal{B}_0)$ be a $(G, \lambda)$-$HGD(\Pi^t_{i=1} n_i^{u_i})$ and $GD(n_i + 1, G, \lambda) = (G \cup \{a\}, \mathcal{B}_G)$ for $G \in \mathcal{G}$, where $a \notin X$ and $|G| = n_i$. Then $\mathcal{B}_0 \cup (\cup_{G \in \mathcal{G}} \mathcal{B}_G)$ is a block set of the $GD(1 + \Sigma_{1 \le i \le t} u_i n_i, G, \lambda)$.

The following theorem gives a powerful construction for $G$-$HGD$ from $k$-$HGD$.

**Theorem 2.7:** (weighting) Let $(X, \mathcal{G}, \mathcal{B})$ be a $k$-$HGD$ and let $w : X \to Z^+ \cup \{0\}$ be a weight function on $X$. Suppose that for each block $B \in \mathcal{B}$, there exists a $G$-$HGD$ of type $\{w(x) : x \in B\}$. Then there exists a $G$-$HGD$ of type $\{\Sigma_{x \in G} w(x) : G \in \mathcal{G}\}$.

**Theorem 2.8:** Suppose that there exists a resolvable $TD(t, m)$. For $i \in [1, m]$, suppose that there exists an integer $n_i \ge 0$ such that an $IGD(n_i + t, n_i, G)$ exists. Further if an $IGD(m + n_0, n_0, G)$ exists, then

(i) there exists an $IGD(n + tm, n, G)$;

(ii) there exists a $GD(n + tm, G)$ when a $GD(n, G)$ exists.

Where $n = \Sigma_{0 \le i \le m} n_i$.

**Proof:** A resolvable $TD(t, m)$ admits $m$ parallel classes of blocks of size $t$, say $\{B_{ri} | i \in [1, m]\}$ $r \in [1, m]$, and one parallel class (of groups) of size $m$, say $\{B_{0i} | i \in [1, t]\}$. Suppose that the point set of $TD(t, m)$ is $X$, and $X, X_0, X_1, ..., X_m$ are pairwise disjoint with $|X_i| = n_i$, $i \in [0, m]$. For each $0 \le r, i \le m$, let $(B_{ir} \cup X_i, \mathcal{B}_{ir})$ be an $IGD(n_i + t, n_i, G)$. Then $(X \cup (\cup_{0 \le r \le m} X_r), \cup_{0 \le r, i \le m} \mathcal{B}_{ri})$ is an $IGD(n + tm, n, G)$. Indeed, every pair $\{x, y\}$ of points not contained in $\cup_{0 \le r \le m} X_r$ occurs in exactly one block. Let $G$ be a $(p, q)$ graph. The number of blocks in $\cup_{0 \le r, i \le m} \mathcal{B}_{ri}$ is

$$m\left[\frac{t(t-1) + 2n_1 t}{2q} + ... + \frac{t(t-1) + 2n_m t}{2q}\right] + t\,\frac{m(m-1) + 2n_0 m}{2q}$$

$$= \frac{mt}{2q}[(t-1)m + 2n + (m-1)]$$

$$= \frac{tm(tm-1)}{2q} + \frac{tmn}{q}.$$

This value is the number of blocks of the $IGD(n + tm, n, G)$.

Part (ii) can be obtained by above (i) and Theorem 2.5.

## 3. RECURSIVE CONSTRUCTIONS OF $GD(v, Q_n, \lambda)$

In this section, we consider the existence problem of $GD(v, Q_n, \lambda)$ for odd number $n$. A quasigroup $(Q, \circ)$ is a set $Q$ and a binary operation "o" such that for every $a, b \in Q$, the equation $a \circ x = b$ and $y \circ a = b$ have unique solutions $x, y \in Q$. A quasigroup $(Q, \circ)$ is commutative if $x \circ y = y \circ x$ for all $x, y \in Q$ and a quasigroup $(Q, \circ)$ is idempotent if $x \circ x = x$ for all $x \in Q$. A quasigroup $(Q, \circ)$ has holes of size 2 if there are subquasigroups, each of size 2, that partition $Q$. It is easy to see that a quasigroup is idempotent if and only if it has holes of size 1.

**Theorem 3.1:** (see [10]) Idempotent quasigroups exist for all orders except 2 and idempotent, commutative quasigroups exist for all odd orders. Quasigroups with holes of size 2, and commutative quasigroups with holes of size 2, exist for all even orders greater than or equal to 6.

**Theorem 3.2:** There exists a $Q_n$-$HGD((4n)^t)$ for any $t \geq 3$.

**Proof:** By Theorem 3.1, the commutative quasigroups with holes of size 2 exist for all even orders greater than or equal to 6. Let $(G, \circ)$ be a commutative quasigroup with holes $\{2i - 1, 2i\}$ for $i \in [1, t]$ on the set $[1, 2t]$, where $t \geq 3$. On the set $X = Z_{2n} \times [1, 2t]$, for each unordered pair $\{a, b\}$, $a, b \in [1, 2t]$ and $a$, $b$ not in the same hole of $G$, define $(g(u_1), g(u_2), \ldots, g(u_n); g(v_1), g(v_2), \ldots, g(v_n))$ mod $(2n, -)$, where

$$g(u_k) = \begin{cases} ((k-1)/2, a), & k \in [1, n-2]_2 \\ (n+1-k)/2, b), & k \in [2, n-1]_2 \\ ((n-1)/2, a \circ b), & k = n \end{cases}$$

$$g(v_k) = \begin{cases} (k-1, a \circ b), & k \in [1, n-2]_2 \\ (2n+1-k, a), & k \in [2, n-1]_2 \\ ((n+1)/2, a), & k = n \end{cases}$$

Since the number of unordered pair not in the same hole of $[1, 2t]$ is $t(2t - 2)$, the construction yields $2nt(2t - 2)$ blocks. The $2nt(2t - 2)$ blocks form a $Q_n$-$HGD((4n)^t)$.

**Theorem 3.3:** If there exist an $IGD(4n + w, w, Q_n, \lambda)$ and a $GD(4n + w, Q_n, \lambda)$, then for any $t \geq 3$, there exists a $GD(4nt + w, Q_n, \lambda)$.

**Proof:** Since a $Q_n$-$HGD((4n)^t)$ exists, a $(Q_n, \lambda)$-$HGD((4n)^t)$ exists by Theorem 2.4. Since, also, an $IGD(4n + w, w, Q_n, \lambda)$ exists, an $IGD(4nt + w, 4n + w, Q_n, \lambda)$ exists from Theorem 2.3. Filling $GD(4n + w, Q_n, \lambda)$ in the hole of length $4n + w$ of the $IGD(4nt + w, 4n + w, Q_n, \lambda)$ form a $GD(4nt + w, Q_n, \lambda)$.

**Theorem 3.4:** If $v$ is an odd number and $v(v-1) \equiv 0 \pmod{2n}$, then there exists an $IGD\left(\frac{3v-1}{2}, \frac{v-1}{2}, Q_n, \lambda\right)$ for any positive integer $\lambda$.

**Proof:** When $v$ is an odd number and $v(v-1) \equiv 0 \pmod{2n}$, a $GD(v, C_n, 1)$ exists by Theorem 1.2. Let $GD(v, C_n, 1) = (V, \mathcal{B})$, $X = V \cup U$, where $|V| = v$ and $|U| = (v-1)/2$. We partition the set $\{xy \mid x \in U, y \in V\}$ into $\frac{v(v-1)}{2n}$ sets, let $\mathcal{A}$ consist of the $\frac{v(v-1)}{2n}$ sets, such that $\{x_1 y_1, x_2 y_2, ..., x_n y_n\} \in \mathcal{A}$ if and only if $(y_1, y_2, ..., y_n) \in \mathcal{B}$. Together $\{x_1 y_1, x_2 y_2, ..., x_n y_n\}$ with the cycle $(y_1, y_2, ..., y_n)$ form a crown graph $Q_n$. Since $|\mathcal{B}| = |\mathcal{A}| = \frac{v(v-1)}{2n}$, the $\frac{v(v-1)}{2n}$ crown graphs generate the required block set of $IGD\left(\frac{3v-1}{2}, \frac{v-1}{2}, Q_n, 1\right)$.

By Theorem 2.4 there exists an $IGD\left(\frac{3v-1}{2}, \frac{v-1}{2}, Q_n, \lambda\right)$ for any positive integer $\lambda$.

Given a graph $G = (V, E)$ with points in $Z_v$, if $V = \{v_1, v_2, ..., v_n\}$, then $G$ is denoted by $[v_1, v_2, ..., v_n]$. The list of differences from $G$ is the multiset $\Delta G = \{\pm(x-y) \mid x, y \in V, xy \in E\}$. We call $(K_v, G)$-difference system ($DS$ in short) any set $\mathcal{F} = \{G_1, G_2, ..., G_t\}$ of $G$-blocks (base blocks) with points $Z_v$ such that the multiset $\Delta\mathcal{F} = \cup_{i=1}^{t} \Delta G_i$ covers each nonzero element of $Z_v$ exactly once. The terminology is justified by the fact:

**Theorem 3.5:** When $v$ is odd, then any $(K_v, G)$-$DS$ generates a $GD(v, G, 1)$.

**Proof:** If $(Z_v, +)$ is a group and $B$ is a base block, we denote $\{B + x \mid x \in Z_v\}$ by $dev\, B$. Then $dev\, \mathcal{F} = \cup_{i=1}^{t} dev\, G_i$ is the block set of $GD(v, G, 1)$. □

When $C_n$ is a $n$-cycle $[v_1, v_2, ..., v_n]$, the $\Delta C_n = \{\pm(v_i - v_{i-1}) \mid i \in [1, n]\}$ where $v_0 = v_n$. The sequence $(v_2 - v_1, v_3 - v_2, ..., v_n - v_{n-1}, v_1 - v_n)$ is called difference sequence of the cycle $C_n$.

**Theorem 3.6:** Let $n \geq 5$ be an odd number. When $v \equiv 0, 1 \pmod{4n}$, there exists a $GD(v, Q_n, \lambda)$ for any positive integer $\lambda$.

**Proof:** We construct the difference sequence as follows:

$$A_i = \begin{pmatrix} (n-1)m+i, m+i, -(2m+i), ..., (-1)^{j-1}(jm+i), ..., [(n-2)m+i], \\ -\left[\dfrac{3(n-1)m}{2} + 2i\right] \end{pmatrix}, \ i \in [1, m].$$

It is not difficult to see that these differences are pairwise distinct because of $mn < \frac{3(n-1)m}{2} + 2i \leq 2mn$ for any $i \in [1, m]$.

When $v \equiv 1 \pmod{4n}$, let $v = 4mn + 1$ and point set be $Z_{4mn+1}$. Since $[(n-1)m + i] + (m + i) - (2m + i) + \ldots + (-1)^{j-1}(jm + i) + \ldots + [(n-2)m + i] - \left[\frac{3(n-1)m}{2} + 2i\right] = 0$, above each $A_i$ ($i \in [1, m]$) form a difference sequence of $n$-cycle. The set of absolute value of these differences is $[m + 1, mn] \cup \left\{\frac{3(n-1)m}{2} + 2i \mid i \in [1, m]\right\}$. We partition differences in $[1, m] \cup \left([mn+1, 2mn] \setminus \left\{\frac{3(n-1)m}{2} + 2i \mid i \in [1, m]\right\}\right)$ into $m$ sets, $B_i$, $i \in [1, m]$. Then every $\pm(A_i \cup B_i)$ ($i \in [1, m]$) forms a list of differences from $Q_n$. Let the $m$ base blocks be $Q_n^i$ satisfying $\Delta Q_n^i = \pm(A_i \cup B_i)$ for $i \in [1, m]$, and $\mathcal{F} = \{Q_n^i \mid i \in [1, m]\}$. It is not difficult to see that the multiset $\Delta\mathcal{F} = \cup_{i=1}^{m} \Delta Q_n^i$ covers each nonzero element of $Z_v$ exactly once. Therefore, the $\Delta\mathcal{F}$ is a $(K_v, Q_n)$-*DS*. This *DS* generates a $GD(v, Q_n, 1)$ by Theorem 3.5.

When $v \equiv 0 \pmod{4n}$, let $v = 4mn$ and point set be $Z_{4mn-1} \cup \{\infty\}$. $A_i$, $i \in [1, m]$ are same with above. Let $B_1 = \{1, 2, \ldots, m-1, \infty\}$. We partition $[1, 2mn-1] \setminus \left([1, m-1], \cup\ [m+1, mn] \cup \left\{\frac{3(n-1)m}{2} + 2i \mid i \in [1, m]\right\}\right)$ into $m - 1$ sets $B_i$, $i \in [2, m]$. Let the $m$ base blocks be $Q_n^i$ satisfying $\Delta Q_n^i = \pm(A_i \cup B_i)$ for $i \in [1, m]$, and $F = \{Q_n^i \mid i \in [1, m]\}$. It is not difficult to see that the multiset $\Delta\mathcal{F} = \cup_{i=1}^{m} \Delta Q_n^i$ covers each nonzero element of $Z_{v-1}$ exactly once. Then the *DS* forms the block set of $GD(v, Q_n, 1)$, where $\infty + x = \infty$ for any $x \in Z_{4mn-1}$. It follows from Theorem 2.4 that a $GD(v, Q_n, \lambda)$ exists for any positive integer $\lambda$ and $v \equiv 0, 1 \pmod{4n}$.

**Theorem 3.7:** (see [6])When integer $n \geq 4$ and positive integer $\lambda \not\equiv 0 \pmod 4$, there does not exist $GD(2n, Q_n, \lambda)$.

**Theorem 3.8:** Let $n$ be an odd number and $n \geq 5$. Then a $GD(2n, Q_n, \lambda)$ exists if and only if $\lambda \equiv 0 \pmod 4$.

**Proof:** By Theorem 3.7, there does not exist $GD(2n, Q_n, \lambda)$ when $\lambda \not\equiv 0 \pmod 4$. When $\lambda = 4$, on the point set $Z_{2n-1} \cup \{\infty\}$, we construct two base blocks, $B_i = (f(u_{i,1}), f(u_{i,2}), \ldots, f(u_{i,n}); f(u_{i,n+1}), f(u_{i,n+2}), \ldots, f(u_{i,2n}))$, $i = 1,2$, as follows:

$$f(u_{1,i}) = \begin{cases} (i-1)/2, & i \in [1, n]_2 \\ n - i/2, & i \in [2, n-1]_2 \end{cases}$$

$$f(u_{1,n+i}) = \begin{cases} \infty, & i = 1 \\ 2n - 1 - (i-1)/2, & i \in [3, n]_2 \setminus \{(n+1)/2, (n+3)/2\} \\ n - 1 + i/2, & i \in [2, n-1]_2 \setminus \{(n+1)/2, (n+3)/2\} \end{cases}$$

When $n \equiv 1 \pmod 4$

$$f(u_{1,n+i}) = \begin{cases} (5n-1)/4, & i = (n+1)/2 \\ (7n-3)/4, & i = (n+3)/2 \end{cases}$$

When $n \equiv 3 \pmod 4$

$$f(u_{1,n+i}) = \begin{cases} (7n-5)/4, & i = (n+1)/2 \\ (5n-3)/4, & i = (n+3)/2 \end{cases}$$

$$f(u_{2,i}) = \begin{cases} n-2-(i-1)/2, & i \in [1, n-2]_2 \\ i/2, & i \in [2, n-3]_2 \\ \infty, & i = n-1 \\ 0, & i - n \end{cases}$$

$$f(u_{2,n+i}) = \begin{cases} n-1+(i-1)/2, & i \in [1, n]_2 \\ 2n-1-i/2, & i \in [2, n-1]_2 \end{cases}$$

By above constructions, we can obtain $\{f(u_{1,i}) \mid i \in [1, n]\} \cup \{f(u_{1,n+i}) \mid i \in [1, n]\} = Z_{2n-1} \cup \{\infty\}$. This shows that the points contained in first base block are pairwise distinct. Similarly, the point set of second base block is also $Z_{2n-1} \cup \{\infty\}$.

