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INDEPENDENT SET POLYNOMIALS $I(G; x)$ AND INDEPENDENCE POLYNOMIALS $I_\alpha(G; x)$ (Series 3)

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ABSTRACT: In graph theory, independent set polynomials $I(G; x)$ and independence polynomials $I_\alpha(G; x)$ are NP-hard (see [1] and [2]). In this paper, our ways are combinatorial counting methods. In the use of counting theory of $S^{(n)}$-factors with exactly k components, the author gains the representing formula of independent set polynomials $I(G; x)$ and independence polynomials $I_\alpha(G; x)$,where let $b_k(G)$ be exactly k-independent sets of G, and presents the explicit formulas of independent set polynomials $I(G; x)$ and independence polynomials $I_\alpha(G; x)$ for a great deal of graphs.

KEYWORDS: Component; $N(G; k)$, independent set polynomials, independence polynomials.

AMS(2000) SUBJECT CLASSIFICATION: 05A18 05C10.

1. INTRODUCTION

In this paper, the author will solve independent set polynomials $I(G; x)$ and Independence polynomials $I_\alpha(G; x)$ by means of counting theory of $S^{(n)}$-factors.

Definition 1.1: For $S^{(n)} = \{K_i : 1 \le i \le n\}$; $n \ge 1$, K_i is a complete graph with i vertices, if M is a subgraph of any graph G, and each component of M is all isomorphic to some element of $S^{(n)} = \{K_i : 1 \le i \le n\}$, then M is called one $S^{(n)}$-subgraph, if M is a spanning subgraph of G, then M is called one $S^{(n)}$-factor of G.

Let $N(G, k)$ denote the number of $S^{(n)}$-factors with exactly k components. $A(G)$ is the number of all $S^{(n)}$-factors, namely, $A(G) = \sum_{k=1}^{n} N(G, k)$.

Definition 1.2: Independent set polynomials $I(G; x)$ are defined as

$$I(G; x) = \sum_{k=1}^{n} b_k(G)x^k = \sum_{I \subset v(G)} \prod_{v \in I} x,$$

where let $b_k(G)$ be exactly k-independent sets of G.

Complexity: It is easy to see that $I(G; x)$ is NP-hard to compute. (see [1])

Definition 1.3: If s_k denotes the number of stable sets of cardinality k in graph G, and $\alpha(G)$ is the size of a maximum stable set, then

$$I_{\alpha}(G;x)=\sum_{k=1}^{\alpha(G)} s_k x^k,$$

is called the independence polynomial of G. (Also see[2])

In the paper [8], LiMin Yang gived the recurrence relation of $A(G)$. In the paper [9], LiMin Yang derived the recurrence formula of regular m-furcating tree. So far, we have solved counting problems of $N(G, k)$ (see [10]), involving the representing formula of $N(G, k)$ and counting formulas of a great deal of graphs, for examples, any path, cycle, complete graph, $O\odot C_n$, windgraph K_n^d, complete d-partite graph, $n-2$-regular graph and $n-3$-regular graph. In the paper [3], we have solved the number of exactly k independent sets of graphs. In the paper [4], we have completed enumeration of all independent sets of graphs. In this paper, the author present independent set polynomials $I(G; x)$ and independence polynomials and the explicit formulas of classes of graphs by means of counting theory of $S^{(n)}$-factors.

2. LEMMAS

Here we will denote that $\alpha(G, k)$ is the number of partitions of $V(G)$ into exactly k non-empty independent sets of any graph G.

Lemma 2.1 ([3]): If $N(G, k)$ is the number of $S^{(n)}$-factors with exactly k components in G, and the chromatic polynomial of graph G is $f(G,t)=\sum_{p=1}^{n} Y_p t^p$, then the representing formula of $\alpha(G, k)$ is the following:

$$\alpha(G,k)=\sum_{p=k}^{n} N\left(K_p,k\right)Y_p,$$

where

$$N\left(K_p,k\right)=\sum_{\sum_{i=1}^{p} ib_i=p\sum_{i=1}^{p} b_i=k}\frac{p!}{b_1!}\prod_{i\ge 2}^{p}\frac{1}{b_i!(i!)^{b_i}}.$$

Lemma 2.2 ([3]): There exists the equality $\alpha(G,k)=N\left(\overline{G},k\right)$.

