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BOUNDED CONFLICT FREE PETRI NETS RELATIVE TO (Z_n, A)

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Abstract: Petri nets concepts provide additional tool for the modeling of metabolic networks. Especially the concepts of siphon and trap are used to study the role of trios phosphate isomerase's (TPI) in trypanosome brucei metabolism. In this paper we show that the bounded conflict free Petri net based on groups with generating set has some structural properties. In particular if the group is (Z_n, A) then the corresponding bounded conflict free Petri net has a subset of places whose input transitions are equal to the output transitions and both of them equal to the set of all transitions of the bounded conflict free Petri net. This leads us to show that the underlying directed graph of this bounded conflict free Petri net is **Hamiltonian**. Also we prove that there exists a decomposition $\pi = \{\pi_1, \pi_2, \pi_3\}$ for the set of places of P such that each block π_i is both siphon and trap and hence the underlying directed graph of this bounded conflict free Petri net is **Eulerian**. We prove some behavioral properties of this bounded conflict free Petri net.

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1. INTRODUCTION

Petri nets are one of the most popular models for the representation and analysis of parallel processes. It has a wide range of application including information systems, operation systems, databases, communication protocols, computer hardware architectures, security systems, manufacturing systems, defence command and control, business processes, banking systems, chemical processes, nuclear waste systems, system biology, graph theory and telecommunications [1, 2]. There are many sub classes for Petri nets namely marked graph, free choice Petri nets, conflict free Petri nets etc.

A Petri net can be represented as a particular kind of bipartite graph consisting of two kinds of nodes called places and transitions. Directed arcs are used to connect places to transitions (output of places) and to connect transitions to places (input of places).

The study of structural properties and behavioral properties for the bounded conflict free Petri net has been made utilizing siphons and traps [4, 6]. A nonempty subset of places J is called a siphon if every transition having an output place in J

has an input place in J. A nonempty subset of places Q is called a trap if every transition having an input place in Q has an output place in Q.

In this paper we construct a bounded conflict free Petri net for a given group (Z_n, A) with a generating set. We prove that the resulting bounded conflict free Petri net associated with this group has subsets of places which are both siphon and trap whose input transitions equal to output transitions and both of them equal to the set of all transitions. This leads us to establish that the underlying directed graph of the constructed bounded conflict free Petri net is Hamiltonian. In addition to this we prove that there exists a partition for a place set P such that each block in the partition of the set of places of the bounded conflict free Petri net is both siphon and trap. This leads us to establish that the underlying directed graph is Eulerian.

2. PRELIMINARIES

In this section we present some basic Definitions relevant to this paper.

Definition 2.1: A Petri net is triple $N = (P, T, F)$ where P is a finite set of places, T is finite set of transitions such that

(i) $P \cup T \neq \phi$; $P \cap T = \phi$

(ii) $F \subseteq (P \times T) \cup (T \times P)$ is set of directed arcs.

For all $p \in P$, ${}^{\bullet}p = \{t \in T\, (t, p) \in F\}$ and $p^{\bullet} = \{t \in T \mid (p, t) \in F\}$ be the input and output sets of p respectively. Similarly for all $t \in T$, ${}^{\bullet}t = \{p \in P \mid (p, t) \in F\}$ and $t^{\bullet} = \{p \in P/(t, p) \in F\}$ be the input and output sets of t respectively.

Definition 2.2: A Petri net is said to be a marked graph if $|{}^{\bullet}p| = |p^{\bullet}| = 1$ for all $p \in P$. A Petri net is said to be conservative if $|{}^{\bullet}t| = |t^{\bullet}|$ for all $t \in T$. If the number of tokens in a place p is always less than or equal to a constant k, then such a place is called k-bounded. A Petri net is said to be k-bounded if all its places are k-bounded. If the constant $k = 1$ then it is called a safe Petri net.

