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OUTER SUM LABELING OF A GRAPH

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ABSTRACT: An outer sum labeling is a labeling of a graph G is an injective function $f : V(G) \rightarrow Z^+$ with the property that for each vertex $v \in V(G)$, there exists a vertex $w \in V(G)$ such that $f(w) = \sum_{J \in N(v)} f(u)$, where $N(v) = \{x : vx \in E(G)\}$. A graph G which admits an outer sum labeling is called an outer sum graph. If G is not an outer sum graph then the minimum of isolated vertices required to make G a outer sum graph, is called outer sum number of G and is denoted by on (G). In this paper we show that no connected graphs except a star graph is an outer sum graphs and thereby completely determine outer sum number of graphs such as cycles, trees, unicyclic graphs, complete graphs, complete k-partite graphs and Fans.

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1. INTRODUCTION

All the graphs considered here are undirected, finite, connected and simple. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d_G(u, v)$ or simply $d(u, v)$. We use the standard terminology, the terms not defined here may be found in [1].

A sum labeling λ of a graph is a mapping of the vertices of G into distinct positive integers such that for $u, v \in V(G)$, $uv \in E(G)$ if and only if the sum of the labels assigned to u and v equals the label of a vertex w of G. In such a case w is called a working vertex. A graph which has a sum labeling is called a sum graph. Sum graphs were originally proposed by Harary [2] and later extended to include all integers in [3].

Sum graphs cannot be connected graphs since an edge from the vertex with the largest label would necessitate a vertex with a larger label. Graphs which are not sum graphs can be made to support a sum labeling by considering the graph in conjunction with a number of isolated vertices which can bear the labels required by the graph. The fewest number of the additional isolates required by the graph to support a sum labeling is called the sum number of the graph, it is denoted by $\sigma(G)$.

Every edge adjacent to the vertex bearing the largest label requires an isolated vertex to witness the edge. Consequently a lower bound for the number of isolates required for a graph to support a sum labeling is delta (G)-the smallest degree of G. Any graph for which $\sigma(G) = \delta(G)$ is known as a δ-optimal summable.

Similarly, a sum labeling of a graph $G \cup \overline{K_r}$ for some positive integer r is said to be exclusive with respect to G if all of its working vertices are in $\overline{K_r}$. Every graph can be made to support an exclusive sum labeling, by adding a required number of isolates.

In the next section we define a new class of labeling akin to sum labeling and study some of the graphs which admits such a labeling. For the entire survey on sum labeling we refer the latest survey article by Joe Ryan [4] and similar work on sum graphs we refer [5, 6, 7, 8, 9].

2. OUTER SUM GRAPHS

In this section we define an outer sum labeling and compute outer sum number of certain class of graphs. Also obtain an upper bound for outer sum number of certain class of graphs.

A labeling of a graph G is an injective mapping $f : V(G) \to Z^+$. An Outer sum labeling of a graph G is a labeling on G with an added property that for each vertex $v \in V(G)$, there exists a vertex $w \in V(G)$ such that $f(w) = \sum_{u \in N(v)} f(u)$, where $N(v) = \{x : vx \in E(G)\}$. A graph G which admits an outer sum labeling is called an outer sum graph. If G is not an outer sum graph, then by adding certain number of isolated vertices to G, we can make the resultant graph an outer sum graph. The minimum of such isolated vertices required for a graph G, to make the resultant graph an outer sum graph, is called the outer sum number of G and is denoted by on (G). That is the outer sum number of G is the minimum non negative integer n such that $G \cup \overline{K_n}$ is an outer sum graph.

An outer sum labeling f of a graph $G \cup n_i K_1$ is called a minimal outer sum labeling of G if $on(G) = n_i$.

Remark 2.1: A graph G is an outer sum graph if and only if its outer sum number is zero.

Remark 2.2: For any graph G on n vertices with at most $(n - 2)$ pendant vertices, we see that there are at least two non terminal vertices u and v that are adjacent in G. Further $f(u) = \sum_{x \in N(u)} f(x)$, then $f(v) < f(u)$ (since $v \in N(u)$ and $\deg(v) > 2$). And hence

$$\sum_{y \in N(v)} f(y) > f(u). \quad (1)$$

If the strict inequality in equation 1 holds, then G is an outer sum graph only if it should have vertex w not in $N[u] \cup N[v]$ such that $f(w) < \sum_{y \in N(v)} f(y)$.

So we required a new vertex for each non pendant edge $uw \in E(G)$. However, if the equality in equation 1 holds, then $N(v)$ should contain only one vertex namely v. This is so only if v is a pendant edge.

Hence we conclude that for each non-pendant vertex u, the graph G should have a new vertex w. If w is a pendant vertex, then we required w_1 such that $f(w_1) = \sum_{z \in N(a)} f(z)$, where a is the vertex adjacent to w in G. If a is w_1, then we return back to the stating point. Hence without loss of generality we conclude w is not a pendant vertex. Continuing this way we see that G is outer sum graph if and only if it has no non pendant edges. Thus we conclude that;

Theorem 2.3: A connected graph G is an outer sum graph if and only if $G \equiv K_{1, n}$.

3. OUTER SUM NUMBER OF A CYCLES AND UNICYCLIC GRAPHS

Let f be an outer sum labeling of a graph G (need not be connected) and v be an vertex in G. Then we define an f-neighborhood sum of the vertex v, denoted by $N_f(v)$ as,

$$N_f(v) = \sum_{u \in N(v)} f(u). \tag{2}$$

Observation 3.1: Let f be a minimal outer sum labeling of the graph C_n, where $n > 3$ and $n \neq 4$. Then for each vertex $u \in V(C_n)$, there are exactly two vertices say v, w adjacent to it. Further for these vertices v, w, we see that $N_f(v) \neq N_f(w)$. In fact, $N_f(v) = f(u) + x$ and $N_f(w) = f(u) + y$, for some $x, y \in \{f(v) : v \in V(C_n)\}$ and hence $N_f(v) = N_f(w) \Rightarrow x = y$. But then, as f is injective, we get $n = 4$, which is a contradiction. Moreover, if v be the vertex in C_n such that $f(u) < f(v)$ for all $u \in C_n$, then the sum $N_f(v_\alpha)$ and $N_f(v_\beta)$ created by the $f(v)$, where v_α and v_β are adjacent to v in C_n, is more than $N_f(v_\beta)$ for all $v_\beta \in V(C_n)$. Hence the labels of the isolated vertices in $C_n \cup K_{n_1}$ is more than the label of any vertex of C_n for any minimal outer sum labeling f of C_n.

The above observation 3.1 leads the following Lemma;

Lemma 3.2: If f is a minimal outer sum labeling of a cycle C_n, where $n > 3$ and $n \neq 4$. Then $N_f(x) = N_f(y)$ implies that $d(x, y) \neq 2$, for all $x, y \in V(C_n)$.