It can be checked that every difference in the set $[1, n-1]$ and the pair $(x, \infty)$ occur 4 times in the two base blocks. The two base blocks mod $2n-1$ form a block set of $GD(2n, Q_n, 4)$. From Theorem 2.4, there exists a $GD(2n, Qn, \lambda)$ for any positive integer $\lambda \equiv 0$ mod 4.

**Theorem 3.9:** Let $v$ and $\lambda$ be two positive integers. Then there exists a $GD(v, Q_3, \lambda)$ if and only if (1). $v > 6$ and $\lambda v(v-1) \equiv 0 \pmod{12}$; (2). $v = 6$ and $\lambda \equiv 0 \pmod 4$.

**Proof:** When $\lambda = 4$, on the point set $Z_5 \cup \{\infty\}$, we construct two base blocks as follows: $(0, 2, 1; 4, \infty, 3)$ and $(0, 2, \infty; 1, 4, 3)$. They under the action of $Z_5$ form a block set of $GD(v, Q_3, 4)$. Thus a $GD(v, Q_3, 4\lambda)$ exists for any positive integer $\lambda$ from Theorem 2.4. By Theorems 1.4 and 3.7, we can obtain the theorem.

## 4. CONSTRUCTIONS OF $GD(v, Q_5, \lambda)$

**Lemma 4.1:** There exists an $IGD(20 + w, w, Q_5, \lambda)$ for $w \in \{5, 16\}$ and $\lambda \in \{1, 3, 7, 9\}$.

**Proof:** Let $IGD(v, w, Q_n, \lambda) = (V, \mathcal{B})$.

$IGD(25, 5, Q_5, 1)$, $V = (Z_{20})_0 \cup (Z_5)_1$, $\mathcal{B}$:

$(1_0, 4_0, 5_0, 14_0, 7_0; 1_1, 9_0, 8_0, 2_1, 5_1) + i$, $i \in [0, 18]$,
$(0_0, 3_0, 4_0, 13_0, 6_0; 1_1, 19_0, 7_0, 2_1, 5_1)$, $(3_1, 4_0, 12_0, 4_1, 19_0; 6_0, 16_0, 0_0, 18_0, 7_0)$,
$(3_1, 5_0, 13_0, 4_1, 17_0; 2_0, 15_0, 1_0, 6_0, 9_0)$, $(3_1, 18_0, 14_0, 4_1, 0_0; 15_0, 10_0, 6_0, 7_0, 4_0)$,
$(3_1, 7_0, 15_0, 4_1, 1_0; 14_0, 11_0, 19_0, 8_0, 9_0)$, $(3_1, 8_0, 16_0, 4_1, 11_0; 13_0, 4_0, 12_0, 9_0, 15_0)$,
$(3_1, 9_0, 5_0, 4_1, 3_0; 16_0, 19_0, 17_0, 4_0, 7_0)$, $(1_0, 11_0, 3_0, 13_0, 17_0; 5_0, 19_0, 15_0, 9_0, 7_0)$,
$(8_0, 12_0, 2_0, 10_0, 0_0; 3_0, 3_1, 6_0, 4_1, 16_0)$, $(18_0, 6_0, 10_0, 14_0, 2_0; 8_0, 16_0, 3_1, 4_0, 4_1)$.

$IGD(36, 16, Q_5, 1)$, $V = (Z_{20})_0 \cup (Z_{16})_1$, $\mathcal{B}$:

$(0_0, 1_0, 4_0, 8_0, 6_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i$, $i \in [0, 19]$,
$(6_1, 0_0, 7_1, 7_0, 12_0; 6_0, 8_1, 13_0, 9_1, 10_1) + \mathrm{i}$, $i \in [0, 5]$,
$(11_1, 6_0, 12_1, 13_0, 18_0; 12_0, 8_1, 19_0, 9_1, 10_1) + i$, $i \in [0, 5]$,
$(13_1, 12_0, 14_1, 19_0, 4_0; 18_0, 8_1, 5_0, 9_1, 10_1) + i$, $i \in [0, 5]$,
$(15_1, 0_0, 16_1, 5_0, 14_0; 6_0, 9_0, 19_0, 12_1, 4_0)$,
$(15_1, 1_0, 16_1, 8_0, 15_0; 12_0, 14_0, 16_0, 19_0, 5_0)$,
$(15_1, 13_0, 16_1, 9_0, 16_0; 5_0, 1_0, 7_0, 18_0, 6_0)$,
$(15_1, 3_0, 16_1, 10_0, 17_0; 9_0, 16_0, 15_0, 1_0, 7_0)$,
$(15_1, 4_0, 16_1, 11_0, 18_0; 10_0, 13_0, 14_0, 10_1, 14_1)$,
$(15_1, 2_0, 16_1, 12_0, 19_0; 7_0, 11_0, 18_0, 1_0, 8_1)$,
$(5_0, 10_0, 18_0, 6_0, 13_0; 9_1, 10_1, 6_1, 16_1, 3_0)$,
$(11_0, 19_0, 7_0, 15_0, 3_0; 13_1, 16_1, 14_0, 6_0, 10_0)$,
$(1_0, 11_0, 4_0, 16_0, 8_0; 9_0, 15_1, 15_0, 5_0, 8_0)$,
$(0_0, 12_0, 5_0, 18_0, 7_0; 13_0, 12_1, 11_1, 8_1, 16_0)$,
$(2_0, 12_0, 4_0, 17_0, 9_0; 15_0, 3_0, 11_1, 5_0, 19_0)$,
$(19_0, 6_0, 14_0, 2_0, 10_0; 7_1, 9_1, 3_0, 13_0, 13_1)$,
$(0_0, 11_0, 6_0, 17_0, 8_0; 10_0, 14_1, 7_1, 16_1, 15_1)$.

By Theorem 2.4 the result is true. □

**Lemma 4.2:** There exists an $IGD(20 + w, w, Q_5, \lambda)$ for $w \in \{6, 10, 11, 15\}$ and $\lambda \in \{2, 6\}$.

**Proof:** Let $IGD(v, w, Q_n, \lambda) = (V, \mathcal{B})$. $IGD(26, 6, Q_5, 2)$, $V = (Z_{20})_0 \cup (Z_6)_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 9_0, 4_0; 11, 2_1, 3_1, 4_1, 5_1) + i$, $i \in [0, 19]$,
$(0_0, 9_0, 11_0, 3_0, 7_0; 5_0, 3_1, 4_1, 5_1, 6_1) + i$, $i \in [0, 9]$,

$(10_0, 19_0, 1_0, 13_0, 17_0; 11_0, 3_1, 4_1, 5_1, 6_1) + i, i \in [0, 9]$,
$(0_0, 9_0, 10_0, 13_0, 6_0; 8_0, 1_1, 4_0, 2_1, 6_1) + i, i \in \{0, 1, 5, 6, 7, 8, 9, 10, 13, 14, 15\}$,
$(2_0, 11_0, 12_0, 15_0, 8_0; 3_0, 1_1, 6_0, 1_0, 6_1), (3_0, 12_0, 13_0, 16_0, 9_0; 11_0, 1_1, 18_0, 2_0, 2_1)$,
$(4_0, 13_0, 14_0, 17_0, 10_0; 16_0, 1_1, 8_0, 2_1, 6_1), (11_0, 0_0, 1_0, 4_0, 17_0; 16_0, 1_1, 7_0, 2_1, 6_1)$,
$(12_0, 1_0, 2_0, 5_0, 18_0; 4_0, 1_1, 8_0, 2_1, 6_1), (16_0, 5_0, 6_0, 9_0, 2_0; 2_1, 1_1, 0_0, 6_1, 17_0)$,
$(17_0, 6_0, 7_0, 10_0, 3_0; 12_0, 1_1, 13_0, 2_1, 6_1), (18_0, 7_0, 8_0, 11_0, 4_0; 3_0, 1_1, 9_0, 2_1, 6_1)$,
$(19_0, 8_0, 9_0, 12_0, 5_0; 11_0, 1_1, 3_0, 2_1, 6_1), (19_0, 4_0, 5_0, 6_0, 7_0; 14_0, 3_0, 17_0, 18_0, 8_0)$,
$(10_0, 15_0, 0_0, 1_0, 2_0; 9_0, 1_1, 12_0, 16_0, 6_1)$.

$IGD(30, 10, Q_5, 2)$, $V = (\mathrm{Z}_{20})_0 \cup (\mathrm{Z}_{10})_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 4_0, 5_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 6_0, 3_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 4_0, 6_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 8_0, 13_0, 6_0, 6_1; 7_1, 8_1, 9_1, 10_1, 12_0) + i, i \in [0, 5]$,
$(3_0, 19_0, 4_0, 17_0, 10_1; 7_0, 8_1, 9_1, 5_0, 0_0) + i, i \in [0, 1]$,
$(6_0, 14_0, 19_0, 12_0, 7_1; 10_0, 8_1, 7_0, 16_0, 17_0), (7_0, 15_0, 0_0, 13_0, 7_1; 12_0, 8_1, 9_1, 17_0, 19_0)$,
$(2_0, 18_0, 3_0, 16_0, 10_1; 8_1, 6_1, 9_1, 4_0, 13_0), (8_0, 16_0, 1_0, 14_0, 7_1; 12_0, 8_1, 13_0, 10_1, 18_0)$,
$(9_0, 17_0, 2_0, 15_0, 7_1; 13_0, 8_1, 9_1, 10_1, 16_0), (19_0, 6_0, 9_1, 11_0, 15_0; 6_1, 2_0, 7_0, 4_0, 3_0)$,
$(14_0, 10_0, 9_1, 9_0, 2_0; 18_0, 7_1, 8_0, 4_0, 7_0), (0_0, 12_0, 5_0, 8_1, 7_0; 16_0, 10_1, 9_0, 18_0, 11_0)$,
$(5_0, 1_0, 8_1, 3_0, 10_0; 10_1, 17_0, 4_0, 8_0, 18_0), (1_0, 6_0, 11_0, 19_0, 9_1; 8_0, 8_1, 7_1, 10_1, 12_0)$.

$IGD(31, 11, Q_5, 2)$, $V = (Z_{20})_0 \cup (Z_{11})_1$, $B$:

$(0_0, 9_0, 12_0, 8_0, 2_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 3_0, 6_0; 11_1, 1_1, 2_1, 3_1, 4_1) + i, i \in [0, 19]$,
$(14_0, 4_0, 6_0, 18_0, 19_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 18]$,
$(13_0, 1_0, 6_0, 19_0, 11_1; 6_1, 7_1, 8_1, 9_1, 5_0) + i, i \in [0, 3]$,
$(8_0, 16_0, 1_0, 14_0, 10_1; 6_1, 7_1, 8_1, 9_1, 0_0), (7_0, 15_0, 0_0, 16_0, 10_1; 19_0, 7_1, 5_0, 9_0, 18_0)$,
$(13_0, 3_0, 5_0, 17_0, 18_0; 1_1, 2_1, 3_1, 4_1, 2_0), (12_0, 0_0, 8_1, 18_0, 11_1; 6_1, 7_1, 5_0, 3_0, 4_0)$,
$(17_0, 5_0, 10_0, 3_0, 11_1; 6_1, 7_1, 8_1, 16_0, 9_0), (11_0, 19_0, 4_0, 17_0, 10_1; 6_0, 7_1, 8_1, 9_1, 13_0)$,
$(6_0, 14_0, 19_0, 12_0, 10_1; 18_0, 7_1, 8_1, 9_1, 19_0), (1_0, 9_0, 14_0, 7_0, 5_1; 6_1, 7_1, 8_1, 9_1, 13_0)$,
$(10_0, 18_0, 9_1, 3_0, 10_1; 6_1, 7_1, 16_0, 8_1, 2_0), (0_0, 8_0, 13_0, 6_0, 5_1; 6_1, 7_1, 8_1, 9_1, 9_0)$,
$(2_0, 10_0, 15_0, 8_0, 5_1; 6_0, 7_1, 8_1, 9_1, 18_0), (3_0, 11_0, 16_0, 12_0, 5_1; 7_0, 7_1, 8_1, 8_0, 19_0)$,
$(4_0, 12_0, 17_0, 10_0, 5_1; 8_0, 7_1, 8_1, 9_1, 18_0), (5_0, 13_0, 9_1, 11_0, 5_1; 10_1, 7_1, 4_0, 18_0, 16_0)$,

$(5_0, 12_0, 7_0, 11_0, 6_1; 9_1, 8_1, 7_1, 11_1, 7_0), (18_0, 6_1, 6_0, 10_0, 14_0; 13_0, 3_0, 7_1, 11_1, 5_1),$
$(19_0, 15_0, 11_0, 4_0, 6_1; 3_0, 5_1, 8_1, 0_0, 2_0), (9_0, 17_0, 2_0, 15_0, 10_1; 6_1, 7_1, 8_1, 9_1, 4_0),$
$(9_0, 13_0, 17_0, 1_0, 5_0; 9_1, 0_0, 5_1, 10_1, 18_0).$

$IGD(35, 15, Q_5, 2)$, $V = (Z_{20})_0 \cup (Z_{15})_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 8_0, 7_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19],$
$(0_0, 9_0, 2_0, 10_0, 4_0; 11_1, 12_1, 13_1, 14_1, 15_1) + i, i \in [0, 19],$
$(13_0, 2_0, 5_0, 0_0, 19_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 18],$
$(11_0, 16_0, 18_0, 3_0, 15_1; 12_1, 13_1, 15_0, 14_1, 17_0) + i, i \in [0, 5],$
$(13_0, 1_0, 8_1, 7_0, 9_1; 3_1, 4_1, 14_0, 5_1, 19_0) + i, i \in [0, 4],$
$(1_0, 9_0, 1_1, 15_0, 2_1; 3_1, 4_1, 2_0, 5_1, 7_0) + i, i \in [0, 3],$
$(9_0, 17_0, 7_1, 3_0, 6_1; 3_1, 4_1, 10_0, 5_1, 15_0) + i, i \in [0, 2] \cup \{17\},$
$(7_0, 12_0, 10_1, 19_0, 11_1; 12_1, 13_1, 5_0, 14_1, 13_0) + i, i \in [0, 2],$
$(5_0, 10_0, 10_1, 17_0, 11_1; 12_1, 13_1, 9_0, 14_1, 11_0),$
$(6_0, 11_0, 10_1, 18_0, 11_1; 12_1, 13_1, 16_0, 14_1, 12_0),$
$(10_0, 15_0, 10_1, 2_0, 11_1; 7_0, 13_1, 8_0, 14_1, 16_0),$
$(14_0, 11_0, 14_1, 13_0, 6_1; 17_0, 8_0, 10_0, 8_1, 12_0),$
$(13_0, 16_0, 14_1, 12_0, 10_0; 2_1, 4_1, 14_0, 5_1, 15_1),$
$(12_0, 0_0, 8_1, 6_0, 9_1; 3_1, 4_1, 19_0, 5_1, 18_0), (0_0, 8_0, 1_1, 14_0, 2_1; 3_1, 4_1, 7_0, 5_1, 6_0),$
$(5_0, 13_0, 1_1, 19_0, 2_1; 3_1, 4_1, 0_0, 5_1, 12_0), (7_0, 15_0, 7_1, 1_0, 6_1; 3_1, 4_1, 8_0, 5_1, 18_0),$
$(8_0, 16_0, 7_1, 2_0, 6_1; 3_1, 14_0, 13_0, 5_1, 19_0), (19_0, 7_0, 9_0, 4_0, 12_1; 3_1, 4_1, 7_1, 9_1, 10_0),$
$(6_0, 4_0, 7_0, 5_0, 8_0; 4_1, 11_1, 13_1, 9_1, 10_0), (17_0, 12_1, 1_0, 4_0, 2_0; 15_0, 0_0, 1_1, 10_1, 13_1),$
$(3_0, 5_0, 13_1, 9_0, 6_0; 11_1, 2_0, 8_0, 14_1, 7_1), (13_0, 11_0, 9_0, 12_0, 15_0; 5_1, 2_1, 15_1, 8_1, 14_1),$
$(18_0, 6_0, 13_1, 3_0, 12_1; 3_1, 1_1, 4_0, 10_1, 2_0).$