Lemma 2.3: If $S(n, k)$ is the Stirling number of the second kind, then $N(K_n, k) = S(n, k)$, where K_n is a complete graph with n vertices.

Lemma 2.4: *If* $G \cap H = \phi$, then $N(G \cup H, k) = \sum_{l+m=k} N(G, l) N(H, m)$.

3. MAIN THEOREMS

Theorem 3.1: If the chromatic polynomial of any graph G is $f(G, t) = \sum_{p=1}^{n} Y_p t^p$ then independent set polynomial $I(G; x)$

$$I(G; x) = \sum_{k=1}^{n} \sum_{p=k}^{n} N\left(K_p, k\right) Y_p x^k.$$

Proof: Because $b_k(G)$ is exactly k-independent sets of G and $\alpha(G, k)$ is the number of partitions of $V(G)$ into exactly k non-empty independent sets of any graph G, then $b_k(G) = \alpha(G; k)$. By Lemma 2.1

$$\alpha(G, k) = \sum_{p=k}^{n} N\left(K_p, k\right) Y_p,$$

where Y_p are coefficients of the chromatic polynomial of $f(G, t)$. Then independent set polynomial $I(G, x)$

$$I(G; x) = \sum_{k=1}^{n} \sum_{p=k}^{n} N\left(K_p, k\right) Y_p x^k.$$

Theorem 3.2: There exists the equality independent set polynomials

$$I(G; x) = \sum_{k=1}^{n} N\left(\bar{G}, k\right) x^k,$$

where $N\left(\bar{G}, k\right)$ is the number of $S^{(n)}$-factors with exactly k components in the complementary graph $\bar{G}$ of G.

Proof: By Lemma 2.2 $\alpha(G, k) = N\left(\bar{G}, k\right)$, so we gain

$$I(G; x) = \sum_{k=1}^{n} N\left(\bar{G}, k\right) x^k,$$

where $N\left(\bar{G}, k\right)$ is the number of $S^{(n)}$-factors with exactly k components in the complementary graph $\bar{G}$ of G.

4. CLASSES OF GRAPHS INDEPENDENT SET POLYNOMIALS $I(G; x)$

In the section, we will obtain classes of graphs independent set polynomials $I(G; x)$, for examples, any $(n-2)$-regular graph, $(n-3)$-regular graph and complete d-partite graph, tree.

Theorem 4.1: If G is a $(n-2)$-regular graph with n (even $2m$) vertices, then independent set polynomial

$$I(G; x) = \sum_{k=m}^{2m} \binom{m}{k-m} x^k.$$

Proof: Let G be a $(n-2)$-regular graph with n (even $2m$), then $\bar{G}$ is a 1-regular graph, namely, $\bar{G} = K_2 \cup K_2 \cup \ldots \cup K_2$, and the number of K_2 is m. We have

$$N\left(\bar{G}, k\right) = \begin{cases} 0, & 1 \le k < \dfrac{n}{2}, \\ \dbinom{\frac{n}{2}}{k-\frac{n}{2}}, & \dfrac{n}{2} \le k \le n. \end{cases}$$

Finally, by Theorem 3.2 then independent set polynomial

$$I(G; x) = \sum_{k=1}^{n} \binom{\frac{n}{2}}{k-\frac{n}{2}} x^k = \sum_{k=m}^{2m} \binom{m}{k-m} x^k.$$

Theorem 4.2: If G is a $(n-3)$-regular graph with n vertices, $n \ge 6$ and $\bar{G} \cong C_n$, then

$$I(G; x) = \sum_{k=\left[\frac{n}{2}\right]}^{n} \frac{n}{k} \binom{k}{n-k} x^k.$$

Proof: Let G be a $n-3$-regular graph with n vertices, $n \ge 6$ and $\bar{G} \cong C_n$, because

$\overline{G}$ is a 2-regular graph, the graph would be able to join the disjoint cycles, thus assume that C_n, say. Then we have

$$N\left(\overline{G}, k\right) = N\left(C_n, k\right) = \begin{cases} 0, & 1 \le k < \dfrac{n}{2}, \\ \dfrac{n}{k}\begin{pmatrix} k \\ k-k \end{pmatrix}, & \dfrac{n}{2} \le k \le n. \end{cases}$$

