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On the Explicit Analytical Solution of a Slider bearing with Non-Newtonian Fluid

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Abstract: *The analytical study of an infinite, lubricated slider bearing consisting of connected surfaces with a second and third order (non-Newtonian) fluid as lubricated is considered. The homotopy analysis method (HAM) for strongly non linear problems is used to give explicit analytic solution of the problem. The velocity profile and pressure distribution for inclined slider bearing is calculated approximately. The variation of pressure and from that the load carrying capacity of the bearing is presented for a range of fluid and bearing parameters.*
Key words: *Homotopy analysis method, Slider bearing, non-Newtonian fluid, Lubrication theory.*

1. INTRODUCTION

The presence of fluid film greatly reduces the sliding friction between solid objects. The enormous practical importance of this effect has stimulated a great deal of research both theoretical and experimental. The problem of slider bearing with non-Newtonian lubricants is difficult to analyze mathematically because of the nonlinear character of the governing equations of motion. Numerical methods remain available, but are some what more costly. In this paper, we revisit the problem discussed by [1,13] and solved it approximately by homotopy analysis method introduced by Shijun Liao [2, 3]. The homotopy analysis is a powerful new analytic method that remains valid even with strong nonlinearity and with no small or large parameter. The method is successfully applied [4- 9] to discuss different problems of fluid flow. We see from our solution that homotopy analysis method is more general than the perturbation method. In 2002, Muhammet Yürüsoy [1] employed the perturbation method to study the problem by introducing a small parameter. We see from the solution and numerical plots that homotopy analysis is with good agreement with the perturbation method.

Second and third grade (Non-Newtonian) fluids are considered by many researchers [21-23] due to its practical importance and with the development of modern industrial materials. Some relevant studies on non-Newtonian lubrication in bearing have been published. Harnoy and Hanin [10] studied elstico-viscous lubricants in dynamically loaded bearing. Bourgin [11] applied the constitutive relation of second order fluid to study of non-Newtonian lubrication with perturbation approach. Rajagopal [12] carried out a study of the creeping flow. Kacou, Rajagopal & Szeri [13] studied the flow of second and third grade model in journal bearing. J.A. Tichy [14] studied the non-Newtonian lubrication with convected Maxwell model. Yürüsoy [15] has studied the pressure distribution in a slider bearing with Powell-Eyring model and constructed a perturbation solution. Yürüsoy & Pakdemirli [16] studied the flow in a slider bearing with a special third grade fluid. Buckholz [17] used a power law model as a non-Newtonian lubricant in a slider bearing. Agrawal [18] studied the magnetic fluid based porous inclined slider bearing. Bhat and Patel [15] used the magnetic fluid based secant shaped porous slider bearing. Ng. and Saibel [19] used a third grade fluid and studied the flow occurring in the slider bearing. Ng. and Saibel [20] used a third grade fluid and studied the flow occurring in the slider bearing.

2. ANALYSIS

Consider the two dimensional bearing (Fig. 1), in which the plane y = 0 moves with constant velocity U in the x-direction and the top of the bearing (the slider) is fixed. It is assumed that the fluid inertia is small, the side leakage is negligible, and the flow is incompressible and laminar.

The non-dimensional basic lubrication equations for second and third grade fluid flow in the film region [1, 13] are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

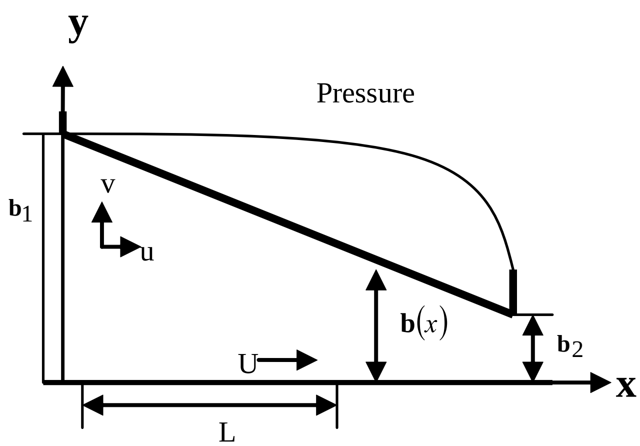

Figure 1: Two Dimensional Bearing

$$0=-\frac{\partial p}{\partial x}+(\lambda_1+2\lambda_2)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial^2 u}{\partial y^2}+\lambda_1\left\{v\frac{\partial^3 u}{\partial y^3}+u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right\}+6\lambda_3\left(\frac{\partial u}{xy}\right)^2\frac{\partial^2 u}{\partial y^2} \quad (2)$$

$$0=-\frac{\partial p}{\partial y}+(2\lambda_1+\lambda_2)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 \quad (3)$$