By Theorem 2.4 the result is true. □

**Lemma 4.3:** There exists an $IGD(20 + w, w, Q_5, 5)$ for $w \in \{4, 8, 9, 12, 13\}$.

**Proof:** Let $IGD(v, w, Q_n, \lambda) = (V, \mathcal{B})$. $IGD(24, 4, Q_5, 5)$, $V = (Z_{20})_0 \cup (Z_4)_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 4_0, 5_0; 1_1, 2_1, 3_1, 4_1, 2_0) + i, i \in [0, 19],$
$(0_0, 10_0, 1_0, 8_0, 2_0; 1_1, 2_1, 3_1, 4_1, 6_0) + i, i \in [0, 19],$
$(0_0, 9_0, 12_0, 5_0, 6_0; 1_1, 1_0, 2_1, 3_1, 4_1) + i, i \in [0, 19],$
$(0_0, 9_0, 11_0, 7_0, 2_0; 5_0, 1_0, 6_0, 4_0, 8_0) + i, i \in [0, 19],$
$(0_0, 9_0, 2_0, 6_0, 4_0; 1_1, 1_0, 2_1, 3_1, 4_1) + i, i \in [0, 19],$

$(13_0, 2_0, 16_0, 19_0, 0_0; 6_0, 14_0, 17_0, 3_0, 1_0) + i, i \in [0, 18]$,
$(0_0, 3_0, 8_0, 14_0, 1_1; 10_0, 2_1, 3_1, 4_1, 6_0) + i, i \in [0, 2] \cup \{5\}$,
$(12_0, 1_0, 15_0, 18_0, 19_0; 5_0, 13_0, 16_0, 4_0, 0_0), (3_0, 6_0, 11_0, 17_0, 1_1; 9_0, 2_1, 3_1, 4_1, 10_0)$,
$(4_0, 7_0, 12_0, 18_0, 1_1; 14_0, 2_1, 3_1, 4_1, 9_0), (6_0, 9_0, 14_0, 0_0, 4_1; 16_0, 2_1, 3_1, 15_0, 4_0)$,
$(7_0, 10_0, 15_0, 1_0, 4_1; 17_0, 2_1, 3_1, 18_0, 12_0), (8_0, 11_0, 16_0, 2_0, 4_1; 18_0, 2_1, 3_1, 17_0, 13_0)$,
$(9_0, 12_0, 17_0, 3_0, 4_1; 19_0, 2_1, 3_1, 13_0, 5_0), (12_0, 15_0, 2_1, 1_0, 6_0; 1_1, 18_0, 17_0, 7_0, 0_0)$,
$(13_0, 16_0, 2_1, 2_0, 7_0; 1_1, 1_0, 18_0, 8_0, 3_1), (2_0, 19_0, 2_1, 0_0, 3_1; 18_0, 5_0, 14_0, 17_0, 1_0)$,
$(19_0, 14_0, 11_0, 5_0, 3_1; 16_0, 17_0, 4_1, 0_0, 3_0), (18_0, 3_1, 4_0, 10_0, 13_0; 3_0, 6_0, 19_0, 4_1, 2_1)$.

$IGD(28, 8, Q_5, 5)$, $V = (Z_{20})_0 \cup (Z_8)_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 4_0, 5_0; 1_1, 2_1, 3_1, 4_1, 5_1) + \mathrm{i}, \mathrm{i} \in [0, 19]$,
$(0_0, 10_0, 7_0, 11_0, 2_0; 6_1, 7_1, 8_1, 1_1, 2_1) + \mathrm{i}, \mathrm{i} \in [0, 19]$,
$(0_0, 9_0, 15_0, 8_0, 5_0; 3_1, 4_1, 5_1, 6_1, 7_1) + \mathrm{i}, \mathrm{i} \in [0, 19]$,
$(0_0, 9_0, 2_0, 8_0, 7_0; 8_1, 1_1, 2_1, 3_1, 4_1) + \mathrm{i}, \mathrm{i} \in [0, 19]$,
$(0_0, 9_0, 5_0, 4_0, 8_0; 5_1, 6_1, 7_1, 8_1, 1_1) + \mathrm{i}, \mathrm{i} \in [0, 19]$,
$(0_0, 8_0, 2_0, 7_0, 4_0; 2_1, 3_1, 4_1, 5_1, 6_1) + \mathrm{i}, \mathrm{i} \in [0, 19]$,
$(0_0, 9_0, 2_0, 7_0, 5_0; 7_1, 8_1, 1_1, 2_1, 3_1) + \mathrm{i}, \mathrm{i} \in [0, 19]$,
$(3_0, 11_0, 5_0, 7_0, 4_0; 10_0, 12_0, 8_0, 4_1, 6_0) + \mathrm{i}, \mathrm{i} \in [0, 14] \cup \{16, 17, 18\}$,
$(1_0, 5_0, 13_0, 7_0, 5_1; 11_0, 6_1, 7_1, 8_1, 14_0) + \mathrm{i}, \mathrm{i} \in \{0, 3, 4\}$,
$(2_0, 10_0, 4_0, 6_0, 3_0; 8_0, 11_0, 7_0, 4_1, 5_0), (18_0, 6_0, 0_0, 2_0, 9_0; 5_0, 10_0, 3_0, 14_0, 1_0)$,
$(0_0, 4_0, 12_0, 6_0, 5_1; 14_0, 6_1, 8_0, 8_1, 13_0), (2_0, 6_0, 14_0, 8_0, 5_1; 4_1, 6_1, 10_0, 8_1, 12_0)$,
$(3_0, 7_0, 15_0, 9_0, 5_1; 11_0, 6_1, 7_1, 8_1, 16_0), (12_0, 16_0, 4_0, 18_0, 8_1; 2_0, 6_1, 7_1, 10_0, 3_0)$,
$(13_0, 17_0, 5_0, 19_0, 8_1; 3_0, 6_1, 7_1, 11_0, 2_0), (14_0, 18_0, 6_0, 0_0, 8_1; 7_1, 6_1, 16_0, 10_0, 4_0)$,
$(15_0, 19_0, 7_0, 1_0, 8_1; 11_0, 6_1, 6_0, 13_0, 5_0), (12_0, 18_0, 7_1, 10_0, 6_1; 0_0, 2_0, 6_0, 4_0, 14_0)$,
$(16_0, 0_0, 8_0, 7_1, 2_0; 8_1, 6_1, 18_0, 12_0, 10_0), (30, 190, 71, 110, 61; 17_0, 5_1, 1_0, 5_0, 2_0)$,
$(3_0, 15_0, 6_1, 13_0, 9_0; 7_1, 5_1, 1_0, 19_0, 2_0), (17_0, 1_0, 9_0, 7_1, 7_0; 8_1, 15_0, 19_0, 0_0, 11_0)$.

$IGD(29, 9, Q_5, 5)$, $V = (Z_{20})_0 \cup (Z_9)_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 4_0, 7_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 8_0, 4_0; 6_1, 7_1, 8_1, 9_1, 1_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 11_0, 6_0, 5_0; 2_1, 3_1, 4_1, 5_1, 6_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 5_0, 6_0; 7_1, 8_1, 9_1, 1_1, 2_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 4_0, 8_0; 3_1, 4_1, 5_1, 6_1, 7_1) + i, i \in [0, 19]$,

$(0_0, 9_0, 2_0, 8_0, 5_0; 8_1, 9_1, 1_1, 2_1, 3_1) + i, i \in [0, 19],$
$(0_0, 8_0, 14_0, 7_0, 3_0; 4_1, 5_1, 6_1, 7_1, 8_1) + i, i \in [0, 19],$
$(0_0, 8_0, 14_0, 5_0, 2_0; 9_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19],$
$(17_0, 4_0, 3_0, 2_0, 1_0; 6_1, 7_1, 8_1, 1_1, 6_0) + i, i \in [0, 19]/\{4, 14, 15, 16, 18, 19\},$
$(8_0, 18_0, 16_0, 14_0, 9_1; 10_0, 0_0, 6_0, 12_0, 17_0) + i, i \in [0, 1],$
$(0_0, 2_0, 40, 6_0, 9_1; 10_0, 7_1, 14_0, 8_0, 3_0), (1_0, 3_0, 5_0, 7_0, 9_1; 16_0, 13_0, 15_0, 9_0, 4_0),$
$(9_1, 11_0, 6_1, 12_0, 10_0; 19_0, 1_0, 13_0, 2_0, 5_0), (1_0, 8_0, 7_0, 6_0, 5_0; 6_1, 7_1, 8_1, 1_1, 9_1),$
$(11_0, 18_0, 17_0, 16_0, 15_0; 13_0, 7_1, 8_1, 1_1, 0_0), (16_0, 3_0, 2_0, 1_0, 0_0; 9_1, 7_1, 8_1, 1_1, 5_0),$
$(13_0, 0_0, 19_0, 18_0, 17_0; 9_1, 7_1, 8_1, 1_1, 2_0), (15_0, 2_0, 1_0, 0_0, 19_0; 6_1, 9_1, 8_1, 1_1, 4_0),$
$(12_0, 19_0, 17_0, 18_0, 16_0; 9_1, 7_1, 8_1, 1_1, 6_1).$

$IGD(32, 12, Q_5, 5)$, $V = (Z_{20})_0 \cup (Z_{12})_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 8_0, 7_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19],$
$(0_0, 9_0, 12_0, 6_0, 5_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19],$
$(0_0, 9_0, 1_0, 6_0, 8_0; 11_1, 12_1, 1_1, 2_1, 3_1) + i, i \in [0, 19],$
$(0_0, 10_0, 3_0, 7_0, 6_0; 4_1, 5_1, 6_1, 7_1, 8_1) + i, i \in [0, 19],$
$(0_0, 9_0, 12_0, 4_0, 6_0; 9_1, 10_1, 11_1, 12_1, 1_1) + i, i \in [0, 19],$
$(0_0, 9_0, 2_0, 6_0, 8_1; 2_1, 3_1, 4_1, 5_1, 1_0) + i, i \in [0, 19],$
$(0_0, 9_0, 11_0, 6_0, 7_1; 1_1, 2_1, 3_1, 4_1, 1_0) + i, i \in [0, 19],$
$(0_0, 8_0, 9_1, 4_0, 10_1; 5_1, 6_1, 2_0, 11_1, 3_0) + i, i \in [0, 19],$
$(0_0, 8_0, 11_0, 4_0, 12_1; 11_1, 3_0, 7_0, 1_0, 2_0) + i, i \in [0, 19],$
$(0_0, 7_0, 11_0, 9_0, 6_0; 1_1, 2_1, 3_1, 8_0, 6_1) + i, i \in [0, 19],$
$(1_0, 2_0, 4_1, 15_0, 5_1; 6_0, 8_0, 9_0, 14_0, 7_0) + i, i \in \{0, 13\},$
$(0_0, 1_0, 4_1, 14_0, 5_1; 5_0, 7_0, 11_0, 13_0, 6_0), (11_0, 12_0, 6_1, 5_0, 11_1; 16_0, 18_0, 1_0, 15_0, 3_0),$
$(3_0, 4_0, 4_1, 17_0, 5_1; 13_0, 10_0, 7_0, 16_0, 9_0), (9_0, 10_0, 6_1, 17_0, 11_1; 14_0, 16_0, 3_0, 2_0, 1_0),$
$(6_0, 7_0, 6_1, 0_0, 11_1; 11_0, 13_0, 4_0, 10_0, 12_0), (7_0, 8_0, 6_1, 19_0, 11_1; 12_0, 4_1, 15_0, 14_0, 4_0),$
$(8_0, 9_0, 6_1, 2_0, 11_1; 14_0, 15_0, 16_0, 12_0, 13_0), (5_0, 6_0, 4_1, 19_0, 5_1; 10_0, 12_0, 0_0, 9_0, 11_0),$
$(19_0, 4_0, 18_0, 8_0, 13_0; 5_0, 14_0, 11_1, 5_1, 4_1), (2_0, 3_0, 4_1, 16_0, 5_1; 7_0, 9_0, 10_0, 15_0, 130),$
$(12_0, 13_0, 18_0, 3_0, 17_0; 5_1, 6_1, 19_0, 8_0, 7_0), (15_0, 1_0, 16_0, 6_0, 0_0; 11_1, 11_0, 2_0, 6_1, 19_0),$
$(10_0, 11_0, 6_1, 14_0, 11_1; 15_0, 17_0, 18_0, 0_0, 16_0).$

$IGD(33, 13, Q_5, 5)$, $\mathrm{V} = (Z_{20})_0 \cup (Z_{13})_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 4_0, 5_1; 1_1, 2_1, 3_1, 4_1, 1_0) + i, i \in [0, 19],$

$(0_0, 9_0, 12_0, 5_0, 6_0; 6_1, 7_1, 8_1, 9_1, 1_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 6_0, 7_0; 11_1, 12_1, 13_1, 1_1, 2_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 5_0, 7_0, 2_0; 3_1, 4_1, 5_1, 6_1, 7_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 7_0, 15_0; 6_1, 9_1, 10_1, 11_1, 12_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 7_0, 4_0; 13_1, 1_1, 2_1, 3_1, 4_1) +i, i \in [0, 19]$,
$(13_1, 0_0, 6_0, 8_1, 2_0; 1_0, 5_1, 6_1, 3_0, 7_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 1_0, 6_0, 12_1; 2_1, 12_0, 9_1, 10_0, 2_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 1_0, 7_0, 11_1; 2_1, 3_1, 4_1, 9_1, 2_0) + i, i \in [0, 19]$,
$(0_0, 6_0, 3_0, 5_0, 10_1; 3_1, 4_1, 6_0, 7_1, 1_0) + i, i \in [0, 19]$,
$(17_0, 4_0, 3_0, 2_0, 1_0; 6_1, 7_1, 8_1, 1_1, 6_0) + i, i \in [0, 19] \setminus \{4, 14, 15, 16, 18, 19\}$,
$(8_0, 18_0, 16_0, 14_0, 9_1; 10_0, 0_0, 6_0, 12_0, 17_0) + i, i \in [0, 1]$,
$(0_0, 2_0, 4_0, 6_0, 9_1; 10_0, 7_1, 14_0, 8_0, 3_0)$, $(1_0, 3_0, 5_0, 7_0, 9_1; 16_0, 13_0, 15_0, 9_0, 4_0)$,
$(9_1, 11_0, 6_1, 12_0, 10_0; 19_0, 1_0, 13_0, 2_0, 5_0)$, $(1_0, 8_0, 7_0, 6_0, 5_0; 6_1, 7_1, 8_1, 1_1, 9_1)$,
$(11_0, 18_0, 17_0, 16_0, 15_0; 13_0, 7_1, 8_1, 1_1, 0_0)$, $(16_0, 3_0, 2_0, 1_0, 0_0; 9_1, 7_1, 8_1, 1_1, 5_0)$,
$(13_0, 0_0, 19_0, 18_0, 17_0; 9_1, 7_1, 8_1, 1_1, 2_0)$, $(15_0, 2_0, 1_0, 0_0, 19_0; 6_1, 9_1, 8_1, 1_1, 4_0)$,
$(12_0, 19_0, 17_0, 18_0, 16_0; 9_1, 7_1, 8_1, 1_1, 6_1)$.

By Theorem 2.4 the result is true.