Lemma 3.3: Let f be a minimal outer sum labeling of a cycle C_n, where $n > 3$ and $n \neq 4$. Then there exists at least three vertices in C_n having distinct f-neighborhood sums.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n such that $v_i v_j$ is an edge of C_n if only if either $j = i + 1$ or $j = 1$ and $i = n$. Then $N_f(v_i) = f(v_{i-1}) + f(v_{i+1})$, for all i, $2 < i < n-1$, $N_f(v_1) = f(v_n) + f(v_2)$ and $N_f(v_n) = f(v_{n-1}) + f(v_1)$. We now suppose that f-neighborhood sums of any vertex is either x or y. Without loss of generality we take $N_f(v_1) = x$. Let $G_1 = C_n^2 - E(C_n)$.

Case 1: n is odd.

In this case G_1 is isomorphic to C_n. Define a coloring $g : V(G_1) \to N_f = \{N_f(v_j) : v_j \in V(C_n)\}$ such that $g(v_i) = N_f(v_i)$, for each i, $1 < i < n$. By Lemma 3.2, we see that g is a proper coloring of the odd cycle G_1. Hence g require at least three colors (since vertex chromatic number of odd cycle is 3), so $|N_f| > 3$, a contradiction.

Case 2: $n = 2k$ and $k > 3$.

In this case $G_1 \equiv 2C_k$. If k is odd, then for each component of G_1, the result follows by the above case 1.

We now take the case k is even. Let $k = 2l$. Then vertices of one of the components of G_1 are $v_2, v_4, ..., v_{4l}$. Let f(v4l) = a. Then by our assumption $N_f(v_1) = x$, it follows that $f(v_2) = x - a$. And by Lemma 3.2, it follows that $N_f(v_3) \neq x$, so $N_f(v_3) = y$ and hence $f(v_4) = y - f(v_2) = y - x + a$, continuing the same argument we get, $f(v_6) = x - f(v_4) = 2x - y - a$, $f(v_8) = y - f(v_6) = 2y - 2x + a$ and so on. In general

$$f(v_{2i}) = (-1)^{i-1}\left\lfloor \frac{i}{2} \right\rfloor x + (-1)^i \left\lfloor \frac{i}{2} \right\rfloor y + (-1)^i a \qquad (3)$$

for all i, $1 < i < 2l$.

The equation 3 shows that $f(v_{4l}) = -lx + ly + a$, but $f(v_{4l}) = a$ implies that $-lx + ly + a = a \Rightarrow x = y$, a contradiction.

Theorem 3.4: For any integer $n > 3$,

$$on(C_n) = \begin{cases} 1, & \text{if } n = 4 \\ 2, & \text{otherwise.} \end{cases}$$

Proof: By Theorem 2.3, we have $on(C_n) > 1$. The case $n = 4$ follows by Figure 1. Let us now consider the case $n \neq 4$. If possible, suppose that $on(C_n) = 1$, then $G = C_n \cup K_1$ is an outer sum graph, so there exists an outer sum labeling f for C_n. Now, By the lemma 3.3 there exists at least three distinct vertices say u_1, u_2 and u_3 such that $N_f(u_i) \neq N_f(u_j)$, $1 < i, j < 3$ and $i \neq j$.

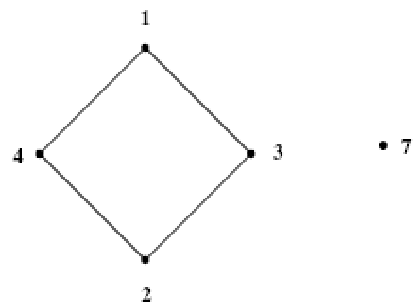

Figure 1: An Outer Sum Labeling of the Graph $C_4 \cup K_1$

Let $x = \max\{N_f(v_i) : v_i \in V(C_n \cup K_1)\}$ and $y = \max\{N_f(v_i) : v_i \in V(C_n)\}$. Let v_j be the vertex of C_n such that $f(v_j) = y$. Then, it is easy to see that the vertex w such that $f(w) = x$ is not in $V(C_n)$. Now for the vertices w_1 and w_2 adjacent to v_j we get $N_f(w_1) > f(v_j) = y$ and $N_f(w_2) > f(v_j) = y$. Since y is the maximum assignment of a vertex in C_n, and G has only one isolated vertex, it follows that $N_f(w_1) = N_f(w_2) = x$, which is a contradiction, by lemma 3.2 as $d(w_1, w_2) = 2$ in C_n. Therefore, on $(C_n) > 2$.

Let $v_0, v_1, ..., v_{n-1}$ be the vertices of the cycle C_n such that v_i is adjacent to v_j only if either $j = i \pm 1 \pmod{n}$. Now to prove the reverse inequality, we define a labeling f as follows:

For Odd Cycles

Step 1: Let $G = C_n \cup 2K_1$. Let w_n and w_{n+1} be the vertices of $2K_1$.

Step 2: For each i, $0 < i < n - 1$, re-label the vertex v_{2i} as w_i and the vertex v_{2i+1} as $w_{\frac{n+1}{2}}$.

Step 3: Let $f(w_0) = 1$ and $f(w_1) = 2$.

Step 4: Define $f(w_i) = f(w_{i-1}) + f(w_{i-2})$, for all i, $2 < i < n$.

Step 5: Define $f(w_{n+1}) = f(w_0) + f(w_n)$.

The function f defined above is clearly an injective function. Further, we now see that f is an outer sum labeling of $C_n \cup 2K_1$. In fact,

(i) For each odd i,

$$N_f(v_i) = f(v_{i-1}) + f(v_{i+1})$$

$$= f(w_{\frac{n+1}{2} + \frac{i+2}{2}}) + f(w_{\frac{n+1}{2} + \frac{i}{2}})$$

$$= \begin{cases} f(w_{\frac{i+3}{2}}), & \text{if } 1 < i < n-4 \\ f(w_{\frac{n+1}{2}}), & \text{if } i = n-2 \end{cases}$$

$$= \begin{cases} f(v_{i+3}), & \text{if } 1 < i < n-4 \\ f(v_{2\times 0+1}), & \text{if } i = n-2 \end{cases}$$

and

(ii) For each even i,

$$N_f(v_i) = f(v_{i-1}) + f(v_{i+1})$$

$$= f(w_{\frac{n+1}{2}+\frac{i-2}{2}}) + f(w_{\frac{n+1}{2}+\frac{i}{2}})$$

$$= \begin{cases} f(w_{\frac{n+1}{2}+\frac{i+2}{2}}), & \text{if } 1 < i < n-5 \\ f(w_n), & \text{if } i = n-3 \\ f(w_{n+1}), & \text{if } i = n-1 \end{cases}$$

$$= \begin{cases} f(v_{i+3}), & \text{if } 1 < i < n-5 \\ f(w_n), & \text{if } i = n-3 \\ f(w_{n+1}), & \text{if } i = n-1 \end{cases}$$

For Even Cycles

Step 1: Let $G = C_n \cup 2K_1$. Let w_n and w_{n+1} be the vertices of $2K_1$.

Step 2: For each i, $0 < i < n-1$, re-label the vertex v_{2i} as w_i and the vertex v_{2i+1} as $w_{\frac{n}{2}+i}$.

Step 3: Let $f(w_0) = 1$ and $f(w_1) = 2$.