On defining the generalized pressure

$$p^*=p-(2\lambda_1+\lambda_2)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 \quad (4)$$

We rewrite the equations of motion as $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

$$\frac{\partial^2 u}{\partial y^2}+\lambda_1\left\{\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+u\frac{\partial^3 u}{\partial x\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right\}+6\lambda_3\left(\frac{\partial u}{xy}\right)^2\frac{\partial^2 u}{\partial y^2}=\frac{\partial p^*}{\partial x} \quad (5)$$

$$0=\frac{\partial p^*}{\partial y} \quad (6)$$

It follows from Eq. (5) and Eq. (6) that

$$p^*=p^*(x) \quad (7)$$

and thus the modified pressure does not vary across the film thickness.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{dp^*}{dx}=\frac{\partial^2 u}{\partial y^2}+\lambda_1\left\{\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+u\frac{\partial^3 u}{\partial x\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right\}+6\lambda_3\left(\frac{\partial u}{xy}\right)^2\frac{\partial^2 u}{\partial y^2} \quad (8)$$

The dimensionless boundary conditions of the problem are

$$u(0)=1,\quad u(b)=0,\quad v(0)=0,\quad v(b)=0 \quad (9)$$

where λ_1, λ_2 and λ_3 are material constants.

We note that the Eq. (1) serves only to determine the vanishing small velocity component v, given the dominant component u by Eq. (8).

In this paper we now employ the homotopy analysis method to solve the viscous flows of non-Newtonian second and third grade fluids in a slider bearing and propose analytic solution of Eq. (8) and Eq. (9).

3. HOMOTOPY ANALYSIS METHOD

3.1. Basic Idea

To explain the basic idea of homotopy analysis method, let us consider the differential equation

$$\aleph[u(y)]=0 \quad (10)$$

in which $\aleph$ is a nonlinear operator, and $u(y)$ is an unknown function of the independent variable. Let $u_\circ(y)$ denote an initial approximation $u(y)$ and ℓ denotes an auxiliary linear operator with the property

$$\ell u=0 \text{ When } u=0 \quad (11)$$

We then construct a family of equations, the so-called homotopy

$$\hat{H}[\phi(y;q);q]=(1-q)\ell[\phi(y;q)-u_\circ(y)]+q\aleph[\phi(y;q)] \quad (12)$$

where $q\in[0,1]$ is an embedding parameter and $\phi(y;q)$ is a function of y and q. When $q=0$ and $q=1$, we have

$$\hat{H}[\phi(y;q);q]_{q=0}=\ell[\phi(y;q)-u_\circ(y)]$$

and

$$\hat{H}[\phi(y;q);q]_{q=1}=\aleph\,\phi(y;1) \quad (13)$$

respectively.

From Eq. (11) it follows that

$$\phi(y;0)=u_\circ(y) \quad (14)$$

is the solution of the equation

$$\hat{H}[\phi(y;q);q]_{q=0}=0 \quad (15)$$

and

$$\phi(y;1)=u(y) \quad (16)$$

is therefore the solution of the equation

$$\hat{H}[\varphi(y;q);q]_{q=1} = 0 \tag{17}$$

Thus when the embedding parameter q increases from 0 to 1, the solution $\phi(y;q)$ of the equation

$$\hat{H}[\phi(y;q);q] = 0 \tag{18}$$

depends upon the embedding parameter q and varies from initial approximation $u_\circ(y)$ to the solution $u(y)$ of Eq. (10). In topology this kind of continuous variation is called deformation.

3.2. Velocity Profile

To find the velocity profile, we define the nonlinear operator $\aleph[\tilde{u}(y,q)]$ as

$$\aleph[\tilde{u}(y,q)] = -\frac{dp^*}{dx} + \frac{\partial^2 \tilde{u}(y,q)}{\partial y^2} + \lambda_1 \left\{ \frac{\partial \tilde{u}(y,q)}{\partial x}\frac{\partial^2 \tilde{u}(y,q)}{\partial y^2} + \tilde{v}(y,q)\frac{\partial^3 \tilde{u}(y,q)}{\partial y^3} + \tilde{u}(y,q)\frac{\partial^3 \tilde{u}(y,q)}{\partial x \partial y^2} - \frac{\partial \tilde{u}(y,q)}{\partial y}\frac{\partial^2 \tilde{u}(y,q)}{\partial x \partial y} \right\} + 6\lambda_3 \left(\frac{\partial \tilde{u}(y,q)}{xy} \right)^2 \frac{\partial^2 \tilde{u}(y,q)}{\partial y^2} \tag{19}$$