**Lemma 4.4:** There exists an $IGD(20 + w, w, Q_5, 10)$ for $w \in \{2, 3, 7, 14, 17, 18, 19\}$.

**Proof:** Let $IGD(v, w, Q_5, 10) = (V, \mathcal{B})$.

$IGD(22, 2, Q_5, 10)$, $V = (Z_{20})_0 \cup (Z_2)_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 4_0, 5_0; 1_1, 2_1, 3_0, 1_0, 11_0) + i, i \in [0, 19]$,
$(0_0, 10_0, 3_0, 7_0, 8_0; 2_0, 2_1, 6_0, 1_0, 4_0) + i, i \in [0, 19]$,
$(0_0, 90_{,} 2_0, 7_0, 5_0; 1_1, 2_1, 1_0, 3_0, 8_0) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 5_0, 7_0; 1_1, 1_0, 2_1, 2_0, 6_0) + i, i \in [0, 19]$,
$(0_0, 9_0, 1_0, 8_0, 4_0; 1_1, 2_1, 2_0, 3_0, 10_0) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 7_0, 4_0; 1_1, 1_0, 2_1, 5_0, 6_0) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 8_0, 7_0; 1_1, 1_0, 2_1, 3_0, 10_0) + i, i \in [0, 19]$,
$(0_0, 10_0, 4_0, 6_0, 7_0; 1_1, 2_0, 1_0, 9_0, 30) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 11_0, 6_0; 1_1, 1_0, 2_1, 4_0, 5_0) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 8_0, 2_0; 5_0, 16_0, 6_0, 7_0, 1_0) + i, i \in [0, 19]$,
$(6_0, 15_0, 7_0, 14_0, 10_0; 1_1, 2_1, 8_0, 9_0, 16_0) + i, i \in [0, 18]$,
$(5_0, 14_0, 6_0, 13_0, 9_0; 1_1, 2_1, 7_0, 8_0, 17_0)$, $(0_0, 8_0, 1_1, 14_0, 2_1; 2_0, 16_0, 1_0, 18_0, 7_0)$,

$(1_0, 9_0, 1_1, 15_0, 2_1; 17_0, 7_0, 2_0, 11_0, 8_0), (2_0, 10_0, 1_1, 19_0, 2_1; 6_0, 14_0, 3_0, 7_0, 9_0),$
$(3_0, 11_0, 1_1, 18_0, 2_1; 19_0, 9_0, 4_0, 6_0, 12_0), (4_0, 12_0, 1_1, 17_0, 2_1; 6_0, 10_0, 5_0, 13_0, 11_0),$
$(5_0, 13_0, 1_1, 16_0, 2_1; 7_0, 9_0, 6_0, 12_0, 10_0), (0_0, 12_0, 8_0, 4_0, 16_0; 1_1, 14_0, 6_0, 0_0, 18_0),$
$(13_0, 15_0, 17_0, 5_0, 1_0; 2_1, 7_0, 19_0, 9_0, 3_0), (18_0, 2_0, 14_0, 6_0, 10_0; 0_0, 4_0, 16_0, 2_1, 8_0),$
$(19_0, 11_0, 7_0, 3_0, 15_0; 1_0, 13_0, 1_1, 5_0, 9_0).$

$IGD(23, 3, Q_5, 10)$, $V = (Z_{20})_0 \cup (Z_3)_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 4_0, 9_0; 1_1, 2_1, 3_1, 1_0, 2_0) + i, i \in [0, 19],$
$(0_0, 10_0, 4_0, 8_0, 9_0; 1_1, 2_1, 3_1, 1_0, 3_0) + i, i \in [0, 19],$
$(0_0, 9_0, 11_0, 3_0, 4_0; 1_1, 2_1, 6_0, 3_1, 1_0) + i, i \in [0, 19],$
$(0_0, 9_0, 2_0, 6_0, 8_0; 1_1, 2_1, 3_1, 1_0, 5_0) + i, i \in [0, 19],$
$(0_0, 8_0, 5_0, 7_0, 6_0; 1_1, 2_1, 3_0, 1_0, 3_1) + i, i \in [0, 19],$
$(0_0, 10_0, 12_0, 5_0, 4_0; 1_1, 1_0, 7_0, 2_1, 3_1) + i, i \in [0, 19],$
$(0_0, 9_0, 2_0, 5_0, 6_0; 4_0, 1_0, 1_1, 2_1, 3_1) + i, i \in [0, 19],$
$(0_0, 9_0, 4_0, 7_0, 8_0; 1_1, 2_0, 2_1, 3_0, 3_1) + i, i \in [0, 19],$
$(0_0, 10_0, 2_0, 6_0, 5_0; 1_1, 2_1, 3_1, 1_0, 3_0) + i, i \in [0, 19],$
$(0_0, 10_0, 1_0, 7_0, 4_0; 5_0, 3_0, 8_0, 9_0, 6_0) + i, i \in [0, 19],$
$(0_0, 9_0, 3_0, 11_0, 7_0; 4_0, 1_0, 5_0, 8_0, 6_0) + i, i \in [0, 19],$
$(7_0, 16_0, 17_0, 2_0, 8_0; 1_0, 19_0, 9_0, 15_0, 3_0) + i, i \in [0, 15] \cup \{18\},$
$(0_0, 3_0, 9_0, 12_0, 1_1; 2_1, 17_0, 15_0, 3_1, 6_0) + i, i \in [0, 5],$
$(3_0, 12_0, 13_0, 18_0, 4_0; 3_1, 2_1, 5_0, 15_0, 19_0), (4_0, 13_0, 14_0, 19_0, 5_0; 18_0, 2_1, 6_0, 12_0, 3_1),$
$(6_0, 15_0, 16_0, 1_0, 7_0; 0_0, 18_0, 19_0, 14_0, 2_0), (8_0, 3_1, 19_0, 2_1, 11_0; 16_0, 4_0, 1_1, 9_0, 18_0),$
$(2_0, 3_1, 7_0, 2_1, 16_0; 19_0, 9_0, 10_0, 8_0, 13_0), (1_0, 3_1, 6_0, 2_1, 15_0; 18_0, 11_0, 9_0, 10_0, 12_0),$
$(17_0, 2_1, 18_0, 3_1, 0_0; 3_0, 14_0, 1_1, 10_0, 5_0).$

$IGD(27, 7, Q_5, 10)$, $V = (Z_{20})_0 \cup (Z_7)_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 8_0, 9_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19],$
$(0_0, 10_0, 2_0, 6_0, 7_0; 6_1, 7_1, 1_1, 2_1, 3_1) + i, i \in [0, 19],$
$(0_0, 9_0, 6_0, 2_0, 4_0; 4_1, 5_1, 6_1, 7_1, 1_1) + i, i \in [0, 19],$
$(0_0, 9_0, 7_0, 4_0, 5_0; 2_1, 3_1, 4_1, 5_1, 6_1) + i, i \in [0, 19],$
$(0_0, 8_0, 2_0, 9_0, 6_0; 7_1, 1_1, 2_1, 3_1, 4_1) + i, i \in [0, 19],$
$(0_0, 10_0, 12_0, 9_0, 4_0; 5_1, 6_1, 7_1, 1_1, 2_1) + i, i \in [0, 19],$
$(0_0, 9_0, 2_0, 4_0, 5_0; 3_1, 4_1, 5_1, 6_1, 7_1) + i, i \in [0, 19],$

$(0_0, 9_0, 1_0, 7_0, 8_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 3_0, 6_0, 8_0; 6_1, 7_1, 1_1, 2_1, 3_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 6_0, 5_0; 4_1, 5_1, 6_1, 7_1, 1_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 7_0, 5_0; 2_1, 3_1, 4_1, 5_1, 6_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 5_0, 7_0; 7_1, 1_1, 2_1, 3_1, 4_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 7_0, 5_0; 5_1, 6_1, 7_1, 4_1, 2_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 11_0, 4_0, 5_0; 5_1, 2_0, 3_0, 6_1, 1_0) + i, i \in [0, 19]$,
$(0_0, 9_0, 11_0, 6_0, 7_0; 4_0, 2_0, 3_0, 7_1, 1_0) + i, i \in [0, 19]$,
$(7_0, 16_0, 17_0, 2_0, 8_0; 1_0, 19_0, 9_0, 15_0, 3_0) + i, i \in [0, 15] \cup \{18\}$,
$(0_0, 3_0, 9_0, 12_0, 1_1; 2_1, 17_0, 15_0, 3_1, 6_0) + i, i \in [0, 5]$,
$(3_0, 12_0, 13_0, 18_0, 4_0; 3_1, 2_1, 5_0, 15_0, 19_0), (4_0, 13_0, 14_0, 19_0, 5_0; 18_0, 2_1, 6_0, 12_0, 3_1)$,
$(6_0, 15_0, 16_0, 1_0, 7_0; 0_0, 18_0, 19_0, 14_0, 2_0), (8_0, 3_1, 19_0, 2_1, 11_0; 16_0, 4_0, 1_1, 9_0, 18_0)$,
$(2_0, 3_1, 7_0, 2_1, 16_0; 19_0, 9_0, 10_0, 8_0, 13_0), (1_0, 3_1, 6_0, 2_1, 15_0; 18_0, 11_0, 9_0, 10_0, 12_0)$,
$(17_0, 2_1, 18_0, 3_1, 0_0; 3_0, 14_0, 1_1, 10_0, 5_0)$.

$IGD(34, 14, Q_5, 10)$, $V = (Z_{20})_0 \cup (Z_{14})_1$, $\mathcal{B}$:

$(0_0, 10_0, 12_0, 8_0, 7_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 11_0, 3_0, 7_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 13_0, 4_0, 6_0; 11_1, 12_1, 13_1, 14_1, 1_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 4_0, 12_0, 3_0; 2_1, 3_1, 4_1, 5_1, 6_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 6_0, 7_0; 7_1, 8_1, 9_1, 10_1, 11_1) + i, i \in [0, 19]$,
$(0_0, 8_0, 2_0, 5_0, 7_0; 12_1, 13_1, 14_1, 1_1, 2_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 14_0, 9_0, 3_0; 3_1, 4_1, 5_1, 6_1, 7_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 4_0, 6_0, 7_0; 8_1, 9_1, 10_1, 11_1, 12_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 5_0, 7_0, 8_0; 13_1, 14_1, 1_1, 2_1, 3_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 13_0, 8_0, 6_0; 4_1, 5_1, 6_1, 7_1, 8_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 5_0, 8_0, 7_0; 9_1, 10_1, 11_1, 12_1, 13_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 1_0, 2_0, 14_1; 1_1, 2_1, 3_1, 4_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 2_0, 7_0, 13_1; 5_1, 6_1, 7_1, 8_1, 1_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 2_0, 6_0, 12_1; 9_1, 10_1, 11_1, 14_1, 1_0) + i, i \in [0, 19]$,
$(0_0, 7_0, 9_0, 6_0, 11_1; 1_1, 2_1, 3_1, 4_1, 1_0) + i, i \in [0, 19]$,
$(0_0, 9_0, 1_1, 2_0, 2_1; 5_1, 6_1, 1_0, 7_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 4_1, 1_0, 5_1; 6_1, 7_1, 2_0, 8_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 7_0, 8_1, 1_0, 9_1; 10_1, 3_1, 2_0, 11_1, 3_0) + i, i \in [0, 19]$,

$(0_0, 10_1, 1_0, 6_0, 2_0; 3_1, 3_0, 6_1, 7_1, 9_1) + i, i \in [0, 19]$,
$(0_0, 7_0, 6_0, 4_0, 1_0; 3_1, 4_1, 5_1, 6_1, 7_1) + i, i \in [0, 19]$,
$(0_0, 6_0, 11_0, 7_0, 12_1; 6_1, 7_1, 8_1, 9_1, 1_0) + i, i \in [0, 19]$,
$(0_0, 4_0, 13_1, 1_0, 14_1; 10_1, 11_1, 2_0, 6_0, 3_0) + i, i \in [0, 19]$,
$(7_0, 16_0, 17_0, 2_0, 8_0; 1_0, 19_0, 9_0, 15_0, 3_0) + i, i \in [0, 15] \cup \{18\}$,
$(0_0, 3_0, 9_0, 12_0, 1_1; 2_1, 17_0, 15_0, 3_1, 6_0) + i, i \in [0, 5]$,
$(3_0, 12_0, 13_0, 18_0, 4_0; 3_1, 2_1, 5_0, 15_0, 19_0), (4_0, 13_0, 14_0, 19_0, 5_0; 18_0, 2_1, 6_0, 12_0, 3_1)$,
$(6_0, 15_0, 16_0, 1_0, 7_0; 0_0, 18_0, 19_0, 14_0, 2_0), (8_0, 3_1, 19_0, 2_1, 11_0; 16_0, 4_0, 1_1, 9_0, 18_0)$,
$(2_0, 3_1, 7_0, 2_1, 16_0; 19_0, 9_0, 10_0, 8_0, 13_0), (1_0, 3_1, 6_0, 2_1, 15_0; 18_0, 11_0, 9_0, 10_0, 12_0)$,
$(17_0, 2_1, 18_0, 3_1, 0_0; 3_0, 14_0, 1_1, 10_0, 5_0)$.

$IGD(37, 17, Q_5, 10)$, $V = (Z_{20})_0 \cup (Z_{17})_1$, $\mathcal{B}$:

$(17_1, 0_0, 1_0, 16_1, 2_0; 3_0, 1_1, 2_1, 4_0, 3_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 15_1, 1_0, 14_1; 4_1, 5_1, 2_0, 6_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 9_0, 13_1, 1_0, 12_1; 7_1, 8_1, 2_0, 9_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 6_0, 2_0, 5_0; 10_1, 11_1, 1_1, 2_1, 3_1) + i, i \in [0, 19]$,
$(0_0, 7_0, 1_0, 2_0, 11_1; 4_1, 5_1, 6_1, 7_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 10_0, 1_0, 8_0, 2_0; 8_1, 9_1, 10_1, 11_1, 12_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 1_0, 6_0, 4_0; 13_1, 14_1, 15_1, 16_1, 17_1) + i, i \in [0, 19]$,
$(0_0, 8_0, 2_0, 7_0, 4_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 7_0, 4_0, 5_0, 8_0; 6_1, 7_1, 8_1, 9_1, 10_1) +i, i \in [0, 19]$,
$(0_0, 10_0, 3_0, 7_0, 5_0; 12_1, 13_1, 14_1, 15_1, 16_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 5_0, 7_0, 6_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 6_0, 8_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 5_0, 6_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 6_0, 5_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 5_0, 2_0; 11_1, 12_1, 13_1, 14_1, 15_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 7_0, 4_0; 16_1, 17_1, 1_1, 2_1, 3_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 7_0, 6_0; 4_1, 5_1, 6_1, 7_1, 8_1) + i, i \in [0, 19]$,
$(0_0, 4_1, 1_0, 3_0, 5_1; 1_1, 2_0, 2_1, 9_1, 4_0) + i, i \in [0, 19]$,
$(6_1, 0_0, 7_0, 7_1, 1_0; 2_0, 8_1, 9_1, 4_0, 10_1) + i, i \in [0, 19]$,
$(8_1, 0_0, 7_0, 9_1, 1_0; 2_0, 1_1, 2_1, 3_0, 3_1) + i, i \in [0, 19]$,
$(0_0, 10_1, 1_0, 3_0, 11_1; 3_1, 2_0, 4_1, 5_1, 4_0) + i, i \in [0, 19]$,
$(12_1, 0_0, 1_0, 13_1, 2_0; 3_0, 6_1, 7_1, 4_0, 10_1) + i, i \in [0, 19]$,

$(0_0, 8_0, 14_1, 1_0, 15_1; 11_1, 12_1, 2_0, 17_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 16_1, 2_0, 17_1, 9_0; 1_1, 1_0, 2_1, 3_0, 3_1) + i, i \in [0, 19]$,
$(0_0, 8_0, 12_0, 7_0, 4_0; 14_1, 15_1, 16_1, 17_1, 13_1) + i, i \in [0, 19]$,
$(7_0, 16_0, 17_0, 2_0, 8_0; 1_0, 19_0, 9_0, 15_0, 3_0) + i, i \in [0, 15] \cup \{18\}$,
$(0_0, 3_0, 9_0, 12_0, 1_1; 2_1, 17_0, 15_0, 3_1, 6_0) + i, i \in [0, 5]$,
$(3_0, 12_0, 13_0, 18_0, 4_0; 3_1, 2_1, 5_0, 15_0, 19_0), (4_0, 13_0, 14_0, 19_0, 5_0; 18_0, 2_1, 6_0, 12_0, 3_1)$,
$(6_0, 15_0, 16_0, 1_0, 7_0; 0_0, 18_0, 19_0, 14_0, 2_0), (8_0, 3_1, 19_0, 2_1, 11_0; 16_0, 4_0, 1_1, 9_0, 18_0)$,
$(2_0, 3_1, 7_0, 2_1, 16_0; 19_0, 9_0, 10_0, 8_0, 13_0), (1_0, 3_1, 6_0, 2_1, 15_0; 18_0, 11_0, 9_0, 10_0, 12_0)$,
$(17_0, 2_1, 18_0, 3_1, 0_0; 3_0, 14_0, 1_1, 10_0, 5_0)$.