Step 4: Define $f(w_i) = f(w_{i-1}) + f(w_{i-2})$, for all i, $2 < i < 2n-1$.

Step 5: Define $f(w_{\frac{n}{2}}) = f(w_{\frac{n}{2}-1}) + f(w_0)$ and $f(w_{\frac{n}{2}+1}) = f(w_{\frac{n}{2}-1}) + f(w_{\frac{n}{2}-2})$

Step 6: Define $f(w_i) = f(w_{i-1}) + f(w_{i-2})$, for all i, $\frac{n}{2} + 2 < i < n-1$.

Step 7: Define $f(w_n) = f(w_{n-1}) + f(w_{\frac{n}{2}})$ and $f(w_{n+1}) = f(w_{n-1}) + f(w_{n-2})$.

The function f defined above is clearly an injective function. Further, we now see that f is an outer sum labeling of $C_n \cup 2K_1$. In fact,

(i) For each odd i,

$$N_f(v_i) = f(v_{i-1}) + f(v_{i+1})$$

$$= f(w_{\frac{i-1}{2}}) + f(w_{\frac{i+1}{2}})$$

$$= \begin{cases} f(w_{\frac{i+3}{2}}), & \text{if } 1 < i < n-5 \\ f(w_{\frac{n}{2}+1}), & \text{if } i = n-3 \\ f(w_{\frac{n}{2}}), & \text{if } i = n-1 \end{cases}$$

$$= \begin{cases} f(v_{i+3}), & \text{if } 1 < i < n-5 \\ f(v_3), & \text{if } i = n-3 \\ f(v_1), & \text{if } i = n-1 \end{cases}$$

and

(ii) For each even i,

$$N_f(v_i) = \begin{cases} f(v_{n-1}) + f(v_1), & \text{if } i = 0 \\ f(v_{i-1}) + f(v_{i+1}), & \text{otherwisw} \end{cases}$$

$$= \begin{cases} f(w_{n-1}) + f(w_{\frac{n}{2}}), & \text{if } i = 0 \\ f(w_{\frac{n}{2}+\frac{i-2}{2}}) + f(w_{\frac{n}{2}+\frac{i}{2}}), & \text{otherwise} \end{cases}$$

$$= \begin{cases} f(w_n), & \text{if } i = 0 \\ f(w_{\frac{n+1}{2}+\frac{i+2}{2}}), & \text{if } 1 < i < n-4 \\ f(w_{n+1}), & \text{if } i = n-2 \end{cases}$$

$$= \begin{cases} f(w_n), & \text{if } i = 0 \\ f(v_{i+3}), & \text{if } 1 < i < n-4 \\ f(w_{n+1}), & \text{if } i = n-2 \end{cases}$$

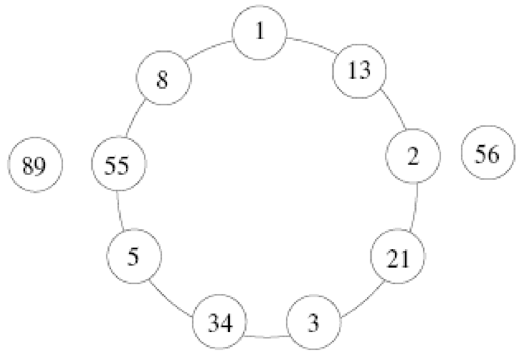

Figure 2: Outer Sum Labeling of the Graph $C_9 \cup 2K_1$

Lemma 3.5: If v is pendant vertex of a connected graph G such that G – v is vertex transitive and on (G – v) = 2, then on (G) = 1.

Proof: Let v be a pendant vertex of a connected graph G such that G – v is vertex transitive and on (G – v) = 2. Let u be a vertex adjacent to v in G. Let f be a minimal outer sum labeling of the graph G. Since on (G – v) = 2, f creates two isolated vertices u_1 and u_2 such that $f(u_1) < f(u_2)$. Let z be the vertex of G – v such that $N_f(z) = f(u_1)$ (such a vertex certainly exists by the definition of f). Since G – v is vertex transitive, we can identify u as z so that the graph G – u is isomorphic to G – z. Now, define a labeling $g : V(G) \rightarrow Z^+$ as;

$$g(x) = \begin{cases} f(x), & \text{if } x \in V(G-v) \\ f(u_1), & \text{if } x = v \\ f(u_1) + f(u_2), & \text{if } x = u_2 \end{cases}$$

since the vertex v in G not in G – v disturbs only the neighborhood sum of z which is now assigned to the vertex u_2 by replacing its older neighborhood sum, it is clear that g is an outer sum labeling for the graph $G \cup \{u_2\}$.

Lemma 3.6: If v is pendant vertex of a connected graph G and on (G – v) = 1, then on (G) = 1.

Proof: Let f be a minimal outer sum labeling of the graph G – v. Since on (G – v) = 1, f creates one isolated vertex say v. Let xv 2 E (G). Then adding an edges from x to y to the graph G – v, we get the graph G which only disturbs the neighborhood sum of the vertices v and x. Now, we see in the graph G that $N_f(v) = f(x)$, so $N_f(v)$ is the label of the vertex x 2 V (G) but $N_f(x)$ is not a label of any vertex of G (since $N_f(x) > f(v) > f(y)$ for any y 2 V (G)), which we can assign for an isolated vertex. So, f can be extended as an outer sum labeling of G with one isolated vertex. Therefore, at most one isolated vertex is sufficient for any outer sum labeling of G. Thus, in view of Theorem 2.3, we conclude on (G) = 1.

Theorem 3.7: Outer sum number of every unicyclic graph containing at least one pendent vertex is 1.

Proof: Let G be a unicyclic graph. Let G_1 be the graph obtained by deleting a pendant vertex from G. Let G_2 be the graph obtained by deleting a pendant vertex from the graph G_1 and so on. The process terminates by yielding a sequence of graph G_1, G_2, ..., G_k such that G_k is a cycle, $G_{i+1} = G_i - v_i$ for some vertex $v_i \in V(G_i) \subset V(G)$. Since the graph G_k is a cycle, we have by Theorem 3.4 that on (G) = 1 or 2. In either of the cases, as cycles are vertex transitive, by Lemma 3.5 or Lemma 3.6, we get on $(G_{k-1}) = 1$. Hence, by the repeated application of the Lemma 3.6, we get on $(G_{k-2}) = 1$, so on $(G_{k-3}) = 1$, ..., on (G) = 1.

