International Journal of Combinatorial Graph Theory and Applications
Vol. 4, No. 1, (January- June, 2019)

THE MATH MODEL AND ALGORITHM FOR THE DYNAMIC MINIMUM TIME PATH PROBLEM WITH CURFEWS

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RECEIVED: July 15, 2017. Revised September 27, 2017

Abstract: Shortest path problem is a basic and important problem in the combinatorial optimization. The goals of the problem always depend on the time. Moreover, there will be curfews on some nodes in the network. We develop the math models for the dynamic minimum time path problem with soft and hard curfews, and give the algorithm for it. We also present a practical case.

Kewords: Shortest Time Path; The Dynamic Problem; Soft Curfews; Hard curfews; Algorithm

1. INTRODUCTION

The shortest path problem is a relatively basic and important problem in the network optimization. Firstly, this problem often appears in real production and life. Many optimization problems can be transformed into the shortest path problem. Secondly, this problem is relatively simple and its effective algorithm often is used as sub-algorithm in other network optimization problem. This problem has been well solved in the static network such as the Dijkstra algorithm in the fifties and the Floyd algorithm in the sixties of twenty century.[1] In some real shortest path problem such as in congesting transportation network, since vehicles traveling times are time-varying quantities so we still need to consider the affecting time to the determination value in solving the shortest path.

At present, there are a few results for dynamic shortest path problem. They focus on the research for the shortest time path problem. Elise D. M. and Hani S. M.[2] studied the minimum time path problem in random dynamic network, put forward two different algorithms and compared the two. R. W. Hall[3] gave a algorithm for the minimum expectation time path in random dynamic network. Daniele P.[4] researched the shortest time path problem under the dispersed random dynamic hyper-graphs network and gave the solving method. G. Gallo, G. Longo, S. Nguyen and S.Pallottion,[5] studied the shortest path problem in hyper-graphs and analyzed its application to the transportation system.

In real situation, vehicle transportation problem with curfews also probably appears. For example, the arrival of the vehicle at some node is limited by time and in some period the vehicle can not arrive or will be punished. Cox R. and Turquist M.[6] studied the shortest path problem with curfews under the static network condition.

This paper is about the dynamic minimum time path problem with curfews. We first transform the dynamic problem with curfews into the static problem with curfews by modeling time expansion network and into the models of the general shortest path problem and desigin the solving algorithm. Finally, we also present a practical case.

2. PROBLEM DESCRIPTION

The dynamic shortest path problem with curfews are mainly classified into two kinds: one is the shortest path problem with soft curfews. The other is the shortest path problem with hard curfews. Their differences are that soft curfews are soft while hard curfews are rigidity. That is, to the shortest path problem with soft curfews if the vehicle drops in some curfews area of this node when it arrives at some node, now the vehicle will be punished in some way and its restarting time is the terminal time of this curfews time area; To the shortest path problem with hard curfews if the time which the vehicle arrives some node lies on some curfews time area of this node then it is a unfeasible solution. Obviously, it is not sure that the problem have feasible solution under the condition of hard curfews.

Let $G = (N, E, t', c')$ be a directed network among which N is the set of nodes, E is the set of directed edges of the each pair nodes. $|N| = n$, $|E| = m$, t: $E \to Z^+$ is the time weight function when the vehicle passes the directed link. It depends on the starting time of the origin point of this edge; c: $E \to R^+$ is the cost needed when the vehicle passes the directed link and it also depends on the starting time of the origin point of this link. $O \in N$ is the origin point. $D \in N$ is a appoint node. We consider the dynamic network under the dispersed time model, that is we take the starting time from origin O from the dispersed time set $T = \{t_1, t_2, ..., t_k\}$ where $t_1 < t_2 < ... < t_k$ are non-negative integers. Z_i denotes the set of the curfews time area at the node $i \in N$, ie, $Z_i = \{[\alpha_{i1}, \beta_{i1}), [\alpha_{i2}, \beta_{i2}), ..., [\alpha_{ip_r}, \beta_{ip_r})\}$ and $t_1 \le \alpha_{i1} < \beta_{i1} < \alpha_{i2} < \beta_{i2} < ... < \alpha_{ip_r} < \beta_{ip_r} \le t_k$ where $[\alpha_{iq}, \beta_{iq})$ is the some curfews time area of the node i.