$IGD(38, 18, Q_5, 10)$, $V = (Z_{20})_0 \cup (Z_{18})_1$, $\mathcal{B}$:

$(17_1, 0_0, 1_0, 16_1, 2_0; 3_0, 1_1, 2_1, 4_0, 3_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 15_1, 1_0, 14_1; 4_1, 5_1, 2_0, 6_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 9_0, 13_1, 1_0, 12_1; 7_1, 8_1, 2_0, 9_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 6_0, 2_0, 5_0; 10_1, 11_1, 1_1, 2_1, 3_1) + i, i \in [0, 19]$,
$(0_0, 7_0, 1_0, 2_0, 11_1; 4_1, 5_1, 6_1, 7_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 10_0, 1_0, 8_0, 2_0; 8_1, 9_1, 10_1, 11_1, 12_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 1_0, 6_0, 18_1; 13_1, 14_1, 15_1, 16_1, 2_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 2_0, 7_0, 4_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 7_0, 4_0, 5_0, 8_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 3_0, 7_0, 5_0; 12_1, 13_1, 14_1, 15_1, 16_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 5_0, 7_0, 6_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 6_0, 8_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 5_0, 6_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 6_0, 5_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 5_0, 2_0; 11_1, 12_1, 13_1, 14_1, 15_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 7_0, 4_0; 16_1, 18_1, 1_1, 2_1, 3_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 7_0, 6_0; 4_1, 5_1, 6_1, 7_1, 8_1) + i, i \in [0, 19]$,
$(0_0, 4_1, 1_0, 3_0, 5_1; 1_1, 2_0, 2_1, 3_1, 4_0) + i, i \in [0, 19]$,
$(6_1, 0_0, 7_0, 7_1, 1_0; 2_0, 8_1, 9_1, 4_0, 10_1) + i, i \in [0, 19]$,
$(8_1, 0_0, 7_0, 9_1, 1_0; 2_0, 1_1, 2_1, 3_0, 3_1) + i, i \in [0, 19]$,
$(0_0, 10_1, 1_0, 3_0, 11_1; 3_1, 2_0, 4_1, 5_1, 4_0) + i, i \in [0, 19]$,
$(12_1, 0_0, 1_0, 13_1, 2_0; 3_0, 6_1, 7_1, 4_0, 10_1) + i, i \in [0, 19]$,
$(0_0, 8_0, 14_1, 1_0, 15_1; 11_1, 12_1, 2_0, 18_1, 3_0) + i, i \in [0, 19]$,

$(0_0, 16_1, 2_0, 17_1, 9_0; 18_1, 1_0, 2_1, 3_0, 3_1) + i, i \in [0, 19]$,
$(0_0, 8_0, 12_0, 7_0, 4_0; 18_1, 15_1, 16_1, 17_1, 13_1) + i, i \in [0, 19]$,
$(0_0, 17_1, 1_0, 3_0, 18_1; 14_1, 4_0, 5_0, 1_1, 2_0) + i, i \in [0, 19]$,
$(7_0, 16_0, 17_0, 2_0, 8_0; 1_0, 19_0, 9_0, 15_0, 3_0) + i, i \in [0, 15] \cup \{18\}$,
$(0_0, 3_0, 9_0, 12_0, 1_1; 2_1, 17_0, 15_0, 3_1, 6_0) + i, i \in [0, 5]$,
$(3_0, 12_0, 13_0, 18_0, 4_0; 3_1, 2_1, 5_0, 15_0, 19_0)$,
$(4_0, 13_0, 14_0, 19_0, 5_0; 18_0, 2_1, 6_0, 12_0, 3_1), (6_0, 15_0, 16_0, 1_0, 7_0; 0_0, 18_0, 19_0, 14_0, 2_0)$,
$(8_0, 3_1, 19_0, 2_1, 11_0; 16_0, 4_0, 1_1, 9_0, 18_0), (2_0, 3_1, 7_0, 2_1, 16_0; 19_0, 9_0, 10_0, 8_0, 13_0)$,
$(1_0, 3_1, 6_0, 2_1, 15_0; 18_0, 11_0, 9_0, 10_0, 12_0), (17_0, 2_1, 18_0, 3_1, 0_0; 3_0, 14_0, 1_1, 10_0, 5_0)$.

$IGD(39, 19, Q_5, 10)$, $V = (Z_{20})_0 \cup (Z_{19})_1$, $\mathcal{B}$:

$(17_1, 0_0, 1_0, 16_1, 2_0; 3_0, 1_1, 2_1, 4_0, 3_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 15_1, 1_0, 14_1; 4_1, 5_1, 2_0, 6_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 9_0, 13_1, 1_0, 12_1; 7_1, 8_1, 2_0, 9_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 6_0, 2_0, 5_0; 10_1, 11_1, 1_1, 2_1, 3_1) + i, i \in [0, 19]$,
$(0_0, 7_0, 1_0, 2_0, 11_1; 4_1, 5_1, 6_1, 7_1, 3_0) + i, i \in [0, 19]$,
$(0_0, 10_0, 1_0, 8_0, 2_0; 8_1, 9_1, 10_1, 11_1, 12_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 19_1, 6_0, 18_1; 13_1, 14_1, 3_0, 15_1, 2_0) + i, i \in [0, 19]$,
$(0_0, 8_0, 2_0, 7_0, 4_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 7_0, 4_0, 5_0, 8_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 3_0, 7_0, 5_0; 12_1, 13_1, 14_1, 15_1, 19_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 5_0, 7_0, 6_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 6_0, 8_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 5_0, 6_0; 1_1, 2_1, 3_1, 4_1, 5_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 6_0, 5_0; 6_1, 7_1, 8_1, 9_1, 10_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 3_0, 5_0, 2_0; 11_1, 12_1, 13_1, 14_1, 15_1) + i, i \in [0, 19]$,
$(0_0, 10_0, 2_0, 7_0, 4_0; 19_1, 18_1, 1_1, 2_1, 3_1) + i, i \in [0, 19]$,
$(0_0, 9_0, 2_0, 7_0, 6_0; 4_1, 5_1, 6_1, 7_1, 8_1) + i, i \in [0, 19]$,
$(0_0, 4_1, 1_0, 3_0, 5_1; 1_1, 2_0, 2_1, 3_1, 4_0) + i, i \in [0, 19]$,
$(6_1, 0_0, 7_0, 7_1, 1_0; 2_0, 8_1, 9_1, 4_0, 10_1) + i, i \in [0, 19]$,
$(8_1, 0_0, 7_0, 9_1, 1_0; 2_0, 1_1, 2_1, 3_0, 3_1) + i, i \in [0, 19]$,
$(0_0, 10_1, 1_0, 3_0, 11_1; 19_1, 2_0, 4_1, 5_1, 4_0) + i, i \in [0, 19]$,
$(12_1, 0_0, 1_0, 13_1, 2_0; 3_0, 6_1, 7_1, 4_0, 10_1) + i, i \in [0, 19]$,
$(0_0, 8_0, 14_1, 1_0, 15_1; 11_1, 12_1, 2_0, 18_1, 3_0) + i, i \in [0, 19]$,

$(0_0, 16_1, 2_0, 17_1, 9_0; 18_1, 1_0, 2_1, 3_0, 3_1) + i, i \in [0, 19]$,
$(0_0, 8_0, 12_0, 7_0, 4_0; 18_1, 19_1, 16_1, 17_1, 13_1) + i, i \in [0, 19]$,
$(0_0, 17_1, 1_0, 3_0, 18_1; 14_1, 4_0, 5_0, 1_1, 2_0) + i, i \in [0, 19]$,
$(16_1, 0_0, 5_0, 19_1, 1_0; 3_0, 8_0, 9_1, 2_0, 15_1) + i, i \in [0, 19]$,
$(7_0, 16_0, 17_0, 2_0, 8_0; 1_0, 19_0, 9_0, 15_0, 3_0) + i, i \in [0, 15] \cup \{18\}$,
$(0_0, 3_0, 9_0, 12_0, 1_1; 2_1, 17_0, 15_0, 3_1, 6_0) + i, i \in [0, 5]$,
$(3_0, 12_0, 13_0, 18_0, 4_0; 3_1, 2_1, 5_0, 15_0, 19_0)$,
$(4_0, 13_0, 14_0, 19_0, 5_0; 18_0, 2_1, 6_0, 12_0, 3_1)$, $(6_0, 15_0, 16_0, 1_0, 7_0; 0_0, 18_0, 19_0, 14_0, 2_0)$,
$(8_0, 3_1, 19_0, 2_1, 11_0; 16_0, 4_0, 1_1, 9_0, 18_0)$, $(2_0, 3_1, 7_0, 2_1, 16_0; 19_0, 9_0, 10_0, 8_0, 13_0)$,
$(1_0, 3_1, 6_0, 2_1, 15_0; 18_0, 11_0, 9_0, 10_0, 12_0)$, $(17_0, 2_1, 18_0, 3_1, 0_0; 3_0, 14_0, 1_1, 10_0, 5_0)$.

By Theorem 2.4 the result is true. □

**Lemma 4.5:** There exists a $GD(v, Q_5, \lambda)$ for the $\lambda$ and $v$ in Table A.

**Table A**
***GD*(*v*, *Q*[5], λ)**

| $\lambda$ | $v \equiv$ | $v$, |
|---|---|---|
| 1, 3, 7, 9 | 0, 1, 5, 16 (mod 20) | 16, 20, 21, 25, 36, 40, 41, 45, 56 |
| 2, 6 | 0, 1, 5, 6 (mod 10) | 11, 15, 26, 30, 31, 35, 46, 50, 51, 55 |
| 5 | 0, 1 (mod 4) | 12, 13, 17, 24, 28, 29, 32, 33, 44, 48, 49, 52, 53, 57 |
| 10 | arbitrary | 14, 18, 19, 22, 23, 27, 34, 37, 38, 39, 42, 43, 47, 54, 58, 59 |

**Proof:** It is proved from the Appendix and Theorem 2.4. □

**Theorem 4.6:** Let $v$ and $\lambda$ be two positive integers. Then there exists a $GD(v, Q_5, \lambda)$ if and only if

(1) $v > 10$ and $\lambda v(v-1) \equiv 0 \pmod{20}$;

(2) $v = 10$ and $\lambda \equiv 0 \pmod 4$.

**Proof:** By Lemmas 4.1-4.4, we have IGD for listed in Table B.

**Table B**
***IGD*(*v*, *w*, *Q*[6], λ)**

| $\lambda$ | *1, 3, 7, 9* | | *2, 6* | | | | *5* | | | | | *10* | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | 25 | 36 | 26 | 30 | 31 | 35 | 24 | 28 | 29 | 32 | 33 | 22 | 23 | 27 | 34 | 37 | 38 | 39 |
| $w$ | 5 | 16 | 6 | 10 | 11 | 15 | 4 | 8 | 9 | 12 | 13 | 2 | 3 | 7 | 14 | 17 | 18 | 19 |

By Lemma 4.5 and Theorem 3.3 we can obtain the theorem.

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## APPENDIX
## $GD(v, Q_5, \lambda)$ IN TABLE A

In the following, let $GD(v, Q_5, \lambda) = (V, \mathcal{B})$.

$GD(16, Q_5, 1)$, $V = (Z_3 \times Z_5) \cup \{\infty\}$, $\mathcal{B}$:

$(0_0, 1_1, 0_2, 1_2, 2_3; 1_4, 2_1, 0_3, \infty, 0_1) + i, i \in [0, 2]$,
$(0_1, 2_4, 0_0, 1_2, 0_3; 1_0, 1_3, 2_2, 1_1, 0_4) + i, i \in [0, 2]$,
$(0_0, 1_3, 0_1, 0_4, \infty; 1_0, 0_3, 1_2, 1_4, 1_1) + i, i \in [0, 2]$,
$(0_2, 1_4, 1_0, 1_3, 0_4; 0_0, 0_1, 1_1, \infty, 1_2) + i, i \in [0, 2]$.

From Theorem 3.6, there are $GD(v, Q_5, 1)$ for $v = 20, 21, 40, 41$.

$GD(25, Q_5, 1)$, $V = Z_5 \times Z_5$, $\mathcal{B}$:

$(0_0, 1_1, 2_2, 3_2, 1_0; 3_3, 2_1, 2_0, 0_1, 2_3) + i, i \in [0, 4]$,
$(0_1, 3_3, 3_2, 1_2, 2_0; 3_0, 0_2, 1_1, 0_0, 2_3) + i, i \in [0, 4]$,
$(0_1, 2_3, 3_2, 0_0, 4_3; 2_1, 1_3, 3_1, 2_0, 2_2) + i, i \in [0, 4]$,
$(0_1, 2_4, 4_0, 4_1, 3_4; 4_2, 3_2, 1_3, 4_3, 2_0) + i, i \in [0, 4]$,
$(1_3, 2_4, 4_2, 0_3, 4_4; 1_4, 2_2, 0_4, 2_3, 0_0) + i, i \in [0, 4]$,
$(2_4, 0_4, 0_1, 1_0, 3_4; 1_1, 3_3, 1_3, 1_4, 1_2) + i, i \in [0, 4]$.

$GD(11, Q_5, 2)$, $V = Z_{11}$, $\mathcal{B}$:

$(0, 5, 6, 4, 1; 2, 8, 10, 11, 7) + i, i \in [0, 10]$.

$GD(36, Q_5, 1) \Leftarrow IGD(36, 16, Q_5, 1) \wedge GD(16, Q_5, 1)$.

$GD(45, Q_5, 1)$, $V = (Z_{11} \times Z_4) \cup \{\infty\}$, $\mathcal{B}$:

$(0_1, 1_1, 4_2, 5_1, 2_0; 1_2, 7_2, 3_2, 2_1, 1_0) + i, i \in [0, 10]$,
$(0_1, 4_3, 8_0, 3_0, 6_3; 2_3, 0_0, 7_2, 5_2, 1_2) + i, i \in [0, 10]$,
$(1_0, 5_1, 3_2, 4_3, 1_3; 3_0, 2_2, 5_2, 4_1, 2_0) + i, i \in [0, 10]$,
$(1_3, 5_3, 3_0, 4_2, 3_3; 2_1, 2_2, 0_2, 4_1, 5_0) + i, i \in [0, 10]$,
$(0_1, 1_3, 3_2, 0_2, 6_0; 4_1, 4_2, 7_1, 2_3, 6_1) + i, i \in [0, 10]$,
$(1_0, 3_1, 5_2, 5_3, 7_1; 4_2, 6_3, \infty, 4_3, 2_1) + i, i \in [0, 10]$,
$(8_3, 3_1, 7_2, 3_0, 0_1; 1_2, \infty, 2_2, 0_3, 2_1) + i, i \in [0, 10]$,
$(9_2, 3_0, 2_1, 7_2, 0_0; 4_3, \infty, 9_3, 3_2, 1_1) + i, i \in [0, 10]$,
$(0_0, 6_3, 1_3, 7_0, 3_0; 7_1, \infty, 8_2, 8_3, 3_2) + i, i \in [0, 10]$.