By Theorem 3.2, then the result is given the following

$$I(G; x) = \sum_{k=1}^{n} \frac{n}{k}\begin{pmatrix} k \\ n-k \end{pmatrix} x^k = \sum_{k=\left\lceil \frac{n}{2} \right\rceil}^{2m} \frac{n}{k}\begin{pmatrix} k \\ n-k \end{pmatrix} x^k .$$

Corollary 4.3: If G is a $(n-3)$-regular graph with n vertices, and

$$\overline{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q},$$

$n_1 + n_2 + ... + n_q = n$, $C_{n_i} \cap C_{n_j} = \phi$ for any i and j, $i \ne j$, $3 \le n_j \le n$; $1 \le j \le q$, $q \ge 1$, $n \ge 6$, the number of $n_j = 3$ is l, then independent set polynomial is gained as the following

$$I(G, x) = (x + 3x^2 + x^3)^l \prod_{j=1}^{q-l} \sum_{l_j=\left\lceil \frac{n_j}{2} \right\rceil}^{n_j} \frac{n_j}{l_j}\begin{pmatrix} l_j \\ n_j - l_j \end{pmatrix} x^{l_j},$$

$\sum_{j=1}^{q-l} n_j = n - 3l.$

Proof: For $\overline{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q}$, $n_1 + n_2 + ... + n_q = n$, $C_{n_i} \cap C_{n_j} = \phi$ for any i and j, $i \ne j$, $3 \le nj \le n$, $1 \le j \le q$, $q \ge 1$, $n \ge 6$, by Lemma 2.4 then

$$\begin{aligned} N\left(\overline{G}, k\right) &= N\left(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_q}, k\right) \\ &= \sum_{l_1 + l_2 + \cdots + l_q = k} N\left(C_{n_1}, l_1\right) N\left(C_{n_2}, l_2\right) \cdots N\left(C_{n_q}, l_q\right) \\ &= \sum_{l_1 + l_2 + \cdots + l_q = k} \prod_{j=1}^{q} N\left(C_{n_j}, l_j\right). \end{aligned}$$

By Theorem 3.2 we have

$$I(G;x)=\sum_{k=1}^{n}N\left(\overline{G},k\right)x^{k}=\sum_{k=1}^{n}\sum_{l_1+l_2+\cdots+l_q=k}\prod_{j=1}^{q}N\left(C_{n_j},l_j\right)x^{k},$$

when

$$n_j=3,\ N\left(C_3,l\right)=\begin{cases}1, & l=1,\\ 3, & l=2,\\ 1, & l=3,\end{cases}$$

when

$$n_j\geq 4,\ N\left(C_{n_j},l_j\right)=\begin{cases}0, & 1\leq l_j<\dfrac{n_j}{2},\\ \dfrac{n_j}{l_j}\begin{pmatrix}l_j\\ n_j-l_j\end{pmatrix}, & \dfrac{n_j}{2}\leq l_j\leq n_j.\end{cases}$$

Finally,

$$I(G;x)\ =\ \prod_{j=1}^{q}\sum_{l_j=1}^{n_j}N\left(C_{n_j},l_j\right)x^{l_j}=\left(x+3x^2+x^3\right)^{l}\prod_{j=1}^{q-l}\sum_{l_j=1}^{n_j}N\left(C_{n_j},l_j\right)x^{l_j}$$

$$=\ \left(x+3x^2+x^3\right)^{l}\prod_{j=1}^{q-l}\sum_{l_j=\left[\frac{n_j}{2}\right]}^{n_j}\frac{n_j}{l_j}\begin{pmatrix}l_j\\ n_j-l_j\end{pmatrix}x^{l_j}$$

and $\sum_{j=1}^{q-l}n_j=n-3l$.

Theorem 4.4: If G is a complete d-partite graph $K_{n_1,n_2,\cdots,n_d}$, and $n_1+n_2+...+n_d$ $=n$, then independent set polynomial $I(G;x)=\prod_{j=1}^{d}\sum_{l_j=1}^{n_j}S\left(n_j,l_j\right)x^{l_j}$, where $S(n,k)$ is the Stirling number of the second kind, $n,k\in N$.