Further more we construct the Zeroth-order deformation equation

$$(1-q)\ell[\tilde{u}(y,q) - u_\circ(y)] = -q\aleph[\tilde{u}(y,q)] \tag{20}$$

subject to the boundary conditions

$$\begin{aligned} \tilde{u}(y,q) &= 1 \quad \text{at} \quad y = 0 \\ \tilde{u}(y,q) &= 0 \quad \text{at} \quad y = b \end{aligned} \tag{21}$$

where $u_\circ(y)$ is an initial guess approximation and q is an embedding parameter such that $q \in [0,1]$. We choose the auxiliary linear operator ℓ, (which is the linear part of the Eq. (8))

$$\ell = \frac{d^2}{dy^2} \tag{22}$$

and the initial guess approximation

$$u_\circ(y) = \frac{dp^*}{dx}\left(\frac{y^2}{2} - \frac{yb}{2}\right) + \left(1 - \frac{y}{b}\right) \tag{23}$$

which can be obtained by solving Eq. (8) with $\lambda_1 = \lambda_3 = 0$ subject to the boundary conditions (9). Obviously, when $q = 0$ and $q = 1$ we have

$$\tilde{u}(y,0) = u_\circ(y), \quad y > 0 \tag{24}$$

and

$$\tilde{u}(y,1) = u(y) \tag{25}$$

respectively.

Therefore, according to Eq. (24) and Eq. (25) the variation of q from 0 to 1 is just the continuous variation $\tilde{u}(y,q)$ from the initial guess approximation $u_\circ(y)$ to the unknown solution $u(y)$ of the original Eq. (8). This kind of continuous variation is called deformation in topology. Assume that the deformation $\tilde{u}(y,q)$ governed by Eqs. (19)-(25) is smooth enough so that

$$u_\circ^{(k)}(y) = \left.\frac{\partial^k \tilde{u}(y,q)}{\partial q^k}\right|_{q=0} \quad k \ge 1 \tag{26}$$

namely, the k-th order deformation derivative exists. Then, in view of equation (24) and Taylor's formula, we expand $\tilde{u}(y,q)$ in the power series

$$\tilde{u}(y,q) = u_\circ(y) + \sum_{k=1}^{\infty} \left[\frac{u_\circ^{(k)}(y)}{k!}\right] q^k \tag{27}$$

We note that the convergence region of the above infinite series is independent upon $h(\neq 0)$. We define

$$u_k(y) = \frac{u_\circ^{(k)}(y)}{k!}, \quad k \ge 1 \tag{28}$$

Using Eqs. (25), (27) and (28), we get at $q = 1$, the important relationship

$$u(y) = \sum_{k=0}^{\infty} u_k(y) \tag{29}$$

between the initial guess approximation $u_\circ(y)$ and the unknown solution $u(y)$. Now differentiating the Zeroth-order deformation Eqs. (19) and (20), k-times with respect to q and then setting $q = 0$ we obtain for $k \ge 1$ the k-th order deformation equation

$$\ell[u_k(y) - \chi_k u_{k-1}(y)] = -\Re_k(y) \tag{30}$$

with the boundary conditions

$u_k(0) = u_k(b) = 0$ (31) in which

$$\Re_k(y) = \frac{1}{(k-1)!}\frac{\partial^{k-1}\aleph[\tilde{u}(y,q)]}{\partial q^{k-1}} \tag{32}$$

and

$$\chi_k = \begin{cases} 0, & k \le 1 \\ 1, & k \ge 2 \end{cases} \tag{33}$$

where prime denotes derivatives with respect to y.

By putting $k = 1$ in Eqs. (30) - (32), we obtain first order solution.