4. OUTER SUM NUMBER OF TREES

Theorem 4.1: For any tree T on n vertices,

$$\text{on}(T) = \begin{cases} 1, & \text{if } T \text{ is a star} \\ 2, & \text{otherwise.} \end{cases}$$

Proof: In view of Theorem 2.3, it suffices to establish an outer sum labeling for the graph $T \cup K_1$ whenever T is not a star. Let T_1 be the graph obtained by removing all the pendant vertices of the tree T. Let T_2 be the graph obtained by removing all the pendant vertices of the graph T_1 and so. Continuing this we finally arrive a tree T_k, which is a star. T_k is an outer sum graph. Now, reconstruct T_{k-1} from T_k by considering the pendant edges in T_{k-1}one by one. The insertion of each vertex (pendant vertex of T_{k-1}) to T_k require a new labeling and this new label yields a new neighborhood sum only for the pendant vertex of T_k adjacent to the new vertex of insertion. This new sum can be assigned only to the next vertex of insertion (if exists). Thus, T_{k-1} require one isolated vertex for every outer sum labeling. This isolated vertex can be treated as an end vertex of T_{k-1} while reconstructing the graph T_{k-2} from T_{k-1}. This shows that T required at most one isolated vertex for any outer sum labeling. Thus, on (T) = 1.

Corollary 4.2: For any connected graph G (V, E), on (G) $< 2(|E| - |V|) + 3$.

Proof: Let T be the spanning cycle of the graph G. Then addition of a chord e = xy to T yields two neighborhood sums namely $N_f(x)$ and $N_f(y)$ for any outer sum labeling f of T. Hence on (G) $<$ 2 (number of chords) + on (T) $= 2(|E| - (|V| - 1)) + 1$.

5. OUTER SUM NUMBER OF COMPLETE GRAPHS AND COMPLETE k-PARTITE GRAPHS

Theorem 5.1: For any positive integer n, the outer sum number of a complete graph K_n is given by

$$on(K_n) = \begin{cases} 0, & \text{if } n < 2 \\ n-1, & \text{otherwise.} \end{cases}$$

Proof: If $n < 2$, then the result follows by Theorem 2.3. We now suppose that $n > 3$. For each vertex v of K_n and a labeling f, $N_f(V)$ can be evaluated only after the labeling of all the vertices of K_n except v. Therefore, every minimal outer sum label, assigns labels all n – 1, vertices of K_n arbitrarily and their sum to the vertex v. But then, the label of v creates n – 1 neighborhood sums one each for the arbitrarily labeled vertices, moreover each sum created is greater than the labels assigned for the vertices of K_n. Therefore, we require exactly n – 1 isolated vertices to assign these n – 1 neighborhood sums. Hence $on(K_n) = n - 1$.

Theorem 5.2: For any positive integers $m_1 < m_2 \ldots < m_k$, the outer sum number of a complete k-partite graph K_{m_1, m_2, m_k} is given by

$$on(K_{m_1, m_2, m_k}) = \begin{cases} f(x), & \text{if } x \in V(G-v) \\ f(u_1), & \text{if } x = v \\ f(u_1) + f(u_2), & \text{if } x = u_2 \end{cases}$$

Proof: For $m_1 = 1$ and $k = 2$, the result follows by Theorem 2.3. For other cases, Theorem 2.3 implies that $on(K_{m_1, m_2, m_k}) > 1$. Therefore, to prove the theorem, it suffices to execute an outer sum labeling f for the graph $K_{m_1, m_2, m_k} \cup K_1$, where $m_2 > 2$ or $k > 3$. (The case $m_1 = m_2 = \ldots = m_i = 1$ and $m_{i+1} \neq 1$ follows by noting the fact that the neighborhoods of the vertices in each partition, which is a singleton set, are distinct).

Let $G = K_{m_1, m_2, m_k} \cup K_1$, where either $m_1 \neq 1$ or $k > 3$, $m_2 \geq 2$ and V_i be the i^{th} partition of the set $V(K_{m_1, m_2, m_k})$ of cardinality m_i, for all i, $1 < i < k$. Let $v_{i,j}$ denotes the i^{th} vertex in the set V_j and u be the isolated vertex of G. We note that $v_{i,j}$ is adjacent to $v_{l,m}$ if and only if $j \neq m$, $i < |V_j|$ and $l < |V_m|$.

Define a function $f : V(G) \to Z^+$ recursively as;

Step 1: Let $N = \sum_{i=1}^{k} m_i$.

Step 2: For each i, $1 < i < m_k$,

$$f(v_{i,k}) = iN. \tag{4}$$

Step 3: For each j, $1 < j < k - 1$; and each i, $1 < i < m_{j-1}$,

$$f(v_{i,j}) = iN - (k - j) \tag{5}$$

Step 4: For j = 1, 2, ..., k – 1; let $s_j = m_{j+1} - m_j$ and

$$f(v_{m_{k-j}}, k - j) = \left(\sum_{i = s_j + 1}^{m_k} f(v_{i, j+1}) \right) + m_j - 1. \tag{6}$$

Step 5: For the isolated vertex u,

$$f(u) = \frac{m_k (m_k + 1) N}{2}. \tag{7}$$

To begin with, for each i, $1 < i < m_j$, we note that

$$N_f(v_{i,j}) = \sum_{p=1, p \neq j}^{k} \sum_{t=1}^{m_p} f(v_{t,p}). \tag{8}$$

We also note that the function f defined above labels the vertices of V_j such that $\sum_{i=1}^{m_j} f(v_{i,j})$ is same for every j, $1 < j < k$. In fact,

By equation 4 we get

$$\sum_{i=1}^{m_k} f(v_{i,k}) = \sum_{i=1}^{m_k} iN = \frac{m_k (m_k + 1) N}{2}. \tag{9}$$

For j = k – 1,

$$\sum_{i=1}^{m_{k-1}} f(v_{i,k-1}) = \sum_{i=1}^{m_{k-1}-1} f(v_{i,k-1}) + f(v_{m_{k-1}, k-1})$$

$$= \sum_{i=1}^{m_{k-1}-1} [iN - 1] + \left(\sum_{i = m_{k-1}}^{m_k} f(v_{i,k}) \right) + m_{k-1} - 1$$

$$= \sum_{i=1}^{m_{k-1}-1} [iN - 1] + \left(\sum_{i = m_{k-1}}^{m_k} iN \right) + \sum_{1}^{m_{k-1}-1} = \sum_{i=1}^{m_k} iN.$$

For j = k – 2,

$$\sum_{i=1}^{m_{k-2}} f(v_{i,k-2}) = \sum_{i=1}^{m_{k-2}-1} f(v_{i,k-2}) + f(v_{m_{k-2},k-2})$$

$$= \sum_{i=1}^{m_{k-2}-1} [iN-2] + \left(\sum_{i=m_{k-2}}^{m_{k-1}} f(v_{i,k-1}) \right) + m_{k-2} - 1$$

$$= \sum_{i=1}^{m_{k-2}-1} [iN-2] + \left(\sum_{i=m_{k-2}}^{m_{k-1}-1} f(v_{i,k-1}) + f(v_{m_{k-1},k-1}) \right) + \sum_{i=1}^{m_{k-2}-1}$$

$$= \sum_{i=1}^{m_{k-2}-1} [iN-1] + \sum_{i=m_{k-2}}^{m_{k-1}-1} [iN-1] + f(v_{m_{k-1},k-1})$$

$$= \sum_{i=1}^{m_{k-1}-1} [iN-1] + \left(\sum_{i=m_{k-1}}^{m_k} f(v_{i,k}) + m_{k-1} - 1 \right)$$

$$= \sum_{i=1}^{m_{k-1}-1} [iN-1] + \sum_{i=m_{k-1}}^{m_k} iN + \sum_{1}^{m_{k-1}-1} = \sum_{i=1}^{m_k} iN .$$

In the similar manner, for all j, 1 < j < k – 1, we can show that

$$\sum_{i=1}^{m_{k-2}} f(v_{i,k-2}) = \sum_{i=1}^{m_{k-2}} iN .$$

Hence the function f defined above is an outer sum labeling of the graph $G \cup K_1$, so on (G) < 1.