Definition 2.3: A non-empty subset of places J in a bounded conflict free Petri net is called a siphon if ${}^{\bullet}J \subseteq J^{\bullet}$. That is every transition having an output place in J has an input place in J.

Definition 2.4: A nonempty subset of places Q in a bounded conflict free Petri net is called a trap if $Q^{\bullet} \subseteq {}^{\bullet}Q$. That is every transition having an input place in Q has an output place in Q.

Definition 2.5: A non empty subset Z of places in a bounded conflict free Petri net is said to be both siphon and trap if ${}^{\bullet}Z = Z^{\bullet}$. That is, every transition having an input place in Z has an output place in Z and vice versa.

Example 2.6: Consider the bounded conflict free Petri net shown in Fig. 1.

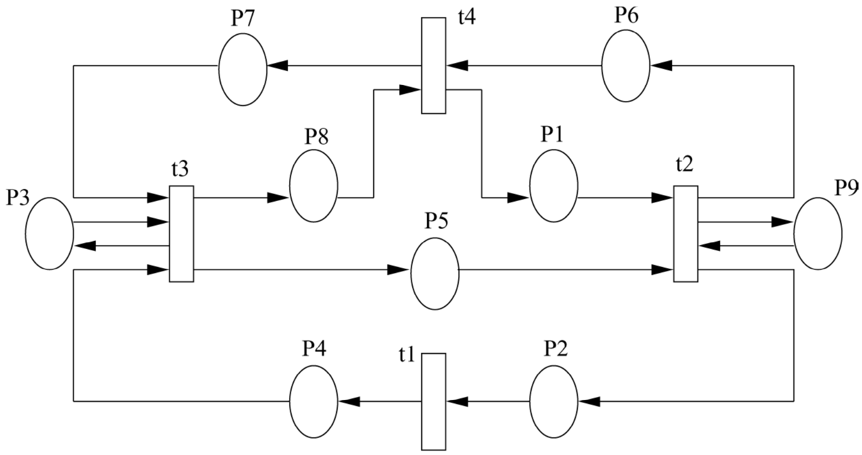

Figure 1: The Petri Net

In this bounded conflict free Petri net let $J = \{p_1, p_6, p_7\}$. Now, ${}^\bullet J = \{t_2, t_4\}$, and $J^\bullet = \{t_2, t_3, t_4\}$, ${}^\bullet J \subseteq J^\bullet$. Therefore J is a siphon. Let $Q = \{p_6, p_7, p_8\}$, ${}^\bullet Q = \{t_2, t_3, t_4\}$, $Q^\bullet = \{t_3, t_4\}$, $Q^\bullet \subseteq {}^\bullet Q$. Therefore Q is a trap. Let $Z = \{p_2, p_4, p_5\}$, ${}^\bullet Z = \{t_1, t_2, t_3\}$, ${}^\bullet Z = \{t_1, t_2, t_3\}$, $Z^\bullet = {}^\bullet Z$. Therefore Z is both siphon and trap.

Definition 2.7: For any natural number n, we use Z_n to denote the additive cyclic group of integers modulo n.

In this paper we consider the group (Z_{2k}, A) where A is a generating set $\{a, b, b + k\}$ with the property that

$$gcd(a, b, k) = 1 \text{ and either}$$
$$gcd(a - b, k) \neq 1 \text{ or}$$
$$gcd(a, 2k) = 1 \text{ or}$$
$$gcd(b, k) = 1 \text{ or}$$

both a&k are even or a is odd and either b or k is odd

Example 2.8: (a) (Z_8, A) is a group consisting of 8 elements where $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and the generating set $A = \{5, 3, 7\}$

(b) (Z_{12}, A) is a group consisting of 12 elements where $Z_{12} = \{0, 1, 2, 3. ..., 11\}$ and the generating set $A = \{7, 5, 11\}$

(c) (Z_{16}, A) is a group consisting of 16 elements where $Z_{16} = \{0, 1, 2, 3, ..., 15\}$ and the generating set $A = \{5, 3, 11\}$

3. CONSTRUCTION OF BOUNDED CONFLICT FREE PETRI NET

In this section we construct a bounded conflict free Petri net for a given group with a generating set and we prove that it has some structural properties.