Our problem is to make out the the minimum time path with the curfews, in which the vehicle starts from the O to the appointed node D in different period, but the cost does not exceed C_0 (C_0 is some real number).

3. MATH MODEL

3.1 we will model a time expansion network G^T and transform the dynamic problem into the static problem.

In the directed network $G = (N, E, t', c')$, let $t'_h(i, j)$ be the cost time traveling from i to j starting at $t_h \cdot t_l$ is the time traveling from i to j starting at t_h. ie., $t_l = t'_h(i \cdot j)$, where all $t_h \in T$, $t'_h(i, j)$ are positive integers and $c'_h(i, j)$ is the traveling cost from i to j starting at $t_h \cdot c'_h(i, j)$ is a non-negative real number.

We model the time expansion network $G^T = (V, A, t, c)$ as follows:

$V = \{i_h : i \in N, 1 \le h \le k\} \cup \{s, v\}$, where i_h is corresponding to the time t_h; s is the origin point, v is the terminal point.

$A = \{(i_h, j_l) : (i, j) \in E, t_h + t'_h(i, j) = t_l; t_h, t_l \in T; 1 \le h \le l \le k, \text{and } t_h \notin Z_i\} \cup \{(i_h, i_{h+1}): t_h \in Z_i, i \in N\} \cup \{(s, O_h) : h = 1, 2, ..., k\} \cup \{(D_h, v) : h = 1, 2, ..., k\}$, in network $G^T = (V, A, t, c)$, $t : A \to Z$ is the non-negative weight function on the arc. The arc $(i, j) \in A$ corresponding to the weight $t(i, j)$ is the cost time the vehicle traveling from i to j. For the soft curfews, let

$$t(i,j) = \begin{cases} t_h, & as \quad i = s, j = O_h, h = 1, 2, ..., k \\ 0, & as \quad i = D_h, j = v, h = 1, 2, ..., k \\ t'_h(i, j), & as \quad i = i_h, j = j_l, i \ne j; h, l = 1, 2, ..., k \\ t_{h+1} - t_h, & as \quad i = i_h, j = i_{h+1}, h = 1, 2, ..., k \end{cases} \tag{1}$$

For the hand curfews, let

$$t(i,j) = \begin{cases} t_h, & \text{as} \quad i = s, j = O_h, h = 1, 2, ..., k \\ 0, & \text{as} \quad i = D_h, j = v, h = 1, 2, ..., k \\ t'_h(i, j), & \text{as} \quad i = i_h, j = j_l, i \ne j; h, l = 1, 2, ..., k \\ M, & \text{as} \quad i = i_h, j = i_{h+1}, h = 1, 2, ..., k \end{cases} \tag{2}$$

where M is a large positive integer.

We assume that $c : A \to R$ is the non-negative weight function and $c(i, j)$ is the cost time the vehicle traveling from i to j, therefore, for the soft curfews, let

$$c(i,j) = \begin{cases} 0, & \text{as} \quad i = s, j = O_h, h = 1, 2, ..., k \\ 0, & \text{as} \quad i = D_h, j = v, h = 1, 2, ..., k \\ c'_h(i, j), & \text{as} \quad i = i_h, j = j_l, i \ne j, t_l \notin Z_j; h, l = 1, 2, ..., k \\ c'_h(i, j) + \alpha(t_l - \alpha_{jq}), & \text{as} \quad i = i_h, j = j_l, t_l \in [\alpha_{jq}, \beta_{jq}) \\ \beta(t_{h+1} - t_h), & \text{as} \quad i = i_h, j = i_{h+1}, h = 1, 2, ..., k \end{cases} \tag{3}$$

where α is the punishment value in the unit time when the vehicle arrives j in curfews time area and β is the waiting cost in the unit time when the vehicle arrives in curfews time area, but need to wait at j until the end of curfews time area.