Therefore on (G) = 1.

6. OUTER SUM NUMBER OF SUM OF GRAPHS

In the pervious section we obtained that on $(K_n + K_1) = n$, for all n > 1. We now compute outer sum number of $G + K_1$, for $G = K_{1,n}$ and G = Pn.

Theorem 6.1: For any integer n > 1,

$$on(K_{1,n} + K_1) = 2.$$

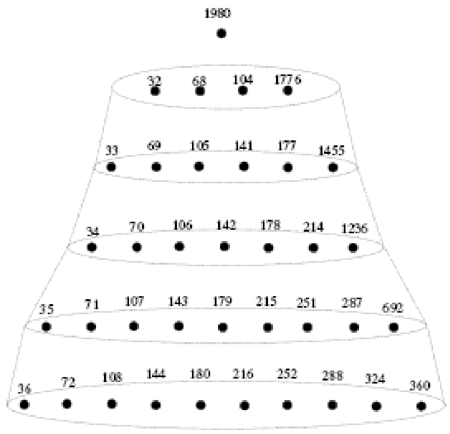

Figure 3: Outer Sum Labeling of the Graph K4, 6, 7, 9, 10 $\cup$ K_1

Proof: If n = 1, then the result follows by Theorem 3.4. We now suppose that $n > 2$. Let G = $K_{1, n} + K_1$. Then G contains exactly two vertices of degree n + 1, say x and y and other vertices are of degree 2. Let $v_1, v_2, ..., v_n$ be the vertices of degree 2 in G and f be a minimal outer sum labeling of G. f creates only the following neighborhood sums;

$$N_f(x) = \sum_{i=1}^{n} f(v_i) + f(y) \tag{10}$$

$$N_f(y) = \sum_{i=1}^{n} f(v_i) + f(x) \tag{11}$$

$$N_f(v_i) = f(x) + f(y). \tag{12}$$

Let z be the vertex of G such that $f(z) = \max\{f(v) : v \in V(G)\}$. We now show that f require at least two isolated vertices. If not, suppose that f require exactly one isolated vertex (at least one is required by Theorem 2.3).

Case 1: z = x (similarly z = y).

Since f(z) is the maximum label of a vertex of G and $N_f(y) > f(x) = f(z)$, it follows that $N_f(y)$ to be the label of the isolated vertex. Further, as f is an injective function, $N_f(x) \neq N_f(y)$ (otherwise from equations 10 and 11 we get f(x) = f(y)), so $N_f(x)$ is the label of the vertex y or a vertex v_i. If $N_f(x) = f(y)$, then equation 10 yields $\sum_{i=1}^{n} f(v_i) = 0$, which is a contradiction. Else, if $N_f(x) = f(v_k)$ for some v_k, then equation 10 yields $f(y) = -\sum_{i=1, i \neq k}^{n} f(v_i)$, which is again a contradiction.

Case 2: $z = v_i$ for some i, $1 < i < n$.

In this case $N_f(x)$ as well as $N_f(y)$ are greater than f (z), so both of these two be assigned for the isolated vertex, so $N_f(x) = N_f(y)$, which is inadmissible (since f is injective). Hence on (G) > 2.

Now to prove the reverse inequality, consider the graph $G_1 = (K_{1,n} + K_1) \cup 2K_1$. Let u and v be the isolated vertices of G_1. We now define a function $f : V(G_1) \to Z^+$ as;

Step 1: Let f (x) = 1 and f (y) = 2.

Step 2: For each i, $1 < i < n$, define $f(v_i) = i + 2$.

Step 3: Let $f(u) = \frac{n^2+5n+4}{2}$ and $f(v) = \frac{n^2+5n+2}{2}$.

The function f defined above is an outer sum labeling of G_1. In fact, $N_f(v_i) = f(v_1) = 3$, for each i, $1 < i < n$. and $N_f(x) = \sum_{i=2}^{n+2} i = f(u)$ and $N_f(y) = (\sum_{i=2}^{n+2} i) - 2 = f(v)$. Hence on (G) < 2. Therefore, on (G) = 2.

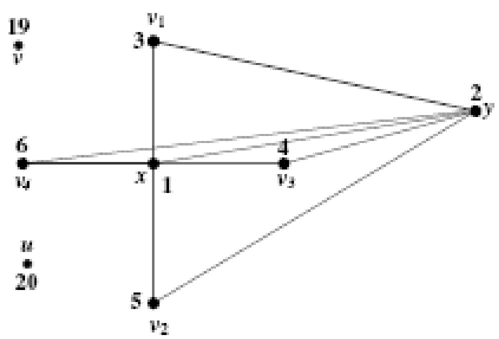

Figure 4: An Outer Sum Labeling of the Graph $(K_{1,4} + K_1) \cup 2K_1$

Theorem 6.2: For any positive integer n,

$$on(P_n + K_1) = \begin{cases} 0, & \text{if } n = 1 \\ 1, & \text{if } n = 3 \\ 2, & \text{otherwise.} \end{cases}$$

Proof: The case n = 1 follows by Theorem 2.3, the case n = 2 follows by Theorem 3.4 and, the case n = 3 follows by Theorem 2.3 and Figure 5 (for any two distinct positive integers a and b).

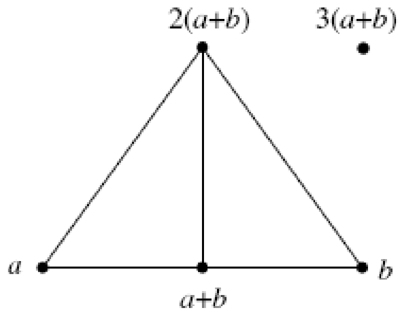

Figure 5: A Minimal Outer Sum Labeling of the Graph $(P_3 + K_1) \cup K_1$

Let us consider the graph $G = P_n + K_1$, where $n > 4$. If possible, suppose that outer sum number of G is 1. Let x be vertex of degree n – 1 and $v_1, v_2, ..., v_n$ be the other vertices (of the path) in G, such that vi adjacent to v_j only if $|j - i| = 1$, for all i, $1 < i < n$. Let f be a minimal outer sum labeling of G and u be the isolated vertex required for f. Now, $N_f(v_i) > f(x)$ for each i, $1 < i < n$ and $Nf(x) = \sum_{i-1}^{n} f(v_i) \neq N_f(v_i)$ for any i, $1 < i < n$. So, $N_f(x) = f(x)$ and $f(u) = N_f(v_i)$ for each i, $1 < i < n$. In particular $N_f(v_1) = N_f(v_2)$ and $N_f(v_2) = N_f(v_3)$.