Theorem 3.1: There exists a bounded conflict free Petri net for every group (Z_n, A) where A a generating set.

Proof: Let (Z_{2k}, A) be a group with generating set $A = \{a, b, b + k\}$ where a, b, k are integers. Take the elements of Z_{2k} as the transitions of the bounded conflict free Petri net. Since Z_{2k} has $2k$ elements, we have $|T| = 2k$. Moreover, $A \subseteq Z_{2k}$. Now let us introduce places as follows. For every $t_i \in Z_{2k}$ and $s_k \in A$ such that $t_i + s_k = t_j$ (mod $2k$), make a place p such that ${}^{\bullet}p = t_i$ and $p^{\bullet} = t_j$. Also deposits tokens in a place p if p is the input of $s_i + s_j$, for every $s_i, s_j \in A$. Since the generating set A has 3 elements, we have each transition has exactly 3 inputs and 3 outputs. Thus we have constructed a bounded conflict free Petri net with initial marking. □

Theorem 3.2: The bounded conflict free Petri net for the group (Z_n, A) is conservative.

Proof: It is clear from the construction that each transition has exactly three inputs and three outputs. The token deposited in the places neither generated nor destroyed by firing of transitions. Thus by Definition 2.2, the bounded conflict free Petri net for the group (Z_n, A) is conservative. □

Theorem 3.3: The bounded conflict free Petri net for the group (Z_n, A) is not safe.

Proof: Through the reachability tree of this bounded conflict free Petri net for the group (Z_n, A) we observe that some places has more than one token at some stages. Hence the constructed bounded conflict free Petri net for the group (Z_n, A) is not safe. □

Theorem 3.4: The bounded conflict free Petri net for the group (Z_n, A) is bounded.

Proof: Since the bounded conflict free Petri net for the group (Z_n, A) is conservative, the token deposited at initial marking neither destroyed nor created. Suppose we initially deposited 'k' tokens in the initial marking, some place can have a maximum of k-tokens at some level. Hence it is k-bounded. □

Remark: The constructed bounded conflict free Petri net is clearly a marked graph.

Example 3.5: The constructed bounded conflict free Petri net for the group $(Z_8, 5, 3, 7)$ is shown in Fig. 2.