For the hand curfews, let

$$c(i,j) = \begin{cases} 0, & \text{as } \ i = s,\ j = O_h,\ h = 1, 2, ..., k \\ 0, & \text{as } \ i = D_h\ ,\ j = v,\ h = 1, 2, ..., k \\ c'_h(i,j), & \text{as } \ i = i_h,\ j = j_l,\ i \neq j,\ t_l \notin Z_j\ ; h, l = 1, 2, ..., k \\ M, & \text{as } \ i = i_h,\ j = j_l,\ i \neq j,\ t_l \in Z_j\ ; h, l = 1, 2, ..., k \\ M, & \text{as } \ i = i_h\ ,\ j = i_{h+1},\ h = 1, 2, ..., k \end{cases} \tag{4}$$

Thus every path with curfews in network G forms from O to the appointed node D at different time corresponding to one path from s to v in network G^T and the costs the two need to pay are the same.[7] So the problem that we solve is transformed into the static minimum time path problem with curfews, in which the vehicle starts forming from s to the terminal point v in the time expansion network G^T, but the cost does not exceed C_0.

3.2 MATH MODEL

x_{ij} denotes whether arc (i, j) lies on the road from the origin point s to the terminal point v : as $x_{ij} = 1$, it denotes that arc (i, j) lies on the road from the origin point s to the terminal point v ; as $x_{ij} = 0$, it denotes that arc (i, j) does not lies on the road from the origin point s to the terminal point v . We establish the math model of the problem as follows:

$$\begin{aligned} &\min \textstyle\sum_{(i,j)\in A} t(i,j)x_{ij} \\ &s.t. \textstyle\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = 0,\ i \in V \setminus \{s, v\} \\ &\quad \textstyle\sum_{(s,j)\in A} x_{sj} = 1 \\ &\quad \textstyle\sum_{(i,v)\in A} x_{iv} = 1 \\ &\quad \textstyle\sum_{(i,j)\in A} c(i,j)x_{ij} \le C_0 \\ &\quad x_{ij} \ge 0,\ \forall\ (i,j) \in A \end{aligned} \tag{5}$$

For the soft curfews, $t(i, j)$ is defined by (1) and $c(i, j)$ is defined by (3); For the hard curfews, $t(i, j)$ is defined by (2) and $c(i, j)$ is defined by (4).

4. ALGORITHM[8, 9]

By the hypothesis and the construction of G^T we know that G^T is a network of the no circles of non-negative cost. Weight function $t : A \to Z$ and $c : A \to Z$ are non-negative increasing function on the arcs. We can put the top point in network in topology order, such that $\forall\ i \geq j,\ (i, j) \notin A$, we mark the top point set put in order: $V = \{1, 2, ..., nk + 2\}$ where $s = 1$ is the origin point , $v = nk + 2$ is the terminal point, mark $t_{ij} = t(i, j)$, $c_{ij} = c(i, j)$. If $(i, j) \notin A$, definite $t_{ij} = +\infty$, $c_{ij} = +\infty$.

We will give the algorithm on solving the dynamic minimum time path problem with curfews:

Step 1: We use the directed $G = (V, A, t', c')$ to construct the expansion network $G^T = (V, A, t, c)$, weight function $t : A \to Z$ and $c : A \to R$ in expansion network $G^T = (V, A, t, c)$ are given by (1)(4) .The math model of problem that we solve is given by (5). Then we mark the top set in expansion network $G^T = (V, A, t, c)$ by $V = \{1, 2, ..., nk + 2\}$ and note $t_{ij} = t(i, j)$, $c_{ij} = c(i, j)$. If $(i, j) \notin A$, definite $t_{ij} = +\infty$, $c_{ij} = +\infty$.

Step 2: Let $u_1 = 0$, $u_j = t_{1j}$, $(j = 2, 3, ..., nk + 2)$, $w_j = c_{1j}$, $S = \{1\}$, $R = V \backslash S = \{2, 3, ..., nk + 2\}$ and

$$d_j = \sum_{i=2}^{nk+2} b_{ij},\ (j = 2, 3, ..., nk + 2),$$

where

$$b_{ij} = \begin{cases} 1, & \text{if} \quad (i, j) \in A \\ 0, & \text{if} \quad \text{not} \end{cases}$$

Step 3: If $R = \phi$, over; if not, turn to step 4 .