In particular differentiating Eq. (19) with respect to q, we obtain

$$
\begin{aligned}
&(1-q)\ell\left[\frac{\partial\tilde{u}(y,q)}{\partial q}-0\right]-[\tilde{u}(y,q)-u_{\circ}(y)]=-\left[\frac{\partial^2\tilde{u}(y,q)}{\partial y^2}+\lambda_1\left\{\frac{\partial\tilde{u}(y,q)}{\partial x}\frac{\partial^2\tilde{u}(y,q)}{\partial y^2}\right.\right.\\
&\left.+\tilde{v}(y,q)\frac{\partial^3\tilde{u}(y,q)}{\partial y^3}+\tilde{u}(y,q)\frac{\partial^3\tilde{u}(y,q)}{\partial x\partial y^2}-\frac{\partial\tilde{u}(y,q)}{\partial y}\frac{\partial^2\tilde{u}(y,q)}{\partial y\partial x}\right\}+6\lambda_3\left(\frac{\partial\tilde{u}(y,q)}{\partial y}\right)^2\frac{\partial^2\tilde{u}(y,q)}{\partial y^2}\\
&\left.-\frac{dp^*}{dx}\right]-q\left[\frac{\partial^3\tilde{u}(y,q)}{\partial y^2\partial q}+\lambda_1\left\{\frac{\partial\tilde{u}(y,q)}{\partial x}\frac{\partial^3\tilde{u}(y,q)}{\partial y^2\partial q}+\frac{\partial^2\tilde{u}(y,q)}{\partial y\partial q}\frac{\partial^2\tilde{u}(y,q)}{\partial y^2}+\tilde{v}(y,q)\frac{\partial^4\tilde{u}(y,q)}{\partial y^3\partial q}\right.\right.\\
&+\frac{\partial\tilde{v}(y,q)}{\partial q}\frac{\partial^3\tilde{u}(y,q)}{\partial y^3}+\tilde{u}(y,q)\frac{\partial^4\tilde{u}(y,q)}{\partial x\partial y^2\partial q}+\frac{\partial\tilde{u}(y,q)}{\partial q}\frac{\partial^3\tilde{u}(y,q)}{\partial x\partial y^2}-\frac{\partial\tilde{u}(y,q)}{\partial y}\frac{\partial^3\tilde{u}(y,q)}{\partial y\partial x\partial q}\\
&\left.\left.-\frac{\partial^2\tilde{u}(y,q)}{\partial y\partial q}\frac{\partial^3\tilde{u}(y,q)}{\partial y\partial x\partial q}\right\}+6\lambda_3\left\{2\left(\frac{\partial\tilde{u}(y,q)}{\partial y}\right)\frac{\partial^2\tilde{u}(y,q)}{\partial y^2}\frac{\partial^2\tilde{u}(y,q)}{\partial y^2}+\left(\frac{\partial\tilde{u}(y,q)}{\partial y}\right)^2\frac{\partial^3\tilde{u}(y,q)}{\partial y^2\partial q}\right\}\right]
\end{aligned}
\tag{34}
$$

Making use of Eq. (26) and setting $q=0$, we have

$$
\ell\left\{u_{\circ}^{(1)}\right\}=-\left[\frac{\partial^2u_{\circ}}{\partial y^2}+\lambda_1\left\{\frac{\partial u_{\circ}}{\partial x}\frac{\partial^2u_{\circ}}{\partial y^2}+v_{\circ}\frac{\partial^3u_{\circ}}{\partial y^3}+u_{\circ}\frac{\partial^3u_{\circ}}{\partial x\partial y^2}-\frac{\partial u_{\circ}}{\partial y}\frac{\partial^2u_{\circ}}{\partial x\partial y}\right\}+6\lambda_3\left(\frac{\partial u_{\circ}}{\partial y}\right)^2\frac{\partial^2u_{\circ}}{\partial y^2}-\frac{dp^*}{dx}\right]
\tag{35}
$$

and making use of Eq. (22), we have

$$
\begin{aligned}
\frac{du_{\circ}^{(1)}}{dy^2}=-\lambda_1&\left[\frac{d}{dx}\left(\frac{dp^*}{dx}\left(\frac{y^2}{2}-\frac{yb}{2}\right)+\left(1-\frac{y}{b}\right)\left(\frac{dp^*}{dx}\right)+\left(\frac{dp^*}{dx}\left(\frac{y^2}{2}-\frac{yb}{2}\right)\right.\right.\right.\\
&\left.+\left(1-\frac{y}{b}\right)\right)\frac{d}{dx}\left(\frac{dp^*}{dx}\right)-\left(\frac{dp^*}{dx}\left(y-\frac{b}{2}\right)-\frac{1}{b}\right)\frac{d}{dx}\left(\frac{dp^*}{dx}\left(y-\frac{b}{2}\right)-\frac{1}{b}\right)\Bigg]\\
&-6\lambda_3\left[\left(\frac{dp}{dx}\right)^3\left(y^2+\frac{b^2}{4}-by\right)+\frac{1}{b^2}\left(\frac{dp}{dx}\right)-\left(\frac{2y}{b}-1\right)\left(\frac{dp^*}{dx}\right)^2\right]
\end{aligned}
\tag{36}
$$

Now integrating Eq. (36) twice with respect to y, and using the boundary conditions (31), we have