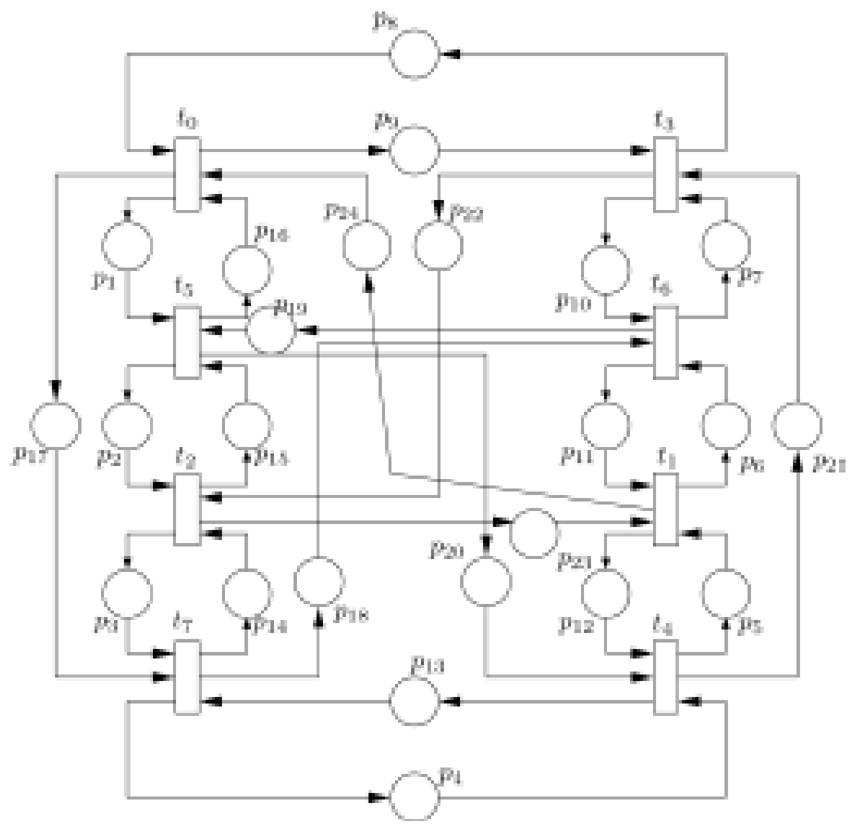

Figure 2: Constructed Bounded Conflict Free Petri Net for the Group (Z_8, 5, 3, 7)

In the bounded conflict free Petri net shown in Fig. 2, there exists a subset of places $Z = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$ such that $Z^\bullet = \{t_0, t_5, t_2, t_7, t_4, t_1, t_6, t_3\}$ and $^\bullet Z = \{t_5, t_2, t_7, t_4, t_1, t_6, t_3, t_0\}$. Hence Z is both siphon and trap.

In [5] it is proved that if there exists a subset J in a constructed marked graph such that $^\bullet Z = Z^\bullet = T$, then the edges corresponding to the places in Z constitute a directed Hamiltonian circuit in the underlying graph. For the bounded conflict free Petri net shown in Fig. 2 the underlying directed graph is shown in Fig. 3. In Fig. 3, the edges $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ corresponding to the places in Z constitutes a directed Hamiltonian Circuit. Hence underlying directed graph of the constructed bounded conflict free Petri net of (Z_8, 5, 3, 7) is Hamiltonian.

In [4] it is proved that if each block π_i of a decomposition $\pi = \{\pi_1, \pi_2, \pi_3, ..., \pi_n\}$ of the set of places is both Siphon and trap then the underlying directed graph of the bounded conflict free Petri net is Eulerian. For the directed graph is shown in Fig. 3,

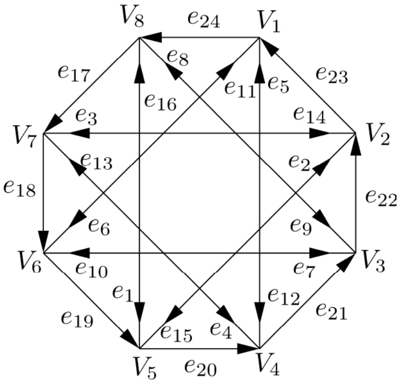

Figure 3: The Underlying Directed Graph for the Bounded Conflict Free Petri net of the Group (Z_8, 5, 3, 7)

let $\pi = \{\pi_1, \pi_2, \pi_3\}$ be the decomposition of the set of places where $\pi_1 = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$, $\pi_2 = \{p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}\}$, $\pi_3 = \{p_{17}, p_{18}, p_{19}, p_{20}, p_{21}, p_{22}, p_{23}, p_{24}\}$ such that ${}^\bullet\pi_1 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$, $\pi_1^\bullet = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$, ${}^\bullet\pi_2 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$, $\pi_2^\bullet = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$, ${}^\bullet\pi_3 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$, $\pi_3^\bullet = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$. Therefore each π_i is both siphon and trap. Hence underlying directed graph of the constructed bounded conflict free Petri net of (Z_8, 5, 3, 7) is Eulerian.

Example 3.6: Consider the group (Z_{12}, A) consisting of 12 elements with a generating set $A = \{7, 5, 11\}$. The constructed bounded conflict free Petri net for this group is shown in Fig. 4.

