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BOUNDS ON THE SIGNED DISTANCE-k-DOMINATION NUMBER OF GRAPHS

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ABSTRACT: Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order n and edge set $E = E(G)$. A k-dominating set of G is a subset $S \subset V$ such that each vertex in $V \setminus S$ has at least k neighbor in S. If v is a vertex of a graph G, the open k-neighborhood of v, denoted by $N_k(v)$, is the set $N_k(v) = \{u \in V : u \neq v \text{ and } d(u, v) < k\}$. $N_k[v] = N_k(v) \cup \{v\}$ is the closed k-neighborhood of v. A function $f : V \to \{-1, 1\}$ is a signed distance-k-dominating function of G, if for every vertex $v \in V$, $f(N_k[v]) = \sum_{u \in N_k[v]} f(u) > 1$. The signed distance-k-domination number, denoted by $\gamma_{k,s}(G)$, is the minimum weight of a signed distance-k-dominating function of G. In this paper we give lower and upper bounds on $\gamma_{k,s}$ of graphs. Also, we determine $\gamma_{k,s}(G \vee H)$ when $k > 2$, and we give a sharp upper bound of $\gamma_{k,s}(G \vee H)$ in term of $\gamma_{k,s}(G)$ and $\gamma_{k,s}(H)$ for any graphs G and H.

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1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order n and edge set $E = E(G)$. For a subset $S \subset V(G)$, we define $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$. If v is a vertex of a graph G, the open k-neighborhood of v, denoted by $N_k(v)$, is the set $N_k(v) = \{u \in V : u \neq v$ and $d(u, v) < k\}$. $N_k[v] = N_k(v) \cup \{v\}$ is the closed k-neighborhood of v. $\delta_k(G) = \min\{|N_k(v)|; v \in V\}$ and $\Delta_k(G) = \max\{|N_k(v)|; v \in V\}$. A k-dominating set of G is a subset $S \subset V$ such that every vertex in $V \setminus S$ has at least k-neighbors in S. The k-domination number $\gamma_k(G)$ is the minimum cardinality among the k-dominating sets of G. A subset $S \subset V$ is a total dominating set, if for every vertex $u \in V$ there exists a vertex $v \in S$, such that u is adjacent to v. Let G be a graph with no isolated vertex. The total domination number $\gamma_t(G)$ is the minimum cardinality among the total dominating sets of G.

A function $f : V \to \{-1, 1\}$ is a signed distance-k-dominating function of G, if for every vertex $v \in V$, $f(N_k[v]) = \sum_{u \in N_k[v]} f(u) > 1$. The signed distance-k-domination number, denoted by $\gamma_{k,s}(G)$, is the minimum weight of a signed distance-k-dominating function on G.

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A signed distance-1-dominating function and signed distance-1-domination number $\gamma_{1,s}(G)$ of a graph G are identified with the usual signed dominating function and signed domination number $\gamma_s(G)$ of a graph G.

Let $k \geq 2$ be a positive integer. A subset $S \subseteq V(G)$ is a k-packing if for every pair of vertices $u, v \in S$, $d(u, v) > k$. The k-packing number $\beta_\kappa(G)$ is the maximum cardinality of a k-packing in G. The join of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$. Let G be a graph of order n with vertex set $\{v_1, v_2, ..., v_n\}$. We construct k^{th} power G^k of a graph G, by $V(G^k) = V(G)$ and u and v are adjacent in G^k if and only if $0 < d_G(u, v) \leq k$.

2. LOWER BOUNDS ON $\gamma_{\kappa,s}(G)$

Observation 1: Let G be a graph of order n, and k be a positive integer. Then $\gamma_{\kappa,s}(G) = \gamma_s(G^k)$.

Proof: Let f be a signed distance-k-dominating function of G. It is easy to see that for every $v \in V(G)$, $N_k[v] = NG_k[v]$. Hence $f(NG_k[v]) = f(N_k[v])$. Therefore f is a signed distance-k-dominating function of G if and only if f is a signed distance dominating set of G^k. Thus $\gamma_s(G^k) = \gamma_{\kappa,s}(G)$.