The first equation implies that

$$f(v_2) + f(x) = f(v_1) + f(v_3) + f(x). \tag{13}$$

The second equation implies that

$$f(v_1) + f(v_3) + f(x) = f(v_2) + f(v_4) + f(x). \tag{14}$$

Equation 13 and equation 14 together imply that $f(v_4) = 0$, which is a contradiction. Thus, f require at least 2 isolated vertices, so $o_n(G) > 2$.

To prove the reverse inequality, we define a labeling $f : V(G \cup 2K_1) \rightarrow Z^+$ as below:

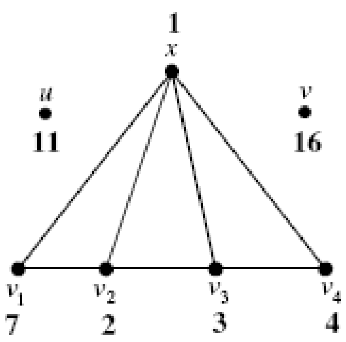

Figure 6: Outer Sum Labeling of the Graph $(P_4 + K_1) \cup 2K_1$

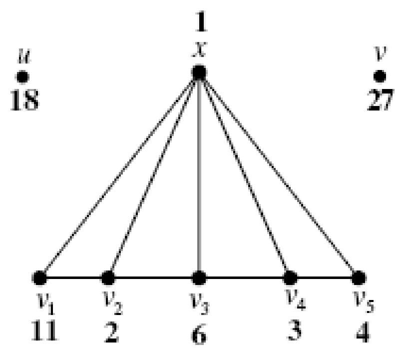

Figure 7: Outer Sum Labeling of the Graph $(P_5 + K_1) \cup 2K_1$

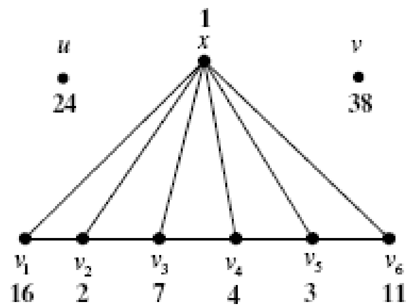

Figure 8: Outer Sum Labeling of the Graph $(P_6 + K_1) \cup 2K_1$

The labeling for the cases n = 4, 5, 6, 7, 8 are shown respectively in Fig. 6, 7, 8, 9 and 10.

When $n > 9$, the labeling in different cases is as follows:

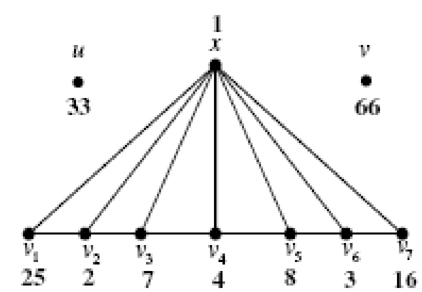

Figure 9: Outer Sum Labeling of the Graph $(P_7 + K_1) \cup 2K_1$

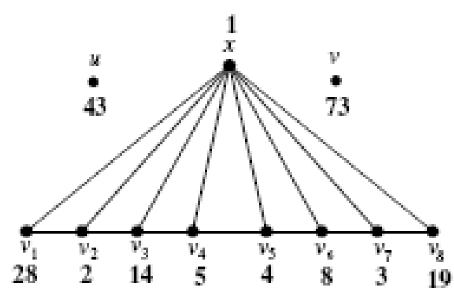

Figure 10: Outer Sum Labeling of the Graph $(P_8 + K_1) \cup 2K_1$

Case 1: When $n \equiv 1 \mod 4$

Step 1: Define $f(x) = 1$, $f(v_2) = 2$, $f(v_4) = 5$, $f(v_{n-1}) = 3$, $f(v_{n-3}) = 4$.

Step 2: For $i = 3, 4, \ldots, \frac{n-1}{4}$; Define

$$f(v_{2i}) = f(v_{2i-2}) + f(v_{2i-4}) + 1 + \frac{1+(-1)^i}{2}.$$

Step 3: For $i = 3, 4, \ldots, \frac{n-1}{4}$; Define

$$f(v_{n-(2i-1)}) = f(v_{2i}) - (-1)^i.$$

Step 4: Define $f(v_{\frac{n+1}{2}}) = f(v_{\frac{n-1}{2}}) + f(v_{\frac{n-5}{2}}) + 1.$

Step 5: Define $f(v_{\frac{n-3}{2}}) = f(v_{\frac{n-1}{2}}) + f(v_{\frac{n+3}{2}}) + 1.$

Step 6: For $i = 2, 3, \ldots, \frac{n-1}{4}$, Define

$$f(v_{\frac{n+1}{2}-2i}) = f(v_{\frac{n+5}{2}-2i}) + f(v_{\frac{n+9}{2}-2i}) + 1.$$

Step 7: Define $f(v_{\frac{n+5}{2}}) = f(v_1) + f(v_3) + 1.$

Step 8: For $i = 2, 3, \ldots, \frac{n-1}{4}$, Define

$$f(v_{\frac{n+1}{2}+2i}) = f(v_{\frac{n-3}{2}+2i}) + f(v_{\frac{n-7}{2}+2i}) + 1.$$

Step 9: For the isolated vertices u and v (say) of $G \cup 2K_1$,

$$f(u) = f(v_{n-2}) + f(v_n) + 1$$

$$f(v) = \sum_{i=1}^{n} f(v_i).$$

The label defined above is clearly an outer sum labeling of the graph $G \cup 2K_1$, because $N_f(v_1) = f(v_{n-1})$; $N_f(v_2) = f(v_{\frac{n+5}{2}})$; $N_f(v_n) = f(v_{n-3})$; $N_f(v_{n-1}) = f(u)$.

For $i = 1, 2, \ldots, \frac{n-9}{4}$

$$N_f(v_{2i+1}) = N_f(v_{n-2i}) = \begin{cases} f(v_{2i+4}), & \text{if } i \text{ odd} \\ f(v_{n-2i-3}), & \text{if } i \text{ even} \end{cases}$$

$$N_f(v_{\frac{n-3}{2}}) = f(v_{\frac{n+1}{2}}) + N_f(v_{\frac{n+5}{2}})$$

$$N_f(v_{\frac{n+1}{2}}) = f(v_{\frac{n-3}{2}}).$$

For each i, $1 < i < \frac{n-5}{4}$;

$$N_f(v_{\frac{n-1}{2}+2i}) = f(v_{\frac{n+5}{2}+2i})$$

$$N_f(v_{\frac{n+3}{2}-2i}) = f(v_{\frac{n-3}{2}-2i})$$

and

$$N_f(x) = f(v).$$

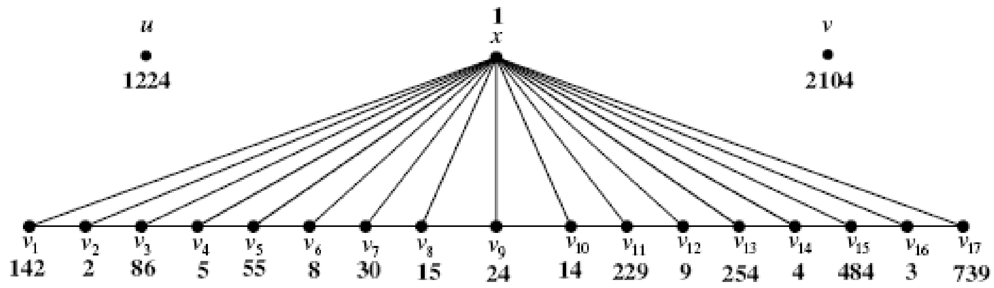

Figure 11: An Outer Sum Labeling of the Graph $(P_{17} + K_1) \cup 2K_1$

Case 2: When $n \equiv 2 \bmod 4$

Step 1: Define $f(x) = 1$, $f(v_2) = 2$, $f(v_4) = 5$, $f(v_{n-1}) = 3$, $f(v_{n-3}) = 4$.