$GD(56, Q_5, 1)$, $V = (Z_{11} \times Z_5) \cup \{\infty\}$, $\mathcal{B}$:

$(0_0, 2_1, 3_2, 6_3, 4_1; 1_2, 5_3, 0_2, 1_0, 2_2) + i, i \in [0, 10]$,
$(0_1, 4_1, 5_0, 4_2, 2_1; 2_0, 2_3, 7_2, 3_3, 1_0) + i, i \in [0, 10]$,
$(1_0, 4_3, 2_0, 6_2, 2_3; 4_2, 1_3, 0_0, 1_1, 1_2) + i, i \in [0, 10]$,
$(3_1, 3_0, 7_3, 2_1, 2_3; 6_2, 3_2, 1_3, 2_2, 0_2) + i, i \in [0, 10]$,
$(0_2, 4_2, 4_3, 5_0, 2_0; 5_3, 0_1, 6_2, 1_0, 10_1) + i, i \in [0, 10]$,
$(4_1, 1_3, 1_0, 6_1, 10_3; 1_1, 2_3, 6_2, 5_2, 2_2) + i, i \in [0, 10]$,
$(7_0, 2_0, 8_1, 3_1, 4_3; 4_2, 1_0, 7_1, 0_0, 0_2) + i, i \in [0, 10]$,
$(0_2, 5_2, 0_3, 2_0, 9_2; 2_4, 3_4, 1_4, 9_1, 6_4) + i, i \in [0, 10]$,
$(0_4, 5_4, 0_0, 2_4, 0_3; 4_2, 1_3, 3_4, 6_4, 2_3) + i, i \in [0, 10]$,
$(6_4, 2_1, 3_4, 0_4, 5_2; \infty, 2_4, 2_0, 5_3, 8_4) + i, i \in [0, 10]$,
$(0_0, 0_4, 1_1, 4_4, 6_4; \infty, 2_0, 7_2, 1_3, 0_1) + i, i \in [0, 10]$,
$(0_0, 7_4, 2_1, 9_2, 8_4; 6_3, 3_0, 9_4, \infty, 0_3) + i, i \in [0, 10]$,
$(5_4, 6_0, 2_3, 6_3, 4_4; 5_2, 1_2, 7_4, \infty, 8_3) + i, i \in [0, 10]$,
$(0_4, 3_1, 0_2, 4_4, 2_1; 6_2, \infty, 1_2, 5_3, 9_3) + i, i \in [0, 10]$.

$GD(15, Q_5, 2)$, $V = (Z_7 \times Z_2) \cup \{\infty\}$, $\mathcal{B}$:

$(0_0, 0_1, 1_0, 4_1, 2_1; 1_1, 3_1, 6_1, 5_1, 2_0) + i, i \in [0, 6]$,
$(0_0, 4_1, 1_0, 2_0, 6_1; 3_0, 1_1, 4_0, \infty, 5_1) + i, i \in [0, 6]$,
$(0_0, 1_0, 3_0, 5_1, \infty; 2_0, 2_1, 1_1, 3_1, 0_1) + i, i \in [0, 6]$.

$GD(26, Q_5, 2)$, $V = Z_{13} \times Z_2$, $\mathcal{B}$:

$(0_0, 6_1, 4_1, 2_0, 3_0; 3_1, 1_0, 1_1, 2_1, 5_0) + i, i \in [0, 12]$,
$(0_1, 1_0, 2_1, 3_1, 5_0; 4_1, 8_1, 6_0, 9_1, 9_0) + i, i \in [0, 12]$,
$(0_0, 5_0, 9_1, 4_1, 6_0; 1_1, 2_0, 2_1, 7_0, 3_1) + i, i \in [0, 12]$,
$(1_0, 3_1, 4_1, 0_0, 8_1; 5_0, 3_0, 0_1, 6_0, 2_0) + i, i \in [0, 12]$,
$(0_1, 1_0, 3_0, 8_1, 5_1; 4_0, 2_0, 8_0, 5_0, 3_1) + i, i \in [0, 12]$.

$GD(30, Q_5, 2)$, $V = Z_{29} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 1, 5, 10, 13; 2, 9, 12, 4, 3) + i, i \in [0, 28]$,
$(0, 14, 2, 11, 6; 9, 3, \infty, 7, 5) + i, i \in [0, 28]$,
$(1, 3, 10, 0, 14; 9, 6, \infty, 11, 2) + i, i \in [0, 28]$.

$GD(31, Q_5, 2) \Leftarrow IGD(31, 11, Q_5, 2) \wedge GD(11, Q_5, 2)$.

GD(35, $Q_5$, 2) $\Leftarrow IGD(35, 15, Q_5, 2) \wedge GD(15, Q_5, 2)$.

$GD(46, Q_5, 2)$, $V = Z_{23} \times Z_2$, $\mathcal{B}$:

$(0_0, 2_1, 3_1, 3_0, 4_1; 1_0, 7_1, 4_0, 6_1, 6_0) + i, i \in [0, 22]$,
$(0_1, 3_0, 5_0, 10_1, 4_0; 11_1, 6_0, 12_1, 2_0, 8_0) + i, i \in [0, 22]$,
$(7_1, 10_1, 0_0, 5_0, 0_1; 5_1, 1_0, 11_1, 2_1, 4_1) + i, i \in [0, 22]$,
$(0_0, 11_0, 0_1, 10_1, 16_1; 7_0, 1_1, 8_1, 2_1, 7_1) + i, i \in [0, 22]$,
$(0_0, 14_1, 5_1, 1_0, 10_0; 15_1, 3_1, 2_0, 7_0, 0_1) + i, i \in [0, 22]$,
$(1_0, 1_1, 2_0, 10_0, 2_1; 3_0, 6_1, 8_1, 0_0, 7_0) + i, i \in [0, 22]$,
$(0_1, 2_0, 3_0, 8_1, 2_1; 4_0, 9_1, 6_0, 14_0, 5_1) + i, i \in [0, 22]$,
$(0_0, 8_1, 7_1, 3_1, 9_0; 11_0, 1_1, 5_0, 15_0, 2_1) + i, i \in [0, 22]$,
$(0_1, 9_0, 5_0, 0_0, 10_1; 11_0, 3_0, 13_0, 7_0, 1_0) + i, i \in [0, 22]$.

$GD(50, Q_5, 2)$, $V = Z_{49} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 24, 1, 20, 3; 5, 2, 7, 4, 10) + i, i \in [0, 48]$,
$(0, 21, 3, 18, 4; 2, 9, 11, 5, 13) + i, i \in [0, 48]$,
$(0, 1, 12, 2, 20; 6, 9, 5, 4, 3) + i, i \in [0, 48]$,
$(0, 24, 1, 20, 4; \infty, 2, 10, 5, 7) + i, i \in [0, 48]$,
$(0, 21, 1, 15, 10; 11, 8, \infty, 3, 9) + i, i \in [0, 48]$.

$GD(51, Q_5, 2)$, $V = Z_{51}$, $\mathcal{B}$:

$(0, 24, 1, 20, 3; 5, 2, 7, 4, 10) + i, i \in [0, 50]$,
$(0, 21, 3, 18, 4; 2, 9, 11, 5, 13) + i, i \in [0, 50]$,
$(0, 1, 12, 2, 20; 6, 9, 5, 4, 3) + i, i \in [0, 50]$,
$(0, 24, 1, 20, 4; 25, 2, 10, 5, 7) + i, i \in [0, 50]$,
$(0, 21, 1, 15, 10; 11, 8, 26, 3, 9) + i, i \in [0, 50]$.

$GD(55, Q_5, 2)$, $V = (Z_{27} \times Z_2) \cup \{\infty\}$, $\mathcal{B}$:

$(0_0, 2_1, 3_1, 4_0, 7_1; 3_0, 4_1, 5_0, 13_0, 2_0) + i, i \in [0, 26]$,
$(1_0, 5_1, 2_1, 5_0, 6_0; 11_0, 10_0, 6_1, 3_0, 4_0) + i, i \in [0, 26]$,
$(0_1, 7_0, 0_0, 8_1, 13_1; 8_0, 3_0, 9_1, 1_1, 0_0) + i, i \in [0, 26]$,
$(1_0, 1_1, 11_1, 0_0, 15_1; 7_0, 5_0, 0_1, 10_1, 3_0) + i, i \in [0, 26]$,
$(0_0, 13_0, 0_1, 9_0, 10_1; 8_0, 1_0, 12_1, 0_1, 2_0) + i, i \in [0, 26]$,
$(0_0, 17_1, 1_0, 10_0, 2_1; 11_1, 8_1, 11_0, 3_0, 10_1) + i, i \in [0, 26]$,
$(7_0, 10_1, 4_1, 5_0, 11_0; 18_0, 0_1, 8_0, 13_0, 5_1) + i, i \in [0, 26]$,
$(0_1, 2_0, 5_0, 9_1, 11_1, 11_0, 7_0, 2_1, 4_0, \infty) + i, i \in [0, 26]$,

$(2_1, 2_0, 8_1, 15_0, 14_0; 6_1, \infty, 3_1, 3_0, 1_0) + i, i \in [0, 26]$,
$(1_0, 10_1, 3_1, 2_1, 14_1; \infty, 3_0, 9_0, 12_0, 1_1) + i, i \in [0, 26]$,
$(0_0, 12_1, 3_1, 9_1, 6_1; 11_0, 4_1, 8_0, \infty, 5_0) + i, i \in [0, 26]$.

$GD(12, Q_5, 5)$, $V = Z_{11} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 5, 7, 3, 4; \infty, 8, 2, 6, 1) + i, i \in [0, 10]$,
$(0, 5, 3, 4, 1; \infty, 9, 7, 6, 2) + i, i \in [0, 10]$,
$(0, 5, 7, 6, \infty; 2, 8, 3, 1, 4) + i, i \in [0, 10]$.

$GD(13, Q_5, 5)$, $V = Z_{13}$, $\mathcal{B}$:

$(0, 6, 7, 9, 4; 2, 1, 3, 8, 5) + i, i \in [0, 12]$,
$(0, 6, 3, 5, 2; 4, 1, 9, 8, 7) + i, i \in [0, 12]$,
$(0, 6, 10, 7, 4; 2, 11, 9, 1, 3) + i, i \in [0, 12]$.

$GD(17, Q_5, 5)$, $V = Z_{17}$, $\mathcal{B}$:

$(0, 8, 10, 7, 1; 5, 4, 3, 6, 9) + i, i \in [0, 16]$,
$(0, 8, 11, 6, 4; 7, 2, 3, 1, 5) + i, i \in [0, 16]$,
$(0, 7, 9, 3, 4; 5, 2, 6, 1, 10) + i, i \in [0, 16]$,
$(0, 8, 11, 4, 6; 3, 1, 7, 5, 2) + i, i \in [0, 16]$.

$GD(24, Q_5, 5)$, $V = Z_{23} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 11, 1, 10, 3; 9, 4, 6, 2, 5) + i, i \in [0, 22]$,
$(0, 11, 3, 7, 2; \infty, 5, 4, 10, 6) + i, i \in [0, 22]$,
$(0, 10, 3, 4, 6; 5, 1, \infty, 7, 16) + i, i \in [0, 22]$,
$(0, 11, 1, 5, 6; 4, 2, \infty, 10, 9) + i, i \in [0, 22]$,
$(0, 11, 3, 10, 2; 7, 5, 8, 9, \infty) + i, i \in [0, 22]$,
$(0, 11, 1, 10, 2; \infty, 7, 4, 16, 3) + i, i \in [0, 22]$.

$GD(28, Q_5, 5)$, $V = Z_{27} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 13, 1, 8, 2; 11, 3, 4, 7, 6) + i, i \in [0, 26]$,
$(0, 13, 1, 10, 2; 5, 3, 8, 4, 6) + i, i \in [0, 26]$,
$(0, 11, 2, 7, 4; 1, 3, \infty, 5, 10) + i, i \in [0, 26]$,
$(0, 13, 1, 10, 2; \infty, 3, 6, 7, 9) + i, i \in [0, 26]$,
$(0, 13, 2, 11, 4; \infty, 12, 1, 5, 9) + i, i \in [0, 26]$,
$(0, 13, 1, 12, 3; \infty, 5, 11, 2, 4) + i, i \in [0, 26]$,
$(0, 12, 1, 9, 4; 2, 5, \infty, 3, 7) + i, i \in [0, 26]$.

$GD(29, Q_5, 5)$, $V = Z_{29}$, $\mathcal{B}$:

$(0, 13, 1, 8, 2; 11, 3, 4, 7, 6) + i$, $i \in [0, 28]$,
$(0, 13, 1, 10, 2; 5, 3, 8, 4, 6) + i$, $i \in [0, 28]$,
$(0, 11, 2, 7, 4; 1, 3, 16, 5, 10) + i$, $i \in [0, 28]$,
$(0, 13, 1, 10, 2; 14, 3, 6, 7, 9) + i$, $i \in [0, 28]$,
$(0, 13, 2, 11, 4; 14, 12, 1, 5, 9) + i$, $i \in [0, 28]$,
$(0, 13, 1, 12, 3; 14, 5, 11, 2, 4) + i$, $i \in [0, 28]$,
$(0, 12, 1, 9, 4; 2, 5, 15, 3, 7) + i$, $i \in [0, 26]$.

$GD(32, Q_5, 5) \Leftarrow IGD(32, 12, Q_5, 5) \wedge GD(12, Q_5, 5)$.
$GD(33, Q_5, 5) \Leftarrow IGD(33, 13, Q_5, 5) \wedge GD(13, Q_5, 5)$.
$GD(44, Q_5, 5)$, $V = Z_{43} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 21, 1, 19, 2; 5, 6, 4, 3, 8) + i$, $i \in [0, 42]$,
$(0, 19, 5, 17, 7; 4, 6, 14, 9, 8) + i$, $i \in [0, 42]$,
$(0, 21, 1, 12, 4; 7, 2, 6, 3, 5) + i$, $i \in [0, 42]$,
$(0, 21, 3, 19, 5; 2, 4, 9, 1, 6) + i$, $i \in [0, 42]$,
$(0, 20, 5, 18, 6; 10, 1, 2, 7, 4) + i$, $i \in [0, 42]$,
$(0, 21, 4, 19, 5; 16, 9, 8, 6, 2) + i$, $i \in [0, 42]$,
$(0, 20, 1, 12, 2; \infty, 3, 10, 4, 9) + i$, $i \in [0, 42]$,
$(0, 18, 3, 15, 2; 10, 7, 4, 1, \infty) + i$, $i \in [0, 42]$,
$(0, 21, 1, 17, 8; 7, 3, \infty, 6, 2) + i$, $i \in [0, 42]$,
$(0, 19, 2, 6, 3; \infty, 5, 9, 11, 4) + i$, $i \in [0, 42]$,
$(0, 16, 1, 13, 4; \infty, 3, 11, 5, 10) + i$, $i \in [0, 42]$.

$GD(48, Q_5, 5)$, $V = Z_{47} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 23, 1, 20, 2; 6, 3, 5, 4, 7) + i$, $i \in [0, 46]$,
$(0, 21, 4, 19, 7; 10, 8, 3, 5, 15) + i$, $i \in [0, 46]$,
$(0, 23, 2, 18, 9; 5, 1, 4, 3, 6) + i$, $i \in [0, 46]$,
$(0, 20, 2, 16, 3; 1, 9, 6, 4, 10) + i$, $i \in [0, 46]$,
$(0, 23, 4, 21, 11; 5, 1, 10, 2, 3) + i$, $i \in [0, 46]$,
$(0, 21, 1, 19, 3; 4, 6, 7, 2, 11) + i$, $i \in [0, 46]$,
$(0, 23, 9, 11, 7; 10, 1, 4, 2, 6) + i$, $i \in [0, 46]$,
$(0, 21, 1, 19, 8; 13, 2, \infty, 7, 9) + i$, $i \in [0, 46]$,

$(0, 17, 1, 13, 2; \infty, 3, 10, 6, 8) + i$, $i \in [0, 46]$,

$(0, 23, 2, 17, 4; \infty, 1, 8, 3, 5) + i$, $i \in [0, 46]$,

$(0, 20, 9, 12, 2; \infty, 1, 6, 5, 10) + i$, $i \in [0, 46]$,

$(0, 18, 2, 14, 5; 13, 1, \infty, 4, 20) + i$, $i \in [0, 46]$.