Proof: Suppose $G=K_{n_1,n_2,\cdots,n_d}$, and $n_1+n_2+...+n_d=n$, then $\overline{G}=K_{n_1}\cup K_{n_2}\cup\cdots$

$\cup K_{n_d}$, $n_1 + n_2 + \ldots + n_d = n$, $K_{n_i} \cap K_{n_j} = \phi$ for any i and j, $i \neq j$, $3 \leq nj < n$, $1 \leq j \leq d$, $d \geq 2$, we have

$$\begin{aligned} N\left(\overline{G}, k\right) &= N\left(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_d}, k\right) \\ &= \sum_{l_1 + l_2 + \cdots + l_d = k} N\left(K_{n_1}, l_1\right) N\left(K_{n_2}, l_2\right) \cdots N\left(K_{n_d}, l_d\right) \\ &= \sum_{l_1 + l_2 + \cdots + l_d = k} \prod_{j=1}^{d} N\left(K_{n_j}, l_j\right). \end{aligned}$$

With Lemma 2.3 $N(K_n, k) = S(n, k)$, then

$$\begin{aligned} N\left(\overline{G}, k\right) &= N\left(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_d}, k\right) \\ &= \sum_{l_1 + l_2 + \cdots + l_d = k} N\left(K_{n_1}, l_1\right) N\left(K_{n_2}, l_2\right) \cdots N\left(K_{n_d}, l_d\right) \\ &= \sum_{l_1 + l_2 + \cdots + l_d = k} \prod_{j=1}^{d} N\left(K_{n_j}, l_j\right). \end{aligned}$$

By Theorem 3.2, then we have

$$\begin{aligned} I(G; x) &= \sum_{k=1}^{n} N\left(\overline{G}, k\right) x^k = \sum_{k=1}^{n} \sum_{l_1 + l_2 + \cdots + l_d = k} \prod_{j=1}^{d} S\left(n_j, l_j\right) x^k \\ &= \prod_{j=1}^{d} \sum_{l_j=1}^{n_j} S\left(n_j, l_j\right) x^{l_j}, \end{aligned}$$

where $S(n, k)$ is the Stirling number of the second kind, $n, k \in N$.

Corollary 4.5: If G is a complete tri-partite graph K_{n_1, n_2, n_3}, and $n_1 + n_2 + n_3 = n$, then $I(G; x) = \prod_{j=1}^{3} \sum_{l_j=1}^{n_j} S\left(n_j, l_j\right) x^{l_j}$, where $S(n_j, l_j)$ is the Stirling number of the second kind, $n_j, l_j \in N$, $j = 3$.

Proof: It is easily proved by Theorem 4.1. Here we omit the proof.

Corollary 4.6: If G is a complete tri-partite graph $K_{n, n, n}$, then $I(G;x) = \prod_{j=1}^{3} \sum_{l_j=1}^{n} S(n, l_j) x^{l_j}$ where S(n, l_j) is the Stirling number of the second kind, $n \in N, j = 3$.

Proof: It is easily proved by Corollary 4.2. Here we omit the proof.

Corollary 4.7: If G is a complete *bi*-partite graph $K_{n, n}$, then I(*G*; *x*) = $\left(\sum_{j=1}^{n} S(n, j) x^j\right)^2$

Proof: It is easily proved by Corollary 4.3. Here we omit the proof.

Corollary 4.8: Let S(n, k) be the Stirling number of the second kind, $h(K_n, x) = \sum_{i=1}^{n} S(n,i)x^i$ (see Brenti [16]), and G is a complete bi-partite graph $K_{n,n}$. Then $I(G, x) = (h(K_n; x))^2$:

Proof: It is easily proved by Corollary 4.4. Here we omit the proof.