Step 2: For $i = 3, 4, ..., \frac{n+2}{4}$; Define

$$f(v_{2i}) = f(v_{2i-2}) + f(v_{2i-4}) + 1 + \frac{1+(-1)^i}{2}$$

Step 3: For $i = 3, 4, ..., \frac{n-2}{4}$; Define

$$f(v_{n-2i+1}) = f(v_{2i}) - (-1)^i.$$

Step 4: Define $f(v_{\frac{n}{2}}) = f(v_{\frac{n+2}{2}}) + 1$.

Step 5: Define $f(v_{\frac{n-4}{2}}) = f(v_{\frac{n}{2}}) + f(v_{\frac{n+4}{2}}) + 1$

Step 6: For $i = 2, 3, ..., \frac{n-2}{4}$; Define

$$f(v_{\frac{n}{2}-2i}) = f(v_{\frac{n}{2}-2i+2}) + f(v_{\frac{n}{2}-2i+4}) + 1$$

Step 7: For $i = 1, 2, 4, ..., \frac{n-2}{4}$; Define

$$f(v_{\frac{n+2}{2}+2i}) = f(v_{\frac{n}{2}-2i}) - (-1)^i.$$

Step 8: For the isolated vertices u and v (say) of $G \cup 2K_1$,

$$f(u) = f(v_{n-2}) + f(v_n) + 1$$

$$f(v) = \sum_{i=1}^{n} f(v_i).$$

The label defined above is clearly an outer sum labeling of the graph $G \cup 2K_1$, because $N_f(v_1) = f(v_{n-1})$; $N_f(v_2) = N_f(v_{n-1}) = f(u)$; $N_f(v_n) = f(v_{n-3})$ and $N_f(x) = f(v)$.

Further for $i = 1, 2, ..., \frac{n-2}{4}$;

$$N_f(v_{2i+1}) = N_f(v_{n-2i}) = \begin{cases} f(v_{2i+4}), & \text{if } i \text{ odd} \\ f(v_{n-2i-3}), & \text{if } i \text{ even} \end{cases}$$

Finally, for $i = 2, 3, ..., \frac{n-2}{4}$;

$$N_f(v_{2i}) = f(v_{2i-3})$$

And for $i = 1, 2, ..., \frac{n-6}{4}$;

$$N_f(v_{n-2i-1}) = f(v_{2i-1})$$

$$f(x) = f(v).$$

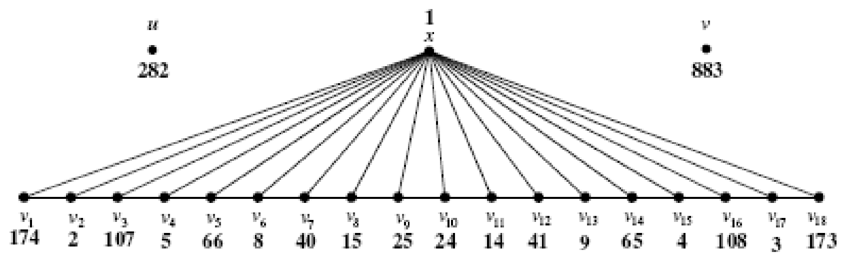

Figure 12: Outer Sum Number of the Graph $(P_{18} + K_1) \cup 2K_1$

Case 3: When $n \equiv 3 \bmod 4$

Step 1: Define $f(x) = 1$, $f(v_2) = 2$, $f(v_4) = 5$, $f(v_{n-1}) = 3$, $f(v_{n-3}) = 4$.

Step 2: For $i = 3, 4, ..., \frac{n-3}{4}$; Define

$$f(v_{2i}) = f(v_{2i-2}) + f(v_{2i-4}) + 1 + \frac{1+(-1)^i}{2}.$$

Step 3: For $i = 3, 4, ..., \frac{n-3}{4}$; Define

$$f(v_{n-(2i-1)}) = f(v_{2i}) - (-1)^i.$$

Step 4: Define $f(v_{\frac{n+1}{2}}) = f(v_{\frac{n-3}{2}}) + f(v_{\frac{n-7}{2}}) + 1.$

Step 5: Define $f(v_{\frac{n-1}{2}}) = f(v_{\frac{n-3}{2}}) + f(v_{\frac{n+1}{2}}) + 1$

$$f(v_{\frac{n+3}{2}}) = f(v_{\frac{n-1}{2}}) + (-1)^{\frac{n+1}{4}}.$$

and

$$f(v_{\frac{n-5}{2}}) = f(v_{\frac{n-1}{2}}) + f(v_{\frac{n+3}{2}}) + 1.$$

Step 6: For $i = 3, 4, \ldots, \frac{n+1}{4}$; Define

$$f(v_{\frac{n+3}{2}-2i}) = f(v_{\frac{n+7}{2}-2i}) + f(v_{\frac{n+11}{2}-2i}) + 1.$$

Step 7: For $i = 2, 4, \ldots, \frac{n+1}{4}$; Define

$$f(v_{\frac{n-1}{2}-2i}) = f(v_{\frac{n+3}{2}-2i}) - (-1)^{\frac{n+1}{4}+i}.$$

Step 8: For the isolated vertices u and v (say) of $G \cup 2K_1$,

$$f(u) = f(v_{n-2}) + f(v_n) + 1$$

$$f(v) = \sum_{i=1}^{n} f(v_i).$$

The label defined above is clearly an outer sum labeling of the graph $G \cup 2K_1$, because $N_f(v_1) = f(v_{n-1})$; $N_f(v_2) = N_f(v_{n-1}) = f(u)$; $N_f(v_{\frac{n-1}{2}}) = f(v_{\frac{n-1}{2}})$; $N_f(v_{\frac{n+3}{2}}) = f(v_{\frac{n+3}{2}})$; $N_f(v_n) = f(v_{n-3})$ and $N_f(x) = f(v)$.