In the constructed bounded conflict free Petri net for (Z_{12}, 7, 5, 11), there exists a subset of places $Z = \{p_1, p_2, p_3, ..., p_{12}\}$ such that $Z^\bullet = \{t_0, t_7, t_2, t_9, t_4, t_{11}, t_6, t_1, t_8, t_3, t_{10}, t_5\}$ and ${}^\bullet Z = \{t_7, t_2, t_9, t_4, t_{11}, t_6, t_1, t_8, t_3, t_{10}, t_5, t_0\}$. The underlying directed graph for the bounded conflict free Petri net given in Fig. 4 is shown Fig. 5. In the directed graph shown in Fig. 5, the edges $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}\}$ corresponding to the places in Z constitutes a directed **Hamiltonian Circuit**. Hence underlying directed graph of the constructed bounded conflict free Petri net of (Z_{12}, 7, 5, 11) is **Hamiltonian** (Fig. 5). For the same bounded conflict free Petri net in Fig. 4, let $\pi = \{\pi_1, \pi_2, \pi_3\}$ be the decomposition of the set of places, where

$\pi_1 = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}\}$,

$\pi_2 = \{p_{13}, p_{14}, p_{15}, p_{16}, p_{17}, p_{18}, p_{19}, p_{20}, p_{21}, p_{22}, p_{23}, p_{24}\}$,

$\pi_3 = \{p_{25}, p_{26}, p_{27}, p_{28}, p_{29}, p_{30}, p_{31}, p_{32}, p_{33}, p_{34}, p_{35}, p_{36}\}$ such that

${}^\bullet\pi_1 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}$, $\pi_1^\bullet = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}$,

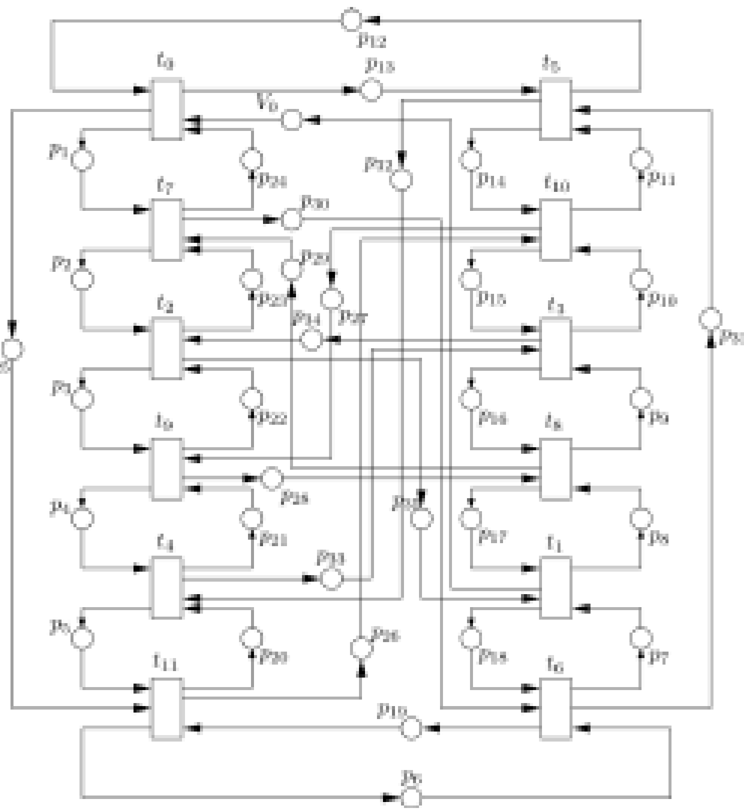

Figure 4. Constructed Bounded Conflict Free Petri Net for the Group (Z_{12}; 7; 5; 11)

$^{\bullet}\pi_2 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}$, $\pi^{\bullet}_2 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}$,

$^{\bullet}\pi_3 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}$, $\pi^{\bullet}_3 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}$.