Theorem 4.9: If *G* is a tree with *n* vertices, then

$$I(G;x) = \sum_{k=1}^{n} \sum_{p=k}^{n} (-1)^{n-p} \binom{n-1}{p-1} N(K_p, k) x^k,$$

where

$$N(K_p, k) = \sum_{\sum_{i=1}^{p} ib_i = p, \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!} \prod_{i \geq 2} \frac{1}{b_i!(i!)^{b_i}}, 2 \leq p\, k \leq n.$$

Proof: If G is a tree with n vertices, then the chromatic polynomial of G is $f(T, t) = t(t-1)^{n-1} = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} t^{k+1}$. Coefficients of the chromatic polynomial of G are $Y_p = (-1)^{n-p} \binom{n-1}{p-1}, 1 \leq p \leq n.$ By Theorem 3.1 $I(G;x) = \sum_{k=1}^{n} \sum_{p=k}^{n} N(K_p, k) Y_p$, then we have

$$I(G;x)=\sum_{k=1}^{n}\sum_{p=k}^{n}(-1)^{n-p}\binom{n-1}{p-1}N(K_p,k)x^k,$$

where

$$N\left(K_p,k\right)=\sum_{\sum_{i=1}^{p} ib_i=p,\ \sum_{i=1}^{p} b_i=k}\frac{p!}{b_1!}\prod_{i\geq 2}\frac{1}{b_i!(i!)^{b_i}},2\leq p\,k\leq n.$$

Corollary 4.10: If P_n is any path with length n, and has n + 1 vertices, then $I(G;x)=\sum_{k=1}^{n+1}\sum_{p=k}^{n+1}(-1)^{n+1-p}\binom{n}{p-1}S(p,k)x^k$, where S(p, k) is the Stirling number of the second kind.

Proof: Because P_n is a special tree with n + 1 vertices, by Theorem 6 we derive the result $I(G;x)=\sum_{k=1}^{n+1}\sum_{p=k}^{n+1}(-1)^{n+1-p}\binom{n}{p-1}N\left(K_p,k\right)x^k$. By Lemma 2.3, then $I(G;x)=\sum_{k=1}^{n+1}\sum_{p=k}^{n+1}(-1)^{n+1-p}\binom{n}{p-1}S(p,k)x^k$, where $S(p, k)$ is the Stirling number of the second kind.

5. INDEPENDENCE POLYNOMIALS $I_\alpha(G; x)$ OF GRAPHS

In the section, the author discusses independence polynomials $I_\alpha(G; x)$ of graphs.

Because s_k denotes the number of stable sets of cardinality k in graph G, and $\alpha(G, k)$ is the number of partitions of $V(G)$ into exactly k non-empty independent sets of any graph G, then $s_k = \alpha(G, k)$.

But in this paper $\alpha(G)$ is the size of a maximum stable set, in [3]$\alpha(G)$is the number of all partitions of $V(G)$ into exactly k non-empty independent sets of any graph G, here the two concepts is not the same.

Theorem 5.1: If G is a $(n – 2)$-regular graph with n (even $2m$) vertices, $\alpha(G)$ is the size of a maximum stable set, then independence polynomial of G

$$I_\alpha(G;x)=\sum_{k=m}^{\alpha(G)}\binom{m}{k-m}x^k.$$

Proof: Let G be a $(n-2)$-regular graph with n (even $2m$). Then $\bar{G}$ is a 1-regular graph, namely, $\bar{G} = K_2 \cup K_2 \cup \cdots \cup K_2$, and the number of K_2 is m. We have

$$N\left(\bar{G}, k\right) = \begin{cases} 0, & 1 \le k < \frac{n}{2}, \\ \begin{pmatrix} \frac{n}{2} \\ k - \frac{n}{2} \end{pmatrix}, & \frac{n}{2} \le k \le n. \end{cases}$$

Finally, $s_k = \alpha(G,k) = N\left(\bar{G}, k\right)$ and by definition 3, then independence polynomial

$$I_\alpha(G;x) = \sum_{k=1}^{\alpha(G)} \begin{pmatrix} \frac{n}{2} \\ k - \frac{n}{2} \end{pmatrix} x^k = \sum_{k=m}^{\alpha(G)} \begin{pmatrix} m \\ k-m \end{pmatrix} x^k.$$

Theorem 5.2: If G is a $n-3$-regular graph with n vertices, $n \ge 6$, and $\bar{G} \cong C_n$, $\alpha(G)$ is the size of a maximum stable set, then independence polynomial of G

$$I_\alpha(G;x) = \sum_{k=\left\lceil \frac{n}{2} \right\rceil}^{\alpha(G)} \frac{n}{k} \begin{pmatrix} k \\ n-k \end{pmatrix} x^k.$$