Let G be a graph of order n, and k be a positive integer. $\delta(G^k) = \delta_\kappa(G)$ and $\Delta(G^k) = \Delta_\kappa(G)$.

Theorem 2 ([2]): For any graph G with $\delta \geq 2$, $\gamma_s(G) \geq n\left(\frac{\left\lceil\frac{\delta}{2}\right\rceil - \left\lfloor\frac{\Delta}{2}\right\rfloor + 1}{\left\lceil\frac{\delta}{2}\right\rceil + \left\lfloor\frac{\Delta}{2}\right\rfloor + 1}\right)$.

As immediate result from Observation 1 and Theorem 2 we have,

Corollary 3: For any graph G with $\delta_\kappa \geq 2$, $\gamma_{\kappa,s}(G) \geq n\left(\frac{\left\lceil\frac{\delta_k}{2}\right\rceil - \left\lfloor\frac{\Delta_k}{2}\right\rfloor + 1}{\left\lceil\frac{\delta_k}{2}\right\rceil + \left\lfloor\frac{\Delta_k}{2}\right\rfloor + 1}\right)$.

Proposition 4: Let G be a graph of order n. Then $2\gamma_2(G) - n \leq \gamma_s(G)$.

Proof: Let f be a minimum signed dominating function of G. Let $V_1 = \{u \in V : f(u) = 1\}$ and $V_{-1} = \{u \in V : f(u) = -1\}$.

If $V_{-1} = \emptyset$, then the proof is clear.

If $v \in V_{-1}$ since $f(N_G[v]) \geq 1$, then v has at least two adjacent in V_1. Therefore V_1 is a 2-dominating set for G and $|V_1| \geq \gamma_2(G)$.

Since $\gamma_s(G) = |V_1| - |V_{-1}|$ and $n = |V_1| + |V_{-1}|$, then $\gamma_s(G) = 2|V_1| - n$ and finally we have $\gamma_s(G) > 2\gamma_2(G) - n$.

Proposition 5: Let G be a graph of order n and with no isolated vertex. Then $2\gamma_t(G) - n < \gamma_s(G)$.

Proof: The proof is similar to the Proposition 4.

3. UPPER BOUNDS ON $\gamma_{\kappa, s}(G)$

Theorem 6: Let k be a positive integer. If G is a simple graph of order n and minimum degree $\delta > 2$ and $\beta_{\kappa+1}$ is a maximum value of k + 1-packing sets. Then $\gamma_{\kappa, s}(G) < n - 2\gamma_{\kappa+1}$, and this bound is sharp.

Proof: Let S be a k + 1-packing set with $|S| = \beta_{\kappa+1}$. We define $f : V \to \{-1, 1\}$ by,

$$f(v) = \begin{cases} -1, & \text{if } v \in S \\ 1 & \text{if } v \in V - S. \end{cases}$$

It is easy to show that $f(V(G)) = n - \beta_{\kappa+1}$. Therefore it is sufficient to show that f is a signed distance-k-dominating function on G. Let v be a vertex in S. Since $\delta > 2$, then $|N_k[v]| > 3$. Since S is a k + 1-packing set. Hence $N_k[v] \cap S = \{v\}$, and $f(N_k[v]) > 1$. Now let v be a vertex in V – S. There are two cases.

Case 1: $N[v] \cap S \neq \emptyset$. Since S is a k + 1-packing set in graph G, then $|N_k[v] \cap S| = 1$ and let $N_k[v] \cap S = \{w\}$. Otherwise let u be a vertex in $N_k[v] \cap S$ different from w. This shows that $d(w, u) < k + 1$. This is a contradiction. Since $\delta > 2$ therefore $f(N_k[v]) > 1$.

Case 2: $N[v] \cap S = \emptyset$. If $N_k[v] \cap S = \emptyset$, then $f(N_k[v]) \geq 1$. Let $N_k[v] \cap S \neq \emptyset$ and let $N_k[v] \cap S = \{s_1, s_2, ..., s_r\}$. Since $d(s_i, s_j) > k + 2$, there exists a vertex v_i on the $v - s_i$ path which is distinct the vertex v_j on the $v - s_j$ path. Thus there exist at least r distinct vertices in $N_k[v] - S$. Suppose v_i be a vertex in $N_k[v]$ such that v_i is adjacent to s_i for each $1 < i < r$.