$GD(49, Q_5, 5)$, $V = Z_{49}$, $\mathcal{B}$:

$(0, 23, 1, 20, 2; 6, 3, 5, 4, 7) + i$, $i \in [0, 48]$,

$(0, 21, 4, 19, 7; 10, 8, 3, 5, 15) + i$, $i \in [0, 48]$,

$(0, 23, 2, 18, 9; 5, 1, 4, 3, 6) + i$, $i \in [0, 48]$,

$(0, 20, 2, 16, 3; 1, 9, 6, 4, 10) + i$, $i \in [0, 48]$,

$(0, 23, 4, 21, 11; 5, 1, 10, 2, 3) + i$, $i \in [0, 48]$,

$(0, 21, 1, 19, 3; 4, 6, 7, 2, 11) + i$, $i \in [0, 48]$,

$(0, 23, 9, 11, 7; 10, 1, 4, 2, 6) + i$, $i \in [0, 48]$,

$(0, 21, 1, 19, 8; 13, 2, 25, 7, 9) +i$, $i \in [0, 48]$,

$(0, 17, 1, 13, 2; 24, 3, 10, 6, 8) + i$, $i \in [0, 48]$,

$(0, 23, 2, 17, 4; 24, 1, 8, 3, 5) + i$, $i \in [0, 48]$,

$(0, 20, 9, 12, 2; 24, 1, 6, 5, 10) + i$, $i \in [0, 48]$,

$(0, 18, 2, 14, 5; 13, 1, 26, 4, 20) + i$, $i \in [0, 48]$.

$GD(52, Q_5, 5)$, $V = Z_{51} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 25, 1, 20, 3; 6, 2, 5, 4, 8) + i$, $i \in [0, 50]$,

$(0, 22, 1, 19, 8; 7, 2, 3, 4, 9) + i$, $i \in [0, 50]$,

$(0, 25, 4, 18, 6; 10, 1, 3, 5, 2) + i$, $i \in [0, 50]$,

$(0, 23, 1, 20, 2; 9, 6, 10, 8, 7) + i$, $i \in [0, 50]$,

$(0, 20, 4, 17, 7; 11, 6, 5, 2, 10) + i$, $i \in [0, 50]$,

$(0, 23, 2, 18, 6; 8, 1, 4, 3, 10) + i$, $i \in [0, 50]$,

$(0, 25, 1, 20, 3; 8, 5, 6, 2, 10) + i$, $i \in [0, 50]$,

$(0, 25, 4, 18, 5; 10, 1, 15, 21, 6) + i$, $i \in [0, 50]$,

$(0, 23, 1, 19, 9; \infty, 3, 6, 2, 5) + i$, $i \in [0, 50]$,

$(0, 19, 5, 17, 8; \infty, 4, 7, 6, 2) + i$, $i \in [0, 50]$,

$(0, 25, 1, 14, 3; 7, 9, \infty, 2, 4) + i$, $i \in [0, 50]$,

$(0, 23, 1, 20, 2; \infty, 3, 8, 4, 11) + i$, $i \in [0, 50]$,

$(0, 21, 4, 18, 8; \infty, 6, 10, 5, 12) + i$, $i \in [0, 50]$.

$GD(53, Q_5, 5)$, $V = Z_{53}$, $\mathcal{B}$:

$(0, 25, 1, 20, 3; 6, 2, 5, 4, 8) + i$, $i \in [0, 50]$,
$(0, 22, 1, 19, 8; 7, 2, 3, 4, 9) + i$, $i \in [0, 50]$,
$(0, 25, 4, 18, 6; 10, 1, 3, 5, 2) + i$, $i \in [0, 50]$,
$(0, 23, 1, 20, 2; 9, 6, 10, 8, 7) + i$, $i \in [0, 50]$,
$(0, 20, 4, 17, 7; 11, 6, 5, 2, 10) + i$, $i \in [0, 50]$,
$(0, 23, 2, 18, 6; 8, 1, 4, 3, 10) + i$, $i \in [0, 50]$,
$(0, 25, 1, 20, 3; 8, 5, 6, 2, 10) + i$, $i \in [0, 50]$,
$(0, 25, 4, 18, 5; 10, 1, 15, 21, 6) + i$, $i \in [0, 50]$,
$(0, 23, 1, 19, 9; 26, 3, 6, 2, 5) + i$, $i \in [0, 50]$,
$(0, 19, 5, 17, 8; 26, 4, 7, 6, 2) + i$, $i \in [0, 50]$,
$(0, 25, 1, 14, 3; 7, 9, 27, 2, 4) + i$, $i \in [0, 50]$,
$(0, 23, 1, 20, 2; 26, 3, 8, 4, 11) + i$, $i \in [0, 50]$,
$(0, 21, 4, 18, 8; 26, 6, 10, 5, 12) + i$, $i \in [0, 50]$.

$GD(57, Q_5, 5)$, $V = Z_{57}$, $\mathcal{B}$:

$(0, 28, 1, 22, 3; 5, 2, 7, 4, 6) + i$, $i \in [0, 56]$,
$(0, 25, 1, 23, 7; 8, 2, 5, 3, 6) + i$, $i \in [0, 56]$,
$(0, 17, 2, 15, 4; 9, 3, 7, 5, 6) + i$, $i \in [0, 56]$,
$(0, 28, 1, 26, 6; 12, 2, 8, 3, 5) + i$, $i \in [0, 56]$,
$(0, 24, 2, 21, 8; 10, 3, 11, 5, 6) + i$, $i \in [0, 56]$,
$(0, 28, 4, 22, 5; 12, 1, 6, 7, 11) + i$, $i \in [0, 56]$,
$(0, 26, 1, 17, 3; 11, 4, 2, 6, 7) + i$, $i \in [0, 56]$,
$(0, 21, 1, 19, 7; 13, 4, 9, 5, 17) + i$, $i \in [0, 56]$,
$(0, 28, 1, 24, 5; 6, 2, 10, 9, 4) + i$, $i \in [0, 56]$,
$(0, 25, 1, 20, 11; 10, 2, 5, 3, 4) + i$, $i \in [0, 56]$,
$(0, 22, 1, 19, 3; 12, 2, 9, 18, 5) + i$, $i \in [0, 56]$,
$(0, 28, 1, 22, 9; 11, 2, 5, 8, 3) + i$, $i \in [0, 56]$,
$(0, 25, 1, 20, 3; 10, 2, 9, 4, 5) + i$, $i \in [0, 56]$,
$(0, 22, 2, 17, 5; 13, 4, 9, 3, 21) + i$, $i \in [0, 56]$.

$GD(14, Q_5, 10)$, $V = Z_{13} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 6, 8, 3, 4; 1, 9, 5, 2, 10) + i$, $i \in [0, 12]$,

$(0, 6, 10, 5, 3; \infty, 8, 4, 1, 2) + i, i \in [0, 12]$,
$(0, 5, 7, 4, 6; \infty, 3, 2, 8, 1) + i, i \in [0, 12]$,
$(0, 6, 10, 7, 5; 1, \infty, 4, 3, 2) + i, i \in [0, 12]$,
$(0, 6, 9, 5, 4; \infty, 1, 7, 8, 3) + i, i \in [0, 12]$,
$(0, 6, 7, 4, \infty; 5, 1, 2, 3, 8) + i, i \in [0, 12]$,
$(0, 6, 10, 8, \infty; 2, 3, 9, 4, 1) + i, i \in [0, 12]$.

$GD(18, Q_5, 10)$, $V = Z_{17} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 8, 10, 7, 6; 4, 1, 3, 2, 5) + i, i \in [0, 16]$,
$(0, 8, 11, 7, 2; \infty, 4, 3, 1, 5) + i, i \in [0, 16]$,
$(0, 8, 10, 5, 6; \infty, 1, 3, 4, 2) + i, i \in [0, 16]$,
$(0, 8, 11, 6, 4; \infty, 1, 5, 3, 2) + i, i \in [0, 16]$,
$(0, 8, 2, 7, 6; 5, 1, \infty, 3, 4) + i, i \in [0, 16]$,
$(0, 8, 11, 6, 7; \infty, 2, 9, 3, 1) + i, i \in [0, 16]$,
$(0, 8, 4, 5, 7; \infty, 6, 3, 2, 1) + i, i \in [0, 16]$,
$(0, 8, 3, 7, 2; 5, 1, \infty, 4, 6) + i, i \in [0, 16]$,
$(0, 8, 9, 6, \infty; 4, 1, 3, 5, 2) + i, i \in [0, 16]$.

$GD(19, Q_5, 10)$, $V = Z_{19}$, $\mathcal{B}$:

$(0, 8, 10, 7, 6; 4, 1, 3, 2, 5) + i, i \in [0, 18]$,
$(0, 8, 11, 7, 2; 9, 4, 3, 1, 5) + i, i \in [0, 18]$,
$(0, 8, 10, 5, 6; 9, 1, 3, 4, 2) + i, i \in [0, 18]$,
$(0, 8, 11, 6, 4; 9, 1, 5, 3, 2) + i, i \in [0, 18]$,
$(0, 8, 2, 7, 6; 5, 1, 11, 10, 4) + i, i \in [0, 18]$,
$(0, 8, 11, 6, 7; 5, 4, 2, 3, 1) + i, i \in [0, 18]$,
$(0, 8, 4, 5, 7; 9, 6, 3, 2, 1) + i, i \in [0, 18]$,
$(0, 8, 3, 7, 2; 9, 1, 12, 4, 11) + i, i \in [0, 18]$,
$(0, 8, 9, 3, 7; 4, 6, 10, 12, 1) + i, i \in [0, 18]$.

$GD(22, Q_5, 10)$, $V = Z_{21} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 10, 1, 4, 5; 6, 2, 9, 8, 3) + i, i \in [0, 20]$,
$(0, 10, 1, 8, 2; \infty, 3, 9, 4, 5) + i, i \in [0, 20]$,
$(0, 7, 2, 3, 10; 1, 5, \infty, 6, 4) + i, i \in [0, 20]$,
$(0, 10, 1, 6, 2; 3, 4, \infty, 5, 7) + i, i \in [0, 20]$,

$(0, 10, 1, 5, 7; \infty, 3, 4, 9, 2) + i$, $i \in [0, 20]$,
$(0, 10, 1, 7, 6; 8, 2, \infty, 5, 3) + i$, $i \in [0, 20]$,
$(0, 10, 1, 8, 4; 7, 3, 6, 2, \infty) + i$, $i \in [0, 20]$,
$(0, 10, 1, 9, 2; 5, 6, 4, 8, \infty) + i$, $i \in [0, 20]$,
$(0, 5, 2, 8, 3; 9, 1, 7, 6, \infty) + i$, $i \in [0, 20]$,
$(0, 10, 1, 9, 3; \infty, 4, 2, 8, 5) + i$, $i \in [0, 20]$,
$(0, 10, 1, 9, 8; \infty, 2, 3, 5, 7) + i$, $i \in [0, 20]$.

$GD(23, Q_5, 10)$, $V = Z_{23}$, $\mathcal{B}$:

$(0, 10, 1, 4, 5; 6, 2, 9, 8, 3) + i$, $i \in [0, 22]$,
$(0, 10, 1, 8, 2; 11, 3, 9, 4, 5) + i$, $i \in [0, 22]$,
$(0, 7, 2, 3, 10; 1, 5, 13, 6, 4) + i$, $i \in [0, 22]$,
$(0, 10, 1, 6, 2; 3, 4, 12, 5, 7) + i$, $i \in [0, 22]$,
$(0, 10, 1, 5, 7; 11, 3, 4, 9, 2) + i$, $i \in [0, 22]$,
$(0, 10, 1, 7, 6; 8, 2, 12, 5, 3) + i$, $i \in [0, 22]$,
$(0, 10, 1, 8, 4; 7, 3, 6, 2, 15) + i$, $i \in [0, 22]$,
$(0, 10, 1, 9, 2; 5, 6, 4, 8, 13) + i$, $i \in [0, 22]$,
$(0, 5, 2, 8, 3; 9, 1, 7, 6, 14) + i$, $i \in [0, 22]$,
$(0, 10, 1, 9, 3; 11, 4, 2, 8, 5) + i$, $i \in [0, 22]$,
$(0, 10, 1, 9, 8; 11, 2, 3, 5, 7) + i$, $i \in [0, 22]$.

$GD(27, Q_5, 10)$, $V = Z_{27}$, $\mathcal{B}$:

$(0, 13, 1, 10, 4; 8, 2, 3, 7, 5) + i$, $i \in [0, 26]$,
$(0, 13, 3, 10, 2; 4, 1, 6, 5, 9) + i$, $i \in [0, 26]$,
$(0, 11, 1, 10, 6; 4, 3, 2, 5, 7) + i$, $i \in [0, 26]$,
$(0, 13, 1, 11, 3; 5, 2, 10, 4, 9) + i$, $i \in [0, 26]$,
$(0, 13, 1, 12, 2; 5, 6, 4, 3, 7) + i$, $i \in [0, 26]$,
$(0, 13, 2, 12, 3; 6, 1, 4, 5, 7) + i$, $i \in [0, 26]$,
$(0, 13, 2, 10, 4; 6, 1, 7, 3, 5) + i$, $i \in [0, 26]$,
$(0, 13, 3, 11, 4; 8, 1, 6, 2, 5) + i$, $i \in [0, 26]$,
$(0, 12, 1, 10, 7; 6, 2, 3, 7, 5) + i$, $i \in [0, 26]$,
$(0, 13, 2, 9, 7; 10, 1, 3, 4, 5) + i$, $i \in [0, 26]$,
$(0, 13, 2, 10, 5; 8, 4, 6, 1, 3)+ i$, $i \in [0, 26]$,

(0, 13, 1, 11, 2; 6, 7, 4, 8, 3) + $i$, $i \in [0, 26]$,
(0, 11, 1, 9, 2; 5, 7, 4, 8, 3) + $i$, $i \in [0, 26]$.

$GD(34, Q_5, 10) \Leftarrow IGD(34, 14, Q_5, 10) \wedge GD(14, Q_5, 10)$.
$GD(37, Q_5, 10) \Leftarrow IGD(37, 17, Q_5, 10) \wedge GD(17, Q_5, 10)$.
$GD(38, Q_5, 10) \Leftarrow IGD(38, 18, Q_5, 10) \wedge GD(18, Q_5, 10)$.
$GD(39, Q_5, 10) \Leftarrow IGD(39, 19, Q_5, 10) \wedge GD(19, Q_5, 10)$.

$GD(42, Q_5, 10)$, $V = Z_{41} \cup \{\infty\}$, $\mathcal{B}$:

(0, 20, 1, 18, 3; 9, 2, 5, 4, 8) + $i$, $i \in [0, 40]$,
(0, 13, 1, 11, 3; 6, 2, 8, 5, 7) + $i$, $i \in [0, 40]$,
(0, 20, 1, 18, 3; 10, 2, 6, 4, 5) + $i$, $i \in [0, 40]$,
(0, 16, 3, 14, 2; 1, 7, 4, 6, 9) + $i$, $i \in [0, 40]$,
(0, 20, 2, 18, 4; 7, 5, 3, 1, 9) + $i$, $i \in [0, 40]$,
(0, 19, 6, 18, 7; 9, 1, 4, 8, 2) + $i$, $i \in [0, 40]$,
(0, 20, 1, 18, 6; 4, 2, 9, 3, 16) + $i$, $i \in [0, 40]$,
(0, 20, 4, 12, 1; 13, 6, 2, 3, 7) + $i$, $i \in [0, 40]$,
(0, 19, 2, 16, 3; 5, 7, 6, 1, 4) + $i$, $i \in [0, 40]$,
(0, 20, 4, 11, 2; 6, 1, 5, 3, 9) + $i$, $i \in [0, 40]$,
(0, 17, 3, 14, 4; 2, 5, 9, 6, 7) + $i$, $i \in [0, 40]$,
(0, 20, 1, 17, 4; ∞, 2, 6, 7, 11) + $i$, $i \in [0, 40]$,
(0, 20, 1, 18, 3; 10, 2, ∞, 7, 5) + $i$, $i \in [0, 40]$,
(0, 16, 2, 14, 5; 8, 3, ∞, 4, 6) + $i$, $i \in [0, 40]$,
(0, 20, 2, 17, 6; ∞, 3, 10, 1, 4) + $i$, $i \in [0, 40]$,
(0, 15, 4, 12, 2; 6, 1, 11, 3, ∞) + $i$, $i \in [0, 40]$,
(0, 19, 2, 15, 4; ∞, 1, 5, 3, 9) + $i$, $i \in [0, 40]$,
(0, 16, 4, 13, 6; ∞, 2, 3, 8, 9) + $i$, $i \in [0, 40]$,
(0, 20, 1, 16, 3; 9, 2, ∞, 4, 7) + $i$, $i \in [0, 40]$,
(0, 17, 1, 14, 3; ∞, 2, 7, 4, 5) + $i$, $i \in [0, 40]$,
(0, 16, 2, 9, 5; ∞, 7, 3, 1, 4) + $i$, $i \in [0, 40]$.