Therefore each π_i is both siphon and trap. Hence underlying directed graph of the constructed bounded conflict free Petri net (Z_{12}, 7, 5, 11) is Eulerian (Fig. 5).

Example 3.7: Consider the group (Z_{16}, A) consisting of 16 elements with a generating set A = {5, 3, 11}. Constructed bounded conflict free Petri net for the group (Z_{16}, 5, 3, 11) is shown in Figure 6. The underlying directed graph shown in Fig. 7.

In the constructed bounded conflict free Petri net of (Z_{16}, 5, 3, 11) , there exists a subset of places $Z = \{p_1, p_2, p_3, ..., p_{16}\}$ such that $Z^{\bullet} = \{t_0, t_5, t_{10}, t_{15}, t_4, t_9, t_{14}, t_3, t_8, t_{13}, t_2, t_7, t_{12}, t_1, t_6, t_{11}\}$ and $^{\bullet}Z = \{t_5, t_{10}, t_{15}, t_4, t_9, t_{14}, t_3, t_8, t_{13}, t_2, t_7, t_{12}, t_1, t_6, t_{11}, t_0\}$. In the underlying directed graph shown in Fig. 7, the edges $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}\}$ corresponding to the places in Z constitutes a directed

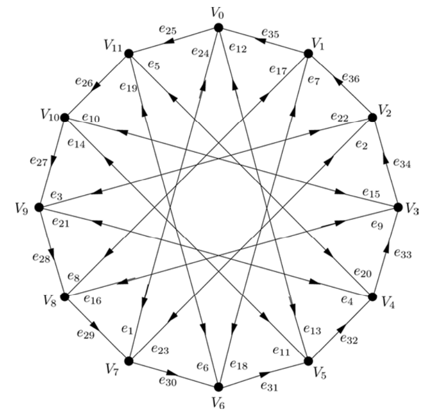

Figure 5: The Underlying Directed Graph for the Bounded Conflict Free Petri net of the Group (Z_{12}, 7, 5, 11)

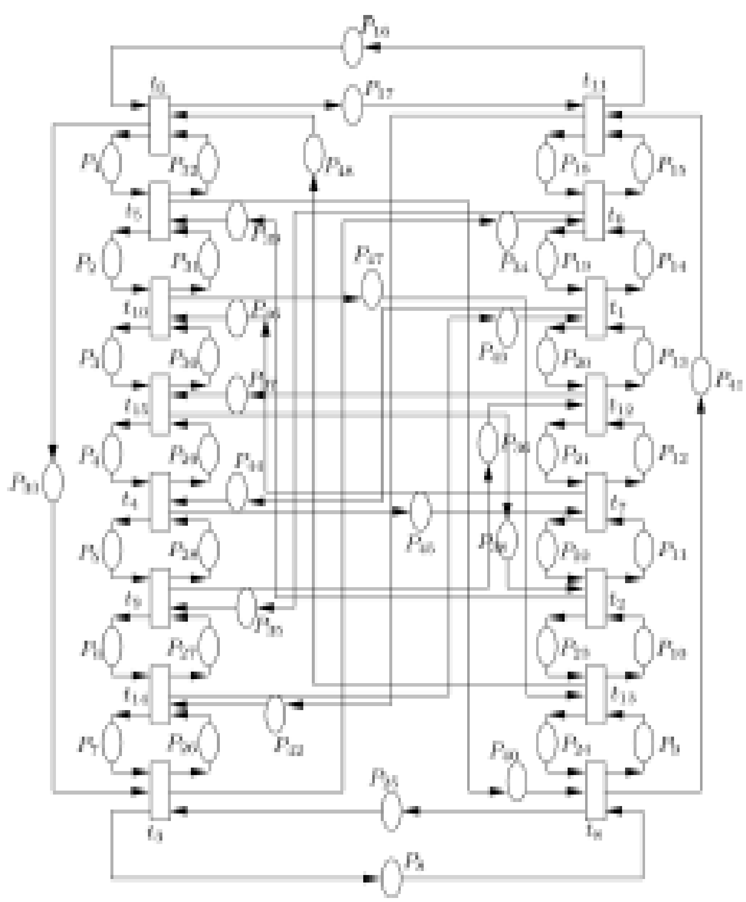

Figure 6: Constructed Bounded Conflict Free Petri Net for the Group (Z_{16}, 5, 3, 11)