Therefore, $f(N_k[v]) > \sum_{i=1}^{r} f(s_i) + \sum_{i=1}^{r} f(v_i) + f(v) = 1$. And f is a signed-k-dominating function on G with weight $|V - S| - |S| = n - 2\beta_{\kappa+1}$. Hence $\gamma_{\kappa, s}(G) < n - 2\beta_{\kappa+1}$.

Now, we show that the bound is sharp. The desired graph G will be the union p copies of C_4. Then $\gamma_{\kappa, s}(G) = 2p$ and $\beta_{\kappa+1} = p$. Therefor $\gamma_{\kappa, s}(G) = 2p = 4p - 2p = n - 2\beta_{\kappa+1}$. This completes the proof.

Corollary 7: Let k be a positive integer. If G is a simple graph of order n and minimum degree $\delta > 2$ and $\beta_{\kappa+1}$ is a maximum value of k + 1-packing sets. Then

$$\beta_{k+1} \leq \frac{n}{2}\left(1-\left(\frac{\left\lceil \frac{\delta_k}{2} \right\rceil - \left\lfloor \frac{\Delta_k}{2} \right\rfloor + 1}{\left\lceil \frac{\delta_k}{2} \right\rceil + \left\lfloor \frac{\Delta_k}{2} \right\rfloor + 1}\right)\right).$$

Proof: By Theorem 6 and Corollary 3 the proof is clear.

Theorem 8: Let G be a connected graph of order n. Let L and S be the sets of vertices degree 1 (leaves) and NG (L) (support vertices) respectively. If D is a maximum 2-packing set in $G-(L \cup S)$, then $\gamma_s(G) < n-2|D|$, and this bound is sharp.

Proof: We define $f: V(G) \to \{-1, 1\}$ by,

$$f(v)=\begin{cases}-1, & \text{if } \; v \in D \\ 1 & \text{if } \; v \in V-D.\end{cases}$$

It is easy to show that $f(V(G)) = n-2|D|$. Therefore it is sufficient to show that f is a signed distance-2-dominating function on G. For each vertex $v \in V(G)$, if v is a vertex in L then $f(N[v]) = 2 \geq 1$. Let $v \in V-(L \cap D)$, if $N[v] \cap D = \emptyset$ then obviously $f(N[v]) > 1$. If $N[v] \cup D \neq \emptyset$, since D is a 2-packing set in $G-(L \cup S)$ then $|N[v] \cap D| = 1$, and let $N[v] \cap D = \{w\}$. Otherwise if u be a vertex in $N[v] \cap D$ different from w then $d(u, w) < 2$. This is a contradiction. Since $\deg(v) > 2$ then $f(N[v]) > 2$. Finally if $v \in D$, since $\deg(v) > 2$ and $N[v] \cap D = \{v\}$ then $f(N[v]) > 1$. Therefore f is a signed dominating function of T with weight $n-2|D|$. Hence $\gamma s < n-2|D|$.

Now, we show that the bound is sharp. Let $G = K_{1, n-1}$ $(n > 2)$. Then $\gamma_s(G) = n$ and $D = \emptyset$. Thus $\gamma_s(G) = n-2|D|$.

In Theorem 6 the graph G can be a simple disconnected graph of order n and $\delta(G) = 1$. Since $H_1, H_2, \ldots, H_m$ are components of G, then $\gamma_s(G) = \gamma_s(H_1) + \gamma_s(H_2) + \ldots + \gamma_s(H_m)$. By a similar reason we can prove $\gamma_s(G) < n-2|D|$, where L and S are the sets of vertices of degree 1 and $N_G(L)$ respectively.

But there exists a natural question here. What's happen, if k > 1? we are going to answer by concept of the G^k of the graph G. At first we have the following lemma.