$GD(43, Q_5, 10)$, $V = Z_{43}$, $\mathcal{B}$:

(0, 20, 1, 18, 3; 9, 2, 5, 4, 8) + $i$, $i \in [0, 42]$,
(0, 13, 1, 11, 3; 6, 2, 8, 5, 7) + $i$, $i \in [0, 42]$,

$(0, 20, 1, 18, 3; 10, 2, 6, 4, 5) + i, i \in [0, 42]$,
$(0, 16, 3, 14, 2; 1, 7, 4, 6, 9) + i, i \in [0, 42]$,
$(0, 20, 2, 18, 4; 7, 5, 3, 1, 9) + i, i \in [0, 42]$,
$(0, 19, 6, 18, 7; 9, 1, 4, 8, 2) + i, i \in [0, 42]$,
$(0, 20, 1, 18, 6; 4, 2, 9, 3, 16) + i, i \in [0, 42]$,
$(0, 20, 4, 12, 1; 13, 6, 2, 3, 7) + i, i \in [0, 42]$,
$(0, 19, 2, 16, 3; 5, 7, 6, 1, 4) + i, i \in [0, 42]$,
$(0, 20, 4, 11, 2; 6, 1, 5, 3, 9) + i, i \in [0, 42]$,
$(0, 17, 3, 14, 4; 2, 5, 9, 6, 7) + i, i \in [0, 42]$,
$(0, 20, 1, 17, 4; 21, 2, 6, 7, 11) + i, i \in [0, 42]$,
$(0, 20, 1, 18, 3; 10, 2, 22, 7, 5) + i, i \in [0, 42]$,
$(0, 16, 2, 14, 5; 8, 3, 23, 4, 6) + i, i \in [0, 42]$,
$(0, 20, 2, 17, 6; 21, 3, 10, 1, 4) + i, i \in [0, 42]$,
$(0, 15, 4, 12, 2; 6, 1, 11, 3, 23) + i, i \in [0, 42]$,
$(0, 19, 2, 15, 4; 21, 1, 5, 3, 9) + i, i \in [0, 42]$,
$(0, 16, 4, 13, 6; 21, 2, 3, 8, 9) + i, i \in [0, 42]$,
$(0, 20, 1, 16, 3; 9, 2, 22, 4, 7) + i, i \in [0, 42]$,
$(0, 17, 1, 14, 3; 21, 2, 7, 4, 5) + i, i \in [0, 42]$,
$(0, 16, 2, 9, 5; 21, 7, 3, 1, 4) + i, i \in [0, 42]$.

$GD(47, Q_5, 10)$, $V = Z_{47}$, $\mathcal{B}$:

$(0, 23, 1, 20, 3; 6, 2, 5, 4, 8) + i, i \in [0, 46]$,
$(0, 20, 2, 17, 7; 8, 6, 3, 4, 5) + i, i \in [0, 46]$,
$(0, 23, 1, 13, 2; 7, 3, 6, 4, 8) + i, i \in [0, 46]$,
$(0, 21, 2, 17, 3; 9, 8, 10, 1, 7) + i, i \in [0, 46]$,
$(0, 23, 5, 6, 12; 10, 2, 7, 9, 1) + i, i \in [0, 46]$,
$(0, 22, 2, 19, 4; 8, 3, 9, 1, 5) + i, i \in [0, 46]$,
$(0, 17, 1, 15, 5; 11, 4, 10, 3, 7) + i, i \in [0, 46]$,
$(0, 23, 1, 19, 4; 9, 2, 5, 3, 7) + i, i \in [0, 46]$,
$(0, 20, 1, 18, 5; 3, 6, 8, 2, 4) + i, i \in [0, 46]$,
$(0, 23, 3, 14, 4; 8, 1, 9, 2, 5) + i, i \in [0, 46]$,
$(0, 21, 2, 19, 4; 10, 3, 13, 5, 9) + i, i \in [0, 46]$,

(0, 13, 1, 10, 6; 8, 2, 7, 3, 4) + $i$, $i \in [0, 46]$,
(0, 23, 4, 18, 8; 9, 1, 7, 2, 3) + $i$, $i \in [0, 46]$,
(0, 21, 1, 18, 3; 16, 2, 4, 6, 5) + $i$, $i \in [0, 46]$,
(0, 18, 4, 17, 7; 15, 1, 6, 2, 8) + $i$, $i \in [0, 46]$,
(0, 23, 2, 10, 8; 6, 1, 4, 3, 7) + $i$, $i \in [0, 46]$,
(0, 20, 2, 15, 5; 7, 9, 1, 3, 8) + $i$, $i \in [0, 46]$,
(0, 23, 9, 15, 5; 21, 1, 2, 6, 3) + $i$, $i \in [0, 46]$,
(0, 21, 1, 19, 9; 11, 5, 6, 4, 2) + $i$, $i \in [0, 46]$,
(0, 23, 1, 20, 4; 16, 3, 10, 8, 9) + $i$, $i \in [0, 46]$,
(0, 21, 3, 16, 4; 15, 2, 6, 5, 10) + $i$, $i \in [0, 46]$,
(0, 23, 1, 18, 6; 8, 10, 2, 5, 20) + $i$, $i \in [0, 46]$,
(0, 20, 1, 18, 17; 8, 2, 10, 7, 3) + $i$, $i \in [0, 46]$.

$GD(54, Q_5, 10)$, $V = Z_{53} \cup \{\infty\}$, $\mathcal{B}$:

(0, 26, 1, 19, 2; 12, 3, 7, 4, 9) + $i$, $i \in [0, 52]$,
(0, 24, 2, 18, 8; 9, 1, 4, 3, 5) + $i$, $i \in [0, 52]$,
(0, 21, 1, 15, 4; 7, 8, 4, 3, 5) + $i$, $i \in [0, 52]$,
(0, 26, 4, 20, 1; 6, 2, 9, 3, 5) + $i$, $i \in [0, 52]$,
(0, 25, 4, 18, 8; 11, 5, 7, 6, 9) + $i$, $i \in [0, 52]$,
(0, 19, 1, 14, 4; 6, 2, 3, 5, 9) + $i$, $i \in [0, 52]$,
(0, 26, 2, 24, 7; 9, 1, 8, 3, 5) + $i$, $i \in [0, 52]$,
(0, 19, 3, 16, 4; 8, 1, 14, 2, 9) + $i$, $i \in [0, 52]$,
(0, 23, 3, 18, 6; 8, 1, 7, 2, 9) + $i$, $i \in [0, 52]$,
(0, 26, 1, 20, 5; 10, 2, 8, 6, 4) + $i$, $i \in [0, 52]$,
(0, 23, 2, 20, 7; 8, 3, 11, 5, 6) + $i$, $i \in [0, 52]$,
(0, 26, 1, 12, 3; 10, 2, 7, 4, 5) + $i$, $i \in [0, 52]$,
(0, 23, 1, 19, 4; 11, 2, 8, 3, 9) + $i$, $i \in [0, 52]$,
(0, 19, 2, 16, 3; 10, 1, 8, 4, 5) + $i$, $i \in [0, 52]$,
(0, 26, 1, 20, 4; 7, 6, 2, 9, 3) + $i$, $i \in [0, 52]$,
(0, 24, 1, 18, 6; 3, 2, 9, 5, 4) + $i$, $i \in [0, 52]$,
(0, 21, 1, 19, 5; 15, 2, 10, 4, 3) + $i$, $i \in [0, 52]$,
(0, 26, 1, 20, 3; $\infty$, 2, 10, 4, 8) + $i$, $i \in [0, 52]$,

$(0, 23, 1, 15, 3; 9, 2, \infty, 4, 5) + i$, $i \in [0, 52]$,

$(0, 18, 5, 15, 7; \infty, 3, 6, 2, 11) + i$, $i \in [0, 52]$,

$(0, 26, 1, 18, 5; 10, 2, \infty, 6, 4) + i$, $i \in [0, 52]$,

$(0, 23, 1, 21, 6; \infty, 2, 9, 5, 8) + i$, $i \in [0, 52]$,

$(0, 26, 2, 16, 6; \infty, 1, 13, 7, 10) + i$, $i \in [0, 52]$,

$(0, 23, 1, 21, 3; 11, 2, \infty, 5, 10) + i$, $i \in [0, 52]$,

$(0, 20, 1, 15, 4; \infty, 3, 17, 2, 5) + i$, $i \in [0, 52]$,

$(0, 26, 1, 23, 5; 10, 2, \infty, 3, 14) + i$, $i \in [0, 52]$,

$(0, 23, 2, 19, 7; \infty, 4, 10, 5, 12) + i$, $i \in [0, 52]$.

$GD(58, Q_5, 10)$, $V = Z_{57} \cup \{\infty\}$, $\mathcal{B}$:

$(0, 28, 1, 20, 4; 7, 2, 6, 3, 5) + i$, $i \in [0, 56]$,

$(0, 25, 1, 21, 6; 8, 2, 16, 3, 15) + i$, $i \in [0, 56]$,

$(0, 28, 6, 20, 8; 10, 7, 4, 9, 1) + i$, $i \in [0, 56]$,

$(0, 28, 1, 21, 3; 5, 2, 7, 4, 12) + i$, $i \in [0, 56]$,

$(0, 25, 1, 22, 10; 11, 2, 5, 3, 8) + i$, $i \in [0, 56]$,

$(0, 22, 6, 7, 13; 10, 8, 1, 3, 2) + i$, $i \in [0, 56]$,

$(0, 27, 1, 22, 3; 8, 2, 10, 7, 5) + i$, $i \in [0, 56]$,

$(0, 24, 1, 21, 3; 12, 2, 5, 8, 4) + i$, $i \in [0, 56]$,

$(0, 28, 1, 23, 4; 10, 2, 8, 6, 5) + i$, $i \in [0, 56]$,

$(0, 21, 1, 17, 2; 5, 3, 7, 4, 8) + i$, $i \in [0, 56]$,

$(0, 25, 1, 18, 3; 11, 2, 10, 4, 8) + i$, $i \in [0, 56]$,

$(0, 28, 1, 17, 3; 7, 2, 11, 5, 1) + i$, $i \in [0, 56]$,

$(0, 25, 1, 21, 8; 11, 4, 10, 5, 7) + i$, $i \in [0, 56]$,

$(0, 22, 4, 19, 5; 7, 1, 3, 2, 9) + i$, $i \in [0, 56]$,

$(0, 28, 1, 21,8;10; 2; 7; 9; 3) + i$, $i \in [0, 56]$,

$(0, 25, 1, 23, 4; 8, 2, 12, 5, 11) + i$, $i \in [0, 56]$,

$(0, 17, 1, 15, 3; 8, 2, 7, 6, 5) + i$, $i \in [0, 56]$,

$(0, 28, 1, 20, 3; 15, 2, 8, 7, 4) + i$, $i \in [0, 56]$,

$(0, 25, 1, 19, 4; 8, 2, 7, 5, 6) + i$, $i \in [0, 56]$,

$(0, 22, 1, 17, 6; 9, 2, \infty, 5, 8) + i$, $i \in [0, 56]$,

$(0, 28, 1, 20, 7; \infty, 2, 4, 3, 6) + i$, $i \in [0, 56]$,

$(0, 25, 1, 21, 3; 10, 2, \infty, 7, 9) + i, i \in [0, 56]$,
$(0, 22, 1, 17, 2; \infty, 3, 5, 6, 7) + i, i \in [0, 56]$,
$(0, 28, 5, 15, 8; \infty, 1, 6, 3, 4) + i, i \in [0, 56]$,
$(0, 25, 1, 23, 5; 10, 2, \infty, 3, 14) + i, i \in [0, 56]$,
$(0, 28, 1, 20, 7; 9, 2, \infty, 3, 6) + i, i \in [0, 56]$,
$(0, 26, 1, 22, 10; \infty, 2, 12, 3, 8) + i, i \in [0, 56]$,
$(0, 23, 1, 21, 13; \infty, 2, 12, 7, 4) + i, i \in [0, 56]$,
$(0, 18, 1, 17, 3; 12, 2, \infty, 4, 8) + i, i \in [0, 56]$.

$GD(59, Q_5, 10)$, $V = Z_{59}$, $\mathcal{B}$:

$(0, 28, 1, 20, 4; 7, 2, 6, 3, 5) + i, i \in [0, 58]$,
$(0, 25, 1, 21, 6; 8, 2, 16, 3, 15) + i, i \in [0, 58]$,
$(0, 28, 6, 20, 8; 10, 7, 4, 9, 1) + i, i \in [0, 58]$,
$(0, 28, 1, 21, 3; 5, 2, 7, 4, 12) + i, i \in [0, 58]$,
$(0, 25, 1, 22, 10; 11, 2, 5, 3, 8) + i, i \in [0, 58]$,
$(0, 22, 6, 7, 13; 10, 8, 1, 3, 2) + i, i \in [0, 58]$,
$(0, 27, 1, 22, 3; 8, 2, 10, 7, 5) + i, i \in [0, 58]$,
$(0, 24, 1, 21, 3; 12, 2, 5, 8, 4) + i, i \in [0, 58]$,
$(0, 28, 1, 23, 4; 10, 2, 8, 6, 5) + i, i \in [0, 58]$,
$(0, 21, 1, 17, 2; 5, 3, 7, 4, 8) + i, i \in [0, 58]$,
$(0, 25, 1, 18, 3; 11, 2, 10, 4, 8) + i, i \in [0, 58]$,
$(0, 28, 1, 17, 3; 7, 2, 11, 5, 1) + i, i \in [0, 58]$,
$(0, 25, 1, 21, 8; 11, 4, 10, 5, 7) + i, i \in [0, 58]$,
$(0, 22, 4, 19, 5; 7, 1, 3, 2, 9) + i, i \in [0, 58]$,
$(0, 28, 1, 21,8;10; 2; 7; 9; 3) + i, i \in [0, 58]$,
$(0, 25, 1, 23, 4; 8, 2, 12, 5, 11) + i, i \in [0, 58]$,
$(0, 17, 1, 15, 3; 8, 2, 7, 6, 5) + i, i \in [0, 58]$,
$(0, 28, 1, 20, 3; 15, 2, 8, 7, 4) + i, i \in [0, 58]$,
$(0, 25, 1, 19, 4; 8, 2, 7, 5, 6) + i, i \in [0, 56]$,
$(0, 22, 1, 17, 6; 9, 2, 30, 5, 8) + i, i \in [0, 58]$,
$(0, 28, 1, 20, 7; 29, 2, 4, 3, 6) + i, i \in [0, 58]$,
$(0, 25, 1, 21, 3; 10, 2, 30, 7, 9) + i, i \in [0, 58]$,

(0, 22, 1, 17, 2; 29, 3, 5, 6, 7) + $i$, $i \in [0, 58]$,
(0, 28, 5, 15, 8; 29, 1, 6, 3, 4) + $i$, $i \in [0, 58]$,
(0, 25, 1, 23, 5; 10, 2, 30, 3, 14) + $i$, $i \in [0, 58]$,
(0, 28, 1, 20, 7; 9, 2, 30, 3, 6) + $i$, $i \in [0, 58]$,
(0, 26, 1, 22, 10; 29, 2, 12, 3, 8) + $i$, $i \in [0, 58]$,
(0, 23, 1, 21, 13; 29, 2, 12, 7, 4) + $i$, $i \in [0, 58]$,
(0, 18, 1, 17, 3; 12, 2, 30, 4, 8) + $i$, $i \in [0, 58]$.