$$
\begin{aligned}
u_{\circ}^{1}=-\lambda_1&\left[\frac{d}{dx}\left(\frac{dp^*}{dx}\left(\frac{y^4}{24}-\frac{by^3}{12}+\frac{b^3y}{24}\right)+\left(\frac{y^2}{2}-\frac{y^3}{6b}-\frac{by}{3}\right)\left(\frac{dp^*}{dx}\right)+\left(\frac{dp^*}{dx}\left(\frac{y^4}{24}-\frac{by^3}{12}+\frac{b^3y}{24}\right)+\right.\right.\right.\\
&\left.\left(\frac{y^2}{2}-\frac{y^3}{6b}-\frac{by}{3}\right)\right)\frac{d}{dx}\left(\frac{dp^*}{dx}\right)-\left(\frac{dp^*}{dx}\left(\frac{y^3}{6}-\frac{by^2}{4}+\frac{b^2y}{12}\right)-\frac{y^2}{2b}+\frac{y}{2}\right)\frac{d}{dx}\left(\frac{dp^*}{dx}\left(\frac{y^3}{6}-\frac{by^2}{4}\right.\right.\\
&\left.\left.\left.+\frac{b^2y}{12}\right)-\frac{y^2}{2b}+\frac{y}{2}\right)\right]-6\lambda_3\left[\left(\frac{dp^*}{dx}\right)^3\left(\frac{y^4}{12}+\frac{b^2y^2}{8}-\frac{by^3}{6}-\frac{b^3y}{24}\right)+\left(\frac{dp^*}{dx}\right)\left(\frac{y^2}{2b^2}-\frac{y}{2b}\right)\right.\\
&\left.+\left(\frac{dp^*}{dx}\right)^2\left(\frac{y^2}{2}-\frac{y^3}{3b}-\frac{by}{6}\right)\right]
\end{aligned}
\tag{37}
$$

Summing up the result, we write

$$u = u_\circ + \frac{u_\circ^1}{1} + \ldots$$

$$u(y) = \frac{dp^*}{dx}\left(\frac{y^2}{2} - \frac{yb}{2}\right) + \left(1 - \frac{y}{b}\right) - \lambda_1\left[\frac{d}{dx}\left(\frac{dp^*}{dx}\left(\frac{y^4}{24} - \frac{by^3}{12} + \frac{b^3 y}{24}\right) + \left(\frac{y^2}{2} - \frac{y^3}{6b} - \frac{by}{3}\right)\left(\frac{dp^*}{dx}\right) + \right.\right.$$
$$\left(\frac{dp^*}{dx}\left(\frac{y^4}{24} - \frac{by^3}{12} + \frac{b^3 y}{24}\right) + \left(\frac{y^2}{2} - \frac{y^3}{6b} - \frac{by}{3}\right)\right)\frac{d}{dx}\left(\frac{dp^*}{dx}\right) - \left(\frac{dp^*}{dx}\left(\frac{y^3}{6} - \frac{by^2}{4} + \frac{b^2 y}{12}\right) - \frac{y^2}{2b} + \frac{y}{2}\right)$$
$$\left.\times \frac{d}{dx}\left(\frac{dp^*}{dx}\left(\frac{y^3}{6} - \frac{by^2}{4} + \frac{b^2 y}{12}\right) - \frac{y^2}{2b} + \frac{y}{2}\right)\right] - 6\lambda_3\left[\left(\frac{dp^*}{dx}\right)^3\left(\frac{y^4}{12} + \frac{b^2 y^2}{8} - \frac{by^3}{6} - \frac{b^3 y}{24}\right) + \right.$$
$$\left.\left(\frac{dp^*}{dx}\right)\left(\frac{y^2}{2b^2} - \frac{y}{2b}\right) + \left(\frac{dp^*}{dx}\right)^2\left(\frac{y^2}{2} - \frac{y^3}{3b} - \frac{by}{6}\right)\right] \tag{38}$$

Eq. (38) is the analytical solution of the problem by using HAM.

4. PRESSURE DISTRIBUTION

Using the continuity equation together with the derived velocity profile, one may find the ordinary differential equation for the pressure distribution. Integrating the continuity equation $v(0) = v(b) = 0$

$$\int_0^b \frac{\partial u}{\partial x} dy = -\int_0^b \frac{\partial v}{\partial x} dy = v(0) - v(b) = 0 \tag{39}$$

with

$$\int_0^b \frac{\partial u}{\partial x} dy = 0 \tag{40}$$

Substituting Eq. (38) into Eq. (40), we get

$$\frac{d}{dx}\left[-\frac{b^3}{12}\frac{dp^*}{dx} + \frac{b}{2} - \lambda_1\left\{\frac{d}{dx}\left(\frac{b^5}{120}\frac{dp^*}{dx} - \frac{b^3}{24}\right)\left(\frac{dp_\circ^*}{dx}\right)\right.\right.$$
$$\left. + \left(\frac{b^5}{120}\frac{dp^*}{dx} - \frac{b^3}{24}\right)\frac{d}{dx}\left(\frac{dp^*}{dx}\right) + \frac{b^2}{144}\frac{db^2}{dx}\right\}$$
$$\left. + 6\lambda_3\left\{\frac{b^5}{240}\left(\frac{dp^*}{dx}\right)^3 + \frac{b}{12}\left(\frac{dp^*}{dx}\right)\right\}\right] = 0 \tag{41}$$