For $i = 1, 2, \ldots, \frac{n-7}{4}$;

$$N_f(v_{2i+1}) = N_f(v_{n-2i}) = \begin{cases} f(v_{2i+4}), & \text{if } i \text{ odd} \\ f(v_{n-2i-3}), & \text{if } i \text{ even} \end{cases}$$

and

$$N_f(v_{2i}) = N_f(v_{n-2i+1}) = \begin{cases} f(v_{2i-3}), & \text{if } i \text{ even} \\ f(v_{n-2i+4}), & \text{if } i \text{ odd} \end{cases}$$

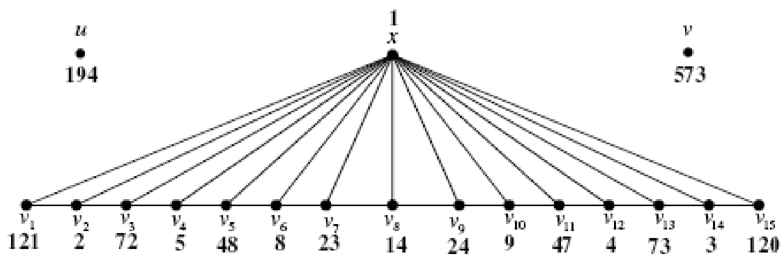

Figure 13: Outer Sum Labeling of the Graph $(P_{15} + K_1) \cup 2K_1$

Case 4: When $n \equiv 0 \bmod 4$

Step 1: Define $f(x) = 1$, $f(v_2) = 2$, $f(v_4) = 5$, $f(v_{n-1}) = 3$, $f(v_{n-3}) = 4$.

Step 2: For $i = 3, 4, \ldots, \frac{n}{4}$; Define

$$f(v_{2i}) = f(v_{2i-2}) + f - (v_{2i-4}) + 1 + \frac{1+(-1)^i}{2}.$$

Step 3: For $i = 1, 2, ..., \frac{n}{4}$; Define

$$f(vn - 2i + 1) = f(v_{2i}) - (-1)^i.$$

Step 4: Define $f(v_{\frac{n}{2}-1}) = f(v_{\frac{n}{2}}) + f(v_{\frac{n}{2}-2}) + 1$

Step 5: For $i = 1, 2, ..., \frac{n-4}{4}$; Define

$$f(v_{\frac{n}{2}-2i-1}) = f(v_{\frac{n}{2}-2i+1}) + f(v_{\frac{n}{2}-2i+3}) + 1$$

Step 6: Define $f(v_{\frac{n}{2}+2}) = f(v_1) + f(v_3) + 1$

Step 7: For $i = 2, 3, ..., \frac{n}{4}$; Define

$$f(v_{\frac{n}{2}+2i}) = f(v_{\frac{n}{2}-2i-2}) + f(v_{\frac{n}{2}-2i-4}) + 1$$

Step 8: For the isolated vertices u and v (say) of $G \cup 2K_1$,

$$f(u) = f(v_{n-2}) + f(v_n) + 1$$

$$f(v) = \sum_{i=1}^{n} f(v_i).$$

Similar to the above cases, we can easily (as it assings the neighbourhoods immidiatly for the next assignment) show that the function f serves as an outer sum labeling of the graph $G \cup 2K_1$.

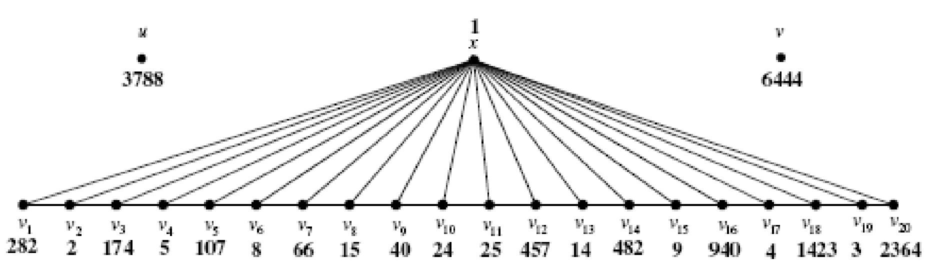

Figure 14: Outer Sum Labeling of the Graph $(P_{20} + K_1) \cup 2K_1$

In the next section we find the outer sum number of sum of a Complete graph with $\overline{K_2}$.

7. OUTER SUM NUMBER OF P_n^{n-2}

Theorem 7.1: For a given integer $n > 2$,

$$on(P_n^{n-2}) = \begin{cases} 1, & \text{if } n=2 \\ 2, & \text{if } n=3 \\ 1, & \text{if } n=4 \\ n-2, & \text{if } n \geq 5 \end{cases}$$

Proof: Let $G = P_n^{n-2}$. When $n = 2$, the graph is a star, so by Theorem 2.3 $on(G) = 1$. When $n = 3$, the graph $G \equiv C_3$, so by Theorem 3.4 we have $on(G) = 2$. When $n = 4$, the graph is isomorphic to $P_3 + K_1$, hence by Theorem 6.2 $on(G) = 1$. Let us now take the case $n > 5$. Let the vertices of square path P_n be $v_1, v_2, ..., v_n$, where v_i adjacent to v_j only if $|j - i| = 1$. In the graph G, only the vertices v_1 and v_n are of degree $n - 2$ and the remaining vertices of degree $n - 1$. Therefore to label/create a neighborhood sum for any vertex in the graph G at least $n - 2$ vertices has to be label arbitrarily. For minimality, we label the vertices $v_2, v_3, .., v_{n-1}$ arbitrarily, this will form a equal neighborhood sum for the vertices v_1 and v_n. By assigning this sum to the vertex v_1 and giving any arbitrary label for v_n we get the neighborhood sum for the remaining vertices. Since only two neighborhood sums are assigned to the vertices of G and all the vertices are labeled. We required minimum $n - 2$ isolated vertices to assign the rest $n - 2$ neighborhood sums created by the vertices other than v_1 and v_n, therefore $on(G) > n - 2$.

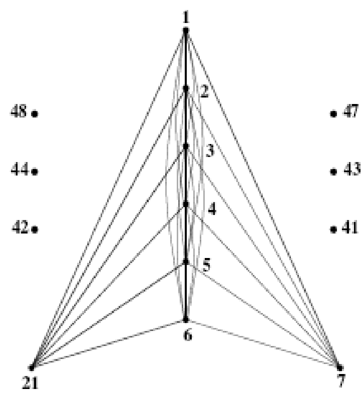

Figure 15: Outer Sum Labeling of the Graph $P_8^6 \cup K_1$

To prove the reverse inequality, let $u_1, u_2, ..., u_{n-2}$ be the isolated vertices of $G \cup (n-2)K_1$ and define a function $f : V(G \cup (n-2)K_1) \rightarrow Z^+$ by $f(v_i) = i - 1$, for all $i, 2 < i < n - 1$, $f(v_1) = \frac{(n-1)(n-2)}{2}$ and $f(v_n) = n - 1$. For each $i, 1 < i < n - 2$, define

$f(u_i) = n^2 - 2n + 2 - i$. Then, it is clear that $N_f(v_i) = f(u_i)$ for each i, $2 \le i \le n - 2$ and $N_f(v_n) = N_f(v_1) = f(v_1)$. Hence f is an outer sum labeling of $G \cup (n - 2) K_1$, so $on(G) \le n - 2$. Thus, $on(G) = n - 2$.

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