Step 4: We look for a top point q in R such that $d_{ij} = 0$. Let $S = S \cup \{q\}$, $R = R \backslash \{q\}$, for all arcs $(q, j) \in A$ starting from top point q, if $u_j > u_q + t_{qj}$, let $u_j = u_q - t_{qj}$, $w_j = w_q + c_{qj}$, $d_j = d_j - b_{qj}$.

Step 5: If $u_j > C_0$, let $S = S \cup FS(q)$, $R = R \cup FS(q)$, where $FS(q)$ is the set of all forward directed nodes of q; if not, let $S = S$, $R = R$ turn to step 3.

Obviously, the analysis and solution above can be applied to look for the minimum time path problem in which the vehicle starts at some time t_h from O to D and the cost can't exceed C_0. This time,we take $x_{sh} = 1$ while $x_{sr} = 1$ $(r \neq h)$.

5. PRACTICAL CASE

The following is a real case. We assume that transportation network as Fig. 1 shows. Table 1 tells the transportation cost and time of each pair nodes under condition of different time. Table 2 gives curfews time area of each nodes where *Y* denotes that there will be curfews in this time area and *N* denotes there is no curfews. Suppose the vehicle can start at time *O* and set off once every two hours. Now we expect to obtain the minimum time path with curfews in which the vehicle starts from *O* to appointed node *D*, but the cost does not exceed 70 at different time.

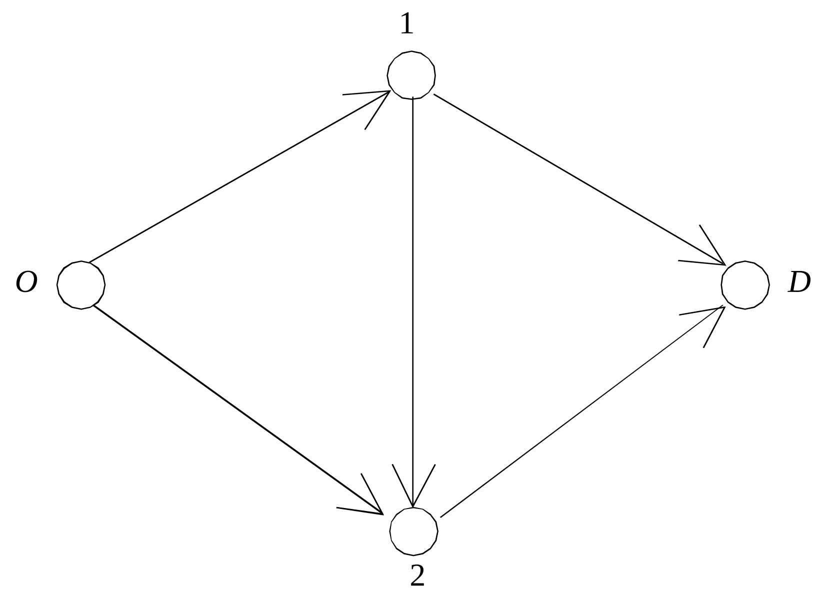

Figure 1: The Transportation Network

Table 1
The Time-Varying Cost and Time of Each Link Arc Time

	Time		
Arc	*[0, 6)*	*[6, 12)*	*[12, 18)*
(0, 1)	20/4	15/4	10/2
(0, 2)	15/6	30/2	35/6
(1, 2)	10/4	10/4	15/4
(1, *D*)	20/2	25/2	30/4
(2, *D*)	10/2	5/4	15/2

Tabble 2
The curfews of each node

Node	*Curfews time area* [2,6)	*Curfews time area* [10,14)
O	*Y*	*N*
1	*N*	*Y*
2	*Y*	*N*
D	*N*	*Y*

Constructing the time expansion network as Fig. 2 shows.