Proof: Let G be a $n-3$-regular graph with n vertices, $n \ge 6$ and $\bar{G} \cong C_n$, because $\bar{G}$ is a 2-regular graph, the graph would be able to join the disjoint cycles, thus assume that C_n, say. Then we have

$$N\left(\bar{G}, k\right) = N\left(C_n, k\right) = \begin{cases} 0, & 1 \le k < \frac{n}{2}, \\ \frac{n}{k} \begin{pmatrix} k \\ n-k \end{pmatrix}, & \frac{n}{2} \le k \le n. \end{cases}$$

Finally, $s_k = \alpha(G, k) = N\left(\overline{G}, k\right)$ and by definition 3, then independence polynomial of G is given the following:

$$I_\alpha(G; x) = \sum_{k=1}^{\alpha(G)} \frac{n}{k}\binom{k}{n-k} x^k = \sum_{k=\left\lceil \frac{n}{2} \right\rceil}^{\alpha(G)} \frac{n}{k}\binom{k}{n-k} x^k.$$

Corollary 5.3: If G is a $(n-3)$-regular graph with n vertices, and

$$\overline{G} = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_q},$$

$n_1 + n_2 + \ldots + n_q = n$, $C_{n_i} \cap C_{n_j} = \phi$ for any i and j, $i \neq j$, $3 \leq n_j \leq n$; $1 \leq j \leq q$; $q \geq 1$, $n \geq 6$, the number of $n_j = 3$ is l, then independence polynomial of G is given as follows

$$I_\alpha(G; x) = \sum_{k=1}^{\alpha(G)} \sum_{l_1 + l_2 + \cdots + l_q = k} \prod_{j=1}^{q} N\left(C_{n_j}, l_j\right) x^k$$

when

$$n_j = 3, N\left(C_3, l\right) = \begin{cases} 1, & l = 1, \\ 3, & l = 2, \\ 1, & l = 3, \end{cases}$$

when

$$n_j \geq 4, N\left(C_{n_j}, l_j\right) = \begin{cases} 0, & 1 \leq l_j < \frac{n_j}{2}, \\ \frac{n_j}{l_j}\binom{l_j}{n_j - l_j}, & \frac{n_j}{2} \leq l_j \leq n_j. \end{cases}$$

Proof: For $\overline{G} = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_q}$, $n_1 + n_2 + \ldots + n_q = n$, $C_{n_i} \cap C_{n_j} = \phi$ for any i and j, $i \neq j$, $3 \leq n_j \leq n$, $1 \leq j \leq q$, $q \geq 1$, $n \geq 6$, by Lemma 4 then

$$\begin{aligned} N\left(\overline{G}, k\right) &= N\left(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_q}, k\right) \\ &= \sum_{l_1 + l_2 + \cdots + l_q = k} N\left(C_{n_1}, l_1\right) N\left(C_{n_2}, l_2\right) \cdots N\left(C_{n_q}, l_q\right) \end{aligned}$$

$$= \sum_{l_1+l_2+\cdots+l_q=k} \prod_{j=1}^{q} N\left(C_{n_j}, l_j\right).$$

Finally, $s_k = \alpha(G, k) = N\left(\overline{G}, k\right)$ and by definition 3, then independence polynomial of G is given the following:

$$I_\alpha\left(G; x\right) = \sum_{k=1}^{\alpha(G)} N\left(\overline{G}, k\right) x^k = \sum_{k=1}^{\alpha(G)} \sum_{l_1+l_2+\cdots+l_q=k} \prod_{j=1}^{q} N\left(C_{n_j}, l_j\right) x^k$$

when

$$n_j = 3, N\left(C_3, l\right) = \begin{cases} 1, & l = 1, \\ 3, & l = 2, \\ 1, & l = 3, \end{cases}$$

when

$$n_j \geq 4, N(C_{n_j}, l_j) = \begin{cases} 0, & 1 \leq l_j < \dfrac{n_j}{2}, \\ \dfrac{n_j}{l_j}\dbinom{l_j}{n_j - l_j}, & \dfrac{n_j}{2} \leq l_j \leq n_j. \end{cases}$$

Remark: (Reviewing the size of maximum independent set)

Because it is NP-hard that $\alpha(G)$ is the size of a maximum stable set (the size of maximum independent set), so far there exact not the explicit formula, a number of mathematicians have studied $\alpha(G)$ is the size of a maximum stable set (the size of maximum independent set).