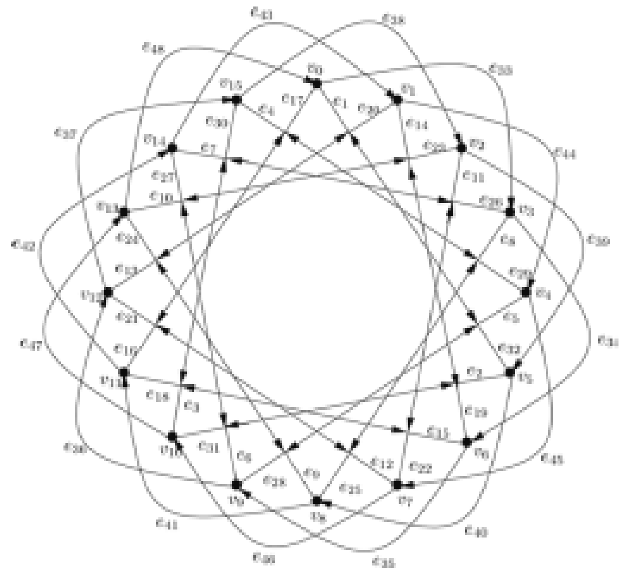

Figure 7: The Underlying Directed Graph for Bounded Conflict Free Petri Net of the Group $\{Z_{16}, 5, 3, 11\}$

Hamiltonian Circuit. Hence underlying directed graph of the constructed bounded conflict free Petri net of $(Z_{12}, 7, 5, 11)$ is **Hamiltonian** (Fig. 7). In the constructed bounded conflict free Petri net of $(Z_{16}, 5, 3, 11)$ let $\pi = \{\pi_1, \pi_2, \pi_3\}$ be the decomposition of the set of places of the constructed bounded conflict free Petri net where

$\pi_1 = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}\}$,

$\pi_2 = \{p_{17}, p_{18}, p_{19}, p_{20}, p_{21}, p_{22}, p_{23}, p_{24}, p_{25}, p_{26}, p_{27}, p_{28}, p_{29}, p_{30}, p_{31}, p_{32}\}$,

$\pi_3 = \{p_{33}, p_{34}, p_{35}, p_{36}, p_{37}, p_{38}, p_{39}, p_{40}, p_{41}, p_{42}, p_{43}, p_{44}, p_{45}, p_{46}, p_{47}, p_{48}\}$ such that

${}^\bullet\pi_1 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}\}$,

$\pi_1^\bullet = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}\}$,

${}^\bullet\pi_2 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}\}$,

$\pi_2^\bullet = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}\}$,

${}^\bullet\pi_3 = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}\}$,

$\pi_3^\bullet = \{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}\}$.

Therefore each π_i is both siphon and trap. Hence underlying directed graph of the constructed bounded conflict free Petri net of $(Z_{16}, 5, 3, 11)$ is Eulerian.

4. CONCLUSION

We constructed a bounded conflict free Petri net for a given group with a generating set. It is proved that the constructed bounded conflict free Petri net has a subsets of places which are both siphon and trap such that the input transitions equal the output transitions and both of them equal the set of all transitions of the constructed bounded conflict free Petri net. It is also proved that the there exists a partition for a place set such that each block of the partition of the set of places of bounded conflict free Petri net is both siphon and trap. As an application the above results we proved that the underlying directed graph of the constructed bounded conflict free Petri net is both Hamiltonian and Eulerian.

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