An approximate solution will be searched for the above equation since it variable coefficient and highly nonlinear differential in p^*. The associated boundary conditions are

$$p^*(0) = p^*(1) = 0 \tag{42}$$

Integrating Eq. (41) w. r. t. x

$$\left[-\frac{b^3}{12}\frac{dp^*}{dx} + \frac{b}{2} - \lambda_1\left\{\frac{d}{dx}\left(\frac{b^5}{120}\frac{dp^*}{dx} - \frac{b^3}{24}\right)\left(\frac{dp_\circ^*}{dx}\right)\right.\right.$$
$$\left. + \left(\frac{b^5}{120}\frac{dp^*}{dx} - \frac{b^3}{24}\right)\frac{d}{dx}\left(\frac{dp^*}{dx}\right) + \frac{b^2}{144}\frac{db^2}{dx}\right\}$$
$$\left. + 6\lambda_3\left\{\frac{b^5}{240}\left(\frac{dp^*}{dx}\right)^3 + \frac{b}{12}\left(\frac{dp^*}{dx}\right)\right\}\right] = C \tag{43}$$

where C is constant of integration.
After simplification we write (43) as

$$\frac{dp^*}{dx} - \frac{6}{b^2} + \frac{12C}{b^3} + \lambda_1\left\{\frac{1}{b^3}\frac{d}{dx}\left(\frac{b^5}{10}\frac{dp^*}{dx} - \frac{b^3}{2}\right)\left(\frac{dp^*}{dx}\right)\right.$$
$$\left. + \left(\frac{b^2}{10}\frac{dp^*}{dx} - \frac{1}{2}\right)\frac{d}{dx}\left(\frac{dp^*}{dx}\right) + \frac{1}{6}\frac{db}{dx}\right\}$$
$$- 6\lambda_3\left\{\frac{b^2}{20}\left(\frac{dp^*}{dx}\right)^3 + \frac{1}{b^2}\left(\frac{dp^*}{dx}\right)\right\} = 0 \tag{44}$$

Again we wish to solve (44) for p^* by using HAM. We construct the Zeroth-order deformation equation as in Eq. (19):

$$(1-q)\ell\left[\tilde{p}^*(x,q) - p_\circ^*(x)\right] = -q\left[\frac{b^3}{12}\frac{d\tilde{p}^*(x,q)}{dx} - \frac{b}{2} + C + \right.$$
$$\lambda_1\left\{\frac{d}{dx}\left(\frac{b^5}{120}\frac{d\tilde{p}^*(x,q)}{dx} - \frac{b^3}{24}\right)\frac{d\tilde{p}^*(x,q)}{dx}\right.$$
$$\left. + \left(\frac{b^5}{120}\frac{d\tilde{p}^*(x,q)}{dx} - \frac{b^3}{24}\right)\frac{d}{dx}\left(\frac{d\tilde{p}^*(x,q)}{dx}\right) + \frac{b^2}{144}\frac{db^2}{144}\right\}$$
$$\left. - 6\lambda_3\left\{\frac{b^5}{240}\left(\frac{d\tilde{p}^*(x,q)}{dx}\right)^5 + \frac{b}{12}\left(\frac{d\tilde{p}^*(x,q)}{dx}\right)\right\}\right] \tag{45}$$

subject to the boundary conditions

$$\left.\begin{aligned}\tilde{p}^*(x,q)=0 \quad \text{at} \quad x=0\\ \tilde{p}^*(x,q)=0 \quad \text{at} \quad x=1\end{aligned}\right\} \tag{46}$$

taking the initial guess approximation as

$$p_\circ^*(x)=\frac{6x(b-r)}{b^2(1+r)}=\left(\frac{6x(1-x+rx-r)}{(1-x+rx)^2(1+r)}\right) \tag{47}$$

where

$$b(x)=(1-x+rx), \quad r=\frac{b_2}{b_1} \tag{48}$$

is the inclined slider in which b_1 is the maximum and b_2 is the minimum value of b. Defining the linear operator as

$$\wp=\frac{d}{dx} \tag{49}$$

and an embedding parameter q such that $q\in[0,1]$.
Setting $q=0$ and $q=1$ in Eq. (38) respectively, we get

$$\tilde{p}^*(x,0)=p_\circ^*(x) \qquad x>0 \tag{50}$$

and

$$\tilde{p}^*(x,1)=p^*(x) \tag{51}$$

Therefore, according to Eq. (50) and Eq. (51), the variation of q from 0 to 1 is just the continuous variation $\tilde{p}^*(x,q)$ from the initial guess approximation $p_\circ^*(x)$ to the unknown solution $p^*(x)$ of Eq. (44). Assume that the deformation $\tilde{p}^*(x,q)$ governed by Eq. (45) and Eq. (51) is smooth enough so that

$$p_\circ^{*(k)}(x)=\left.\frac{\partial^k\tilde{p}^*(x,q)}{\partial q^k}\right|_{q=0} \qquad k\geq 1 \tag{52}$$

Namely, the k-th order deformation derivative exists.