Theorem 5.4: If G is a complete d-partite graph $K_{n_1, n_2, \cdots, n_d}$, and $n_1 + n_2 + \ldots + n_d = n$, then independence polynomial of G

$$I_\alpha(G; x) = \sum_{k=1}^{\alpha(G)} \sum_{l_1+l_2+\cdots+l_d=k} \prod_{j=1}^{d} S\left(n_j, l_j\right) x^k,$$

where $S(n, k)$ is the Stirling number of the second kind, $n, k \in N$.

Proof: Suppose $G = K_{n_1, n_2, \cdots, n_d}$, and $n_1 + n_2 + ... + n_d = n$, then $\overline{G} = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_d}$, $n_1 + n_2 + ... + n_d = n$, $K_{n_i} \cap K_{n_j} = \phi$ for any i and j, $i \neq j$, $3 \leq n_j < n$, $1 \leq j \leq d$, $d \geq 2$, we have

$$
\begin{aligned}
N\left(\overline{G}, k\right) &= N\left(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_d}, k\right) \\
&= \sum_{l_1 + l_2 + \cdots + l_d = k} N\left(K_{n_1}, l_1\right) N\left(K_{n_2}, l_2\right) \cdots N\left(K_{n_d}, l_d\right) \\
&= \sum_{l_1 + l_2 + \cdots + l_d = k} \prod_{j=1}^{d} N\left(K_{n_j}, l_j\right).
\end{aligned}
$$

With Lemma 3 $N(K_n; k) = S(n, k)$, then

$$
\begin{aligned}
N\left(\overline{G}, k\right) &= N\left(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_d}, k\right) \\
&= \sum_{l_1 + l_2 + \cdots + l_d = k} N\left(K_{n_1}, l_1\right) N\left(K_{n_2}, l_2\right) \cdots N\left(K_{n_d}, l_d\right) \\
&= \sum_{l_1 + l_2 + \cdots + l_d = k} \prod_{j=1}^{d} S\left(n_j, l_j\right)
\end{aligned}
$$

Then $I_\alpha(G, x) = \sum_{k=1}^{\alpha(G)} N\left(\overline{G}, k\right) x^k = \sum_{k=1}^{\alpha(G)} \sum_{l_1 + l_2 + \cdots + l_d = k} \prod_{j=1}^{d} S\left(n_j, l_j\right) x^k$ where $S(n, k)$ is the Stirling number of the second kind, $n, k \in N$.

Corollary 5.5: If G is a complete tri-partite graph $K_{n, n, n}$, then independence polynomial of $I_\alpha(G; x) = \sum_{k=1}^{\alpha(G)} \sum_{l_1 + l_2 + l_3 = k} \prod_{j=1}^{3} S\left(n, l_j\right) x^k$, where $S(n, k)$ is the Stirling number of the second kind, $n, k \in N$.

Proof: Let $n_j = n$, $d = 3$, $1 \leq j \leq 3$ and by Theorem 9. Then $I_\alpha(G; x) = \sum_{k=1}^{\alpha(G)} \sum_{l_1 + l_2 + l_3 = k} \prod_{j=1}^{3} S\left(n, l_j\right) x^k$, where $S(n, k)$ is the Stirling number of the second kind, $n, k \in N$.

Corollary 5.6: If G is a complete bi-partite graph $K_{n,n}$, then independence polynomial of $G \to I_\alpha(G; x) = \sum_{k=1}^{\alpha(G)} \sum_{l_1+l_2=k} \prod_{j=1}^{2} S(n, l_j) x^k$, where $S(n, k)$ is the Stirling number of the second kind, $n, k \in N$.

Proof: Let $n_j = n$, $d = 2$, $1 \le j \le 2$ and by Theorem 9. Then $I_\alpha(G; x) = \sum_{k=1}^{\alpha(G)} \sum_{l_1+l_2=k}$ $\prod_{j=1}^{2} S(n, l_j) x^k$, where $S(n, k)$ is the Stirling number of the second kind, $n, k \in N$.