Lemma 9: Let G be a simple graph of order n and G^k be the k^{th} power of the graph G. Then $D \subset V(G)$ is a maximum set of tk-packing vertices if and only if $D \subset V(G^k)$ is a maximum set of t-packing vertices.

Proof: Since every edge in G^k is equal to a path with length $l < k$ we have u and v are two vertices in $V(G^k)$ such that there is not any path between them with length $l < t$ if and only if u and v are two vertices in $V(G)$ such that is not any path between them with length $l < tk$. This shows that $D \subset V(G^k)$ is a set of t-packing vertices if and only if $D \subset V(G)$ is a set of tk-packing vertices. Also, it is easy to see that $D \subset V(G^k)$ is maximum if and only if $D \subset V(G)$ is maximum. This completes the proof.

Theorem 10: Let $k > 2$ be a positive integer. If G is a simple graph and each component is order $n > 3$, with minimum degree $\delta = 1$ and S is a maximum 2k-packing set. Then $\gamma_{\kappa,s}(G) < n - 2\beta_{2k}$, where $\beta_{2k} = |S|$, and this bound is sharp.

Proof: Let G^k be the k^{th} power of the graph G. By Observation 1 we have $\gamma_{\kappa,s}(G) = \gamma_s(G^k)$. Since $n > 3$ then $\delta(G^k) > 2$. Therefore by Theorem 6 we have $\gamma_{\kappa,s}(G) = \gamma_s(G^k) < n - \beta_2(G^k)$. Finally by Lemma 9 we have $\gamma_{\kappa,s}(G) < n - 2\beta_{2k}(G)$.

Now we show that bound is sharp. The desired graph G will be the union t copies of star $k_{1,2}$. Then $\gamma_{\kappa,s}(G) = t$, $n = 3t$, and $\beta_{2k} = t$. Therefore $\gamma_{\kappa,s}(G) = t = 3t - 2t = n - 2\beta_{2k}$.

This completes the proof.

Observation 11: Let G and H be two simple graphs. If $k > 2$ then

$$\gamma_{k,s}(G \vee H) = \begin{cases} 1, & \text{if } |V(G)| + |V(H)| \text{ is odd} \\ 2 & \text{if } |V(G)| + |V(H)| \text{ is even.} \end{cases}$$

Now we show that for any integer k we can find a simple graph G such that $\gamma_s(G) = k$.

Theorem 12: For any integer k, there exists a graph G with $\gamma_s(G) = k$.

Proof: We consider four cases.

Case 1: Let $k < 0$. We consider the star $K_{1,2}|k| + 2$ with vertices $v_1, v_2, ..., v_2|k| + 2$ with central vertex v. We add vertices ui $(1 < i < 2|k| + 2)$ be adjacent to v_i and v_{i+1} in modulo $2|k| + 2$. Then we add edges $v_i v_{i+1}$ $(1 < i < 2|k| + 2)$ in modulo $2|k| + 2$. Finally we add vertices w_i $(1 < i < |k| + 1)$ be adjacent to v_{2i-1} and v_{2i} (when $k = -3$, G is illustrated in Figure 1).

We define $f : V(G) \rightarrow \{1, -1\}$ by,

$$f(u) = \begin{cases} 1, & \text{if } u \in \{v_1, v_2, ..., v_{2|k|+2}\} \\ -1 & \text{if } u \in \{u_1, u_2, ..., u_{2|k|+2}\} \cup \{w_1, w_2, ..., w_{|k|+1}\}. \end{cases}$$

In the following, we prove that f is a signed dominating function of G. By symmetry it is sufficient to show that $f(N[u]) > 1$ for $u \in \{v, v_1, u_1, w_1\}$. $f(N[v]) = f(v) + \sum_{i=1}^{2|k|+2} f(v_i) = 2|k| + 3 > 1$. $f(N[v_1]) = f(v) + f(v_1) + f(v_2) + f(v_3) + f(u_1) + f(u_{2|k|+2}) + f(w_1) = 1 > 1$. $f(N[u_1]) = f(u_1) + f(v_1) + f(v_2) = 1 > 1$. $f(N[w_1]) = f(v_1) + f(v_2) + f(w_1) = 1 > 1$. Therefore f is a signed dominating function of G with weight $f(V(G)) = 1 + 2|k| + 2 - |k| - 1 - 2|k| - 2 = -|k| = k$. Hence $\gamma_s(G) > k$.