Then, according to Eq. (50) and Taylor's formula, we have

$$\tilde{p}^*(x,q)=p_\circ^*(x)+\sum_{k=1}^{\infty}\left[\frac{p_\circ^{*(k)}(x)}{k!}\right]q^k \tag{53}$$

Defining

$$p_k^*(x)=\frac{p_\circ^{*(k)}(x)}{k!} \tag{54}$$

Using Eqs. (51), (53) and (54), we get at $q=1$, the important relationship

$$p^*(x)=\sum_{k=0}^{\infty}p_k^*(x) \tag{55}$$

between the initial guess approximation $p_\circ^*(x)$ and the unknown solution $p^*(x)$. Setting $q=0$ in Eq. (45), we get $\tilde{p}^*(x,0)=p_\circ^*(x)$ (56) In particular, differentiating (45) w. r. t. q, making use of (52) and setting $q=0$, we have

$$\begin{aligned}\wp\left[p_\circ^{*(1)}\right]=-\Bigg[&\frac{dp_\circ^*}{dx}-\frac{6}{b^2}+\frac{12C}{b^3}+\\ &\lambda_1\left\{\frac{1}{b^3}\frac{d}{dx}\left(\frac{b^5}{10}\frac{dp_\circ^*}{dx}-\frac{b^3}{2}\right)\left(\frac{dp_\circ^*}{dx}\right)+\right.\\ &\left.\left(\frac{b^2}{10}\frac{dp_\circ^*}{dx}-\frac{1}{2}\right)\frac{d}{dx}\left(\frac{dp_\circ^*}{dx}\right)+\frac{1}{6}\frac{db}{dx}\right\}\\ &-6\lambda_3\left\{\frac{b^2}{20}\left(\frac{dp_\circ^*}{dx}\right)^3+\frac{1}{b^2}\left(\frac{dp_\circ^*}{dx}\right)\right\}\end{aligned} \tag{57}$$

Making use of Eq. (49), we get,

$$\begin{aligned}\frac{dp_\circ^{*(1)}}{dx}=\Bigg[&\frac{12r}{(r+1)(rx-x+1)^3}-\frac{12C}{(1-x+rx)^3}-\lambda_1\left\{\frac{3(r-1)\begin{pmatrix}2r-2x-2rx-23r^2+x^2+2r^2x\\+2r^3x-2r^2x^2+r^4x^2+1\end{pmatrix}}{5(r+1)^2(rx-x+1)^5}\right.\\ &\left.-\frac{6(r-1)(r^2x-x-2r+1)(r^2x-x-11r+1)}{5(r+1)^2(rx-x+1)^5}+\frac{(r-1)}{6}\right\}+6\lambda_3\left\{\frac{54(r-1)^3(x+rx-1)^3}{5(r+1)^3(rx-x+1)^7}\right.\\ &\left.+\frac{6(r-1)(x+rx-1)}{(r+1)(rx-x+1)^5}\right\}\Bigg]\end{aligned} \tag{58}$$

Integrating w. r. t. x, gives

$$
\begin{aligned}
p_{\circ}^{*(1)} = \Bigg[& \frac{6C}{(r-1)(rx-x+1)^2} + D - \frac{6r}{(r-1)(r+1)(rx-x+1)^2} \\
& -\lambda_1 \frac{\begin{pmatrix} 157rx - 23x - 162r + 261r^2 + 29x^2 - 30x^3 + 20x^4 - 5x^5 + 23r^2x + 30rx^3 - 157r^3x \\ -40rx^4 + 15rx^5 - 58r^2x^2 + 60r^2x^3 - 20r^2x^4 - 60r^3x^3 + 29r^4x^2 - 5r^2x^5 + 80r^3x^4 - 30r^4x^3 \\ -25r^3x^5 - 20r^4x^4 + 30r^5x^3 + 25r^4x^5 - 40r^5x^4 + 5r^5x^5 + 20r^6x^4 - 15r^6x^5 + 5r^7x^5 + 9 \end{pmatrix}}{30(r+1)^2(rx-x+1)^4} \\
& - \frac{24\lambda_3 \begin{pmatrix} 30rx - 105x - 15r + 27r^2 - 13r^3 + 105x^2 - 35x^3 - 15rx^2 + 78r^2x - 30r^3x \\ -210r^2x^2 + 105r^2x^3 + 30r^3x^2 + 105r^4x^2 - 105r^4x^3 - 15r^5x^2 + 35r^6x^3 + 35 \end{pmatrix}}{25(r-1)(r+1)^3(rx-x+1)^6} \Bigg]
\end{aligned}
\tag{59}
$$