Theorem 5.7: If G is a tree with n vertices, then independence polynomial of G $I_\alpha(G; x) = \sum_{k=1}^{\alpha(G)} \sum_{p=k}^{n} (-1)^{n-p} \binom{n-1}{p-1} N(K_p, k) x^k$.

Proof: If G is a tree with n vertices, then the chromatic polynomial of G is $f(T, t) = t(t-1)^{n-1} = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} t^{k+1}$. Coefficients of the chromatic polynomial of G are $Y_p = (-1)^{n-p} \binom{n-1}{p-1}$, $1 \le p \le n$. By Theorem 3.1 $I_\alpha(G; x) = \sum_{k=1}^{\alpha} \sum_{p=k}^{n} N(K_p, k) Y_p$, then we have

$$I_\alpha(G; x) = \sum_{k=1}^{\alpha(G)} \sum_{p=k}^{n} (-1)^{n-p} \binom{n-1}{p-1} N(K_p, k) x^k,$$

where

$$N(K_p, k) = \sum_{\sum_{i=1}^{p} ib_i = p,\ \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!} \prod_{i \ge 2} \frac{1}{b_i!(i!)^{b_i}}, 2 \le p\, k \le n.$$

Theorem 5.8: If P_n is any path with length n, and has $n + 1$ vertices, then independence polynomial of P_n

$$I_\alpha(P_n; x) = \sum_{k=1}^{\alpha(P_n)} \sum_{p=k}^{n+1} (-1)^{n+1-p} \binom{n}{p-1} S(p, k) x^k,$$

where $S(p, k)$ is the Stirling number of the second kind, $p, k, n \in N$.

Proof: The formula from the proving course of Theorem 5.3. Omitted.

REFERENCES

[1] Makowsky, J.A., Mariño, J.P., Farrell Polynomials on Graphs of Bounded Tree Width, *Adv. in Appl. Math.*, **30(1-2)** (2003), 160-176.

[2] Vadim E. Levit and Eugen Mandrescu, Independence Polynomials of Well-covered Graphs: Generic Counterexamples for the Unimodality Conjecture, *European Journal of Combinatorics*, **27(6)** (2006), 931-939.

[3] LiMin Yang and SiHong Nian, The Number of k Independent Sets of Graphs and the Mean Colour Numbers of Graphs, *International Journal of Combinatorial Graph Theory and Applications*, **1(1)** (2008), (Series 1).

[4] LiMin Yang and TianMing Wang, Enumeration of Independent Sets of Graphs, *International Journal of Combinatorial Graph Theory and Applications*, **1(1)** (2008), (Series 2).

[5] TianMing Wang and LiMin Yang, Enumeration of Ideal Subgraphs, SIAM, 1991, 539-545.

[6] J. E. Bartels, D. J. A. Welsh, the Markor Chain of Colouring, Lecture Notes in Computer Science, **920** (1995), Springgor, New York/Berlin, 373-387.

[7] F. M. Dong, Bounds for Mean Colour Numbers of Graphs, *J. of Combinatorial Theory, Series B*, **87** (2003), 348-365.

[8] LiMin Yang, The Recurrence Counting Formula of $S^{(n)}$-factors, *J. of Math. Res. and Expo.*, **11(1)** (1991), 78 (In Chinese).

[9] LiMin Yang and Hong Yao, A Recurrence Formula of the Number of $S^{(n)}$-factors of a Regular m-furcating Tree, *J. of Math. Res. and Expo.*, **23(2)** (2003), 632-636. (In Chinese).

[10] LiMin Yang and TianMing Wang, The Repersenting Formula of $N(G, k)$ *International Journal of Combinatorial Graph Theory and Applications*, **1**(1) (2008).

[11] LiMin Yang, Enumeration of Ideal Graph with Applications, *J. of Dalian Uni. of Tech.*, **29(5)** (1989), 605-609.

[12] Louis Comtet, Advanced Combinatorics, D. Reidel Publishing Company, 1974.

[13] Richard P. Stanley, Enumerative Combinatorics, Cambridge University Press, 1997.

[14] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, The Macmil Press, L.T.D, 1976.

[15] J. Harry, E. Palmer, Graphical Enumeration, New York and London, 1973.

[16] Brenti. F. Expansions of Chromatic Polyomial and Log-convavity. *Trans. Am. Math. Soc.*, **332** (1992), 729-756.

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