On the other hand let g be a minimum signed dominating function on G, such that $\gamma_s(G) = g(V(G))$, we have,

$$
\begin{aligned}
g(V(G) &= \sum_{v \in V(G)} g(u) \\
&= \sum_{i=1}^{2|k|+2} g(N[u_i] + \sum_{i=1}^{|k|+1} g(w_i) + g(v) \\
&\geq \sum_{i=1}^{2|k|+2} 1 + \sum_{i=1}^{|k|+1} (-1) - 1 = |k| \geq k.
\end{aligned}
$$

Therefore $\gamma_s(G) = k$.

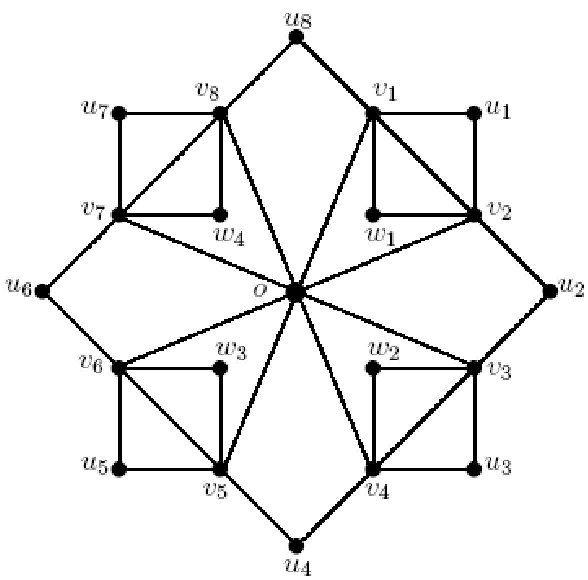

Figure 1

Case 2: If k = 0. We consider the Hajos'graph (Figure 2).

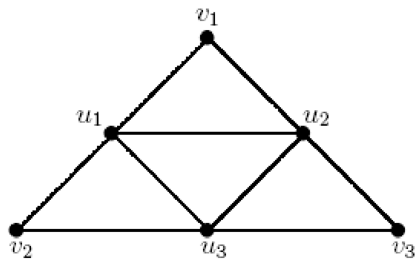

Figure 2

We define $f : V(G) \longrightarrow \{1, -1\}$ by,

$$f(u) = \begin{cases} 1, & \text{if} \quad v \in \{u_1, u_2, ..., u_3\} \\ -1 & \text{if} \quad v \in \{v_1, v_2, ..., v_3\}. \end{cases}$$

It is easy to see that f is a signed dominating function of G_H, with weight 0. Therefore $\gamma_s(G_H) < 0$.

On the other hand let g be a minimum signed dominating function of G_H, such that $\gamma_s(G_H) = g(V(G_H))$. We have, $\gamma_s(G_H) = g(V(G_H)) = \sum_{u \in N_k[G_H]} g(u) = g(N[u_1]) + g(v_3) > 1 - 1 = 0$. Therefor $\gamma_s(G_H) > 0$.

Hnce $\gamma_s(G_H) = 0$.

Case 3: If k = 1. Obviously for the complete graph K_{2n+1} we have $\gamma_s(K_{2n+1}) = 1$.

Case 4: If $k > 2$. We consider the star $K_{1, k-1}$. It is easy to see that $\gamma_s(K_{1, k-1}) = k$. This complete the proof.

REFERENCES

[1] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New york, (1998).

[2] R. Haas, and T. B. Wexler, Bounds on the Signed Domination Number of a Graph, Electronic Notes in Discrete Mathematics, 11, (2002), 742-750.

[3] H. Xing, L. Sun, and X. Chen, On Signed Distance-k-Domination in Graphs, Czechoslovak Mathematical Journal, 56(1), (2006), 229-238.

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