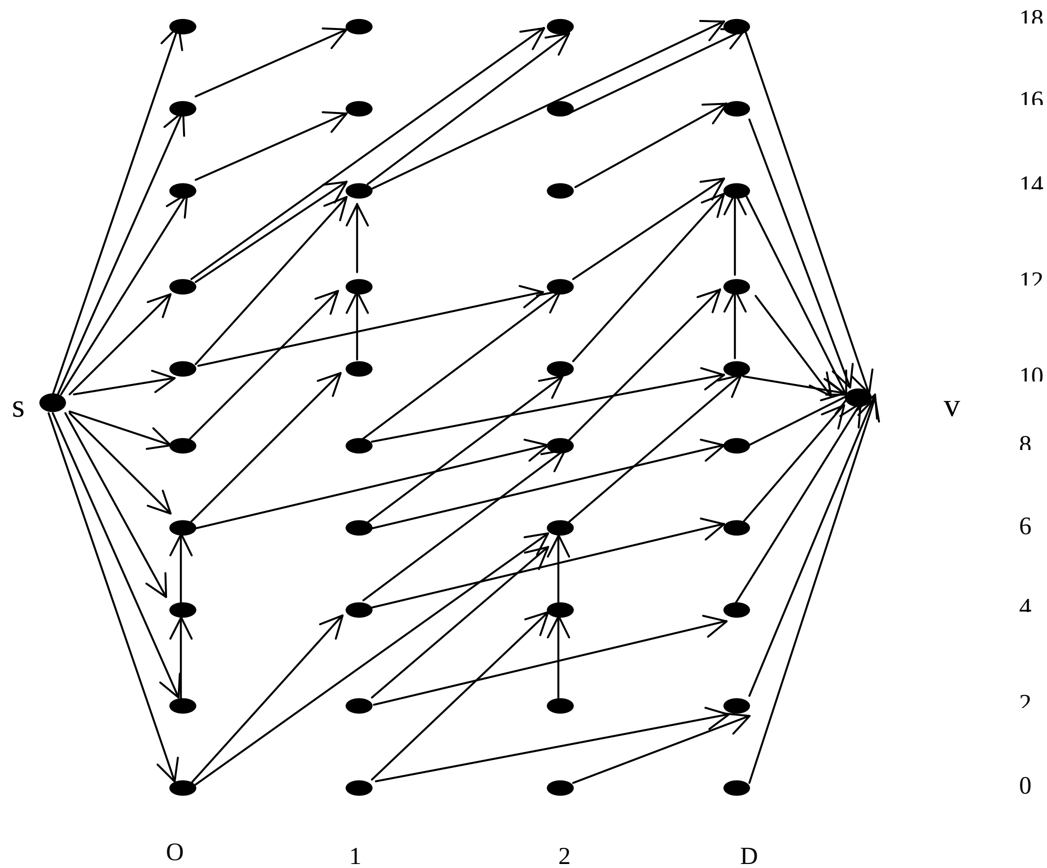

Figure 2: The Time-Expanded Network

As Table 3 shows ,we can obtain the minimum time path in which the vehicle starts at different time from *O* to *D* by using the above algorithm. Here, suppose under the condition of the soft curfews the vehicle arrives at the time over the starting time in one curfews time area, the punishment cost in unit time it gets is $\alpha = 3$. While the waiting cost in unit time that the vehicle waits until the curfews time area end is $\beta = 2$.

Table 3
The minimum time path in different start time

Start time	*The minimum time path with soft curfews*	*Time*
0	O-1-D	6
2	O-2-D	12
4	O-2-D	10
6	O-2-D	8
8	O-1-D	10
10	O-2-D	4
12	O-1-D	6
Start time	*The minimum time path with hard curfews*	*Time*
0	O-1-*D*	6
2	—	—
4	—	—
6	—	—
8	—	—
10	O-2-*D*	4
12	O-1-*D*	6

6. CONCLUSION

As the minimum time path can be seen from Table 3 no matter there is soft curfews or hard curfews. When the vehicle sets off at different time the shortest path is different and so is objective value. Therefore decision maker may choose proper starting time and path according to his own situation to reach the aim of no excess to certain costs and saving time as possibly as one can. This dynamic minimum time path problem with curfews, there are a lot of applications in future, including the transportation, material flow, electric power, communicate by letter etc.

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