where D is constant of integration.
Now using the boundary conditions (46), we get

$$
\begin{aligned}
p_{\circ}^{*(1)} = \Bigg[& -\frac{\begin{pmatrix} 1872\lambda_3 + 1305r\lambda_1 - 3888r\lambda_3 - 900r^2 - 1800r^3 - 900r^4 - 810r^2\lambda_1 - 25r^3\lambda_1 \\ 4032r^2\lambda_3 + 785r^4\lambda_1 - 3888r^3\lambda_3 - 1280r^5\lambda_1 + 1872r^4\lambda_3 + 25r^6\lambda_1 \end{pmatrix}}{150r(r-1)(r+1)^3(rx-x+1)^2} \\
& + \frac{\begin{pmatrix} 1872\lambda_3 + 1260r\lambda_1 + 1152r\lambda_3 - 1285r^3\lambda_1 \\ +1872r^2\lambda_3 - 25r^4\lambda_1 + 25r^5\lambda_1 + 25r^6\lambda_1 \end{pmatrix}}{300r^4 - 300r^2 - 150r + 150r^5} - \frac{6r}{(r-1)(r+1)(rx-r+1)^2} \\
& -\lambda_1 \frac{\begin{pmatrix} 157rx - 23x - 162r + 261r^2 + 29x^2 - 30x^3 + 20x^4 - 5x^5 + 23r^2x + 30rx^3 - 157r^3x - 40rx^4 \\ +15rx^5 - 58r^2x^2 + 60r^2x^3 - 20r^2x^4 - 60r^3x^3 + 29r^4x^2 - 5r^2x^5 + 80r^3x^4 - 30r^4x^3 - 25r^3x^5 \\ -20r^4x^4 + 30r^5x^{3^3} + 25r^4x^5 - 40r^5x^4 + 5r^5x^5 + 20r^6x^4 - 15r^6x^5 + 5r^7x^5 + 9 \end{pmatrix}}{30(r+1)^2(rx-x+1)^4} \\
& - \frac{24\lambda_3 \begin{pmatrix} 30rx - 105x - 15r + 27r^2 - 13r^3 + 105x^2 - 35x^3 - 15rx^2 + 78r^2x - 30r^3x \\ -210r^2x^2 + 105r^2x^3 + 30r^3x^2 + 105r^4x^2 - 105r^4x^3 - 15r^5x^2 + 35r^6x^3 + 35 \end{pmatrix}}{25(r-1)(r+1)^3(rx-x+1)^6} \Bigg]
\end{aligned}
\tag{60}
$$

Therefore, the final pressure distribution would then be

$$p^* = p_\circ^* + \frac{p_\circ^{*(1)}}{1!} + \cdots = p_\circ^* + p_1^* + \cdots = \frac{6x(1-x+rx-r)}{(1-x+rx)^2(1+r)} - \frac{6r}{(r-1)(r+1)(rx-r+1)^2}$$
$$- \frac{\begin{pmatrix} 1872\lambda_3 + 1305r\lambda_1 - 3888r\lambda_3 - 900r^2 - 1800r^3 - 900r^4 - 810r^2\lambda_1 - 25r^3\lambda_1 \\ 4032r^2\lambda_3 + 785r^4\lambda_1 - 3888r^3\lambda_3 - 1280r^5\lambda_1 + 1872r^4\lambda_3 + 25r^6\lambda_1 \end{pmatrix}}{150r(r-1)(r+1)^3(rx-x+1)^2}$$
$$+ \frac{\left(1872\lambda_3 + 1260r\lambda_1 + 1152r\lambda_3 - 1285r^3\lambda_1 + 1872r^2\lambda_3 - 25r^4\lambda_1 + 25r^5\lambda_1 + 25r^6\lambda_1\right)}{300r^4 - 300r^2 - 150r + 150r^5}$$
$$- \lambda_1 \frac{\begin{pmatrix} 157rx - 23x - 162r + 261r^2 + 29x^2 - 30x^3 + 20x^4 - 5x^5 + 23r^2x + 30rx^3 - 157r^3x - 40rx^4 \\ + 15rx^5 - 58r^2x^2 + 60r^2x^3 - 20r^2x^4 - 60r^3x^3 + 29r^4x^2 - 5r^2x^5 + 80r^3x^4 - 30r^4x^3 - 25r^3x^5 \\ - 20r^4x^4 + 30r^5x^{33} + 25r^4x^5 - 40r^5x^4 + 5r^5x^5 + 20r^6x^4 - 15r^6x^5 + 5r^7x^5 + 9 \end{pmatrix}}{30(r+1)^2(rx-x+1)^4} \tag{61}$$
$$- \frac{24\lambda_3 \begin{pmatrix} 30rx - 105x - 15r + 27r^2 - 13r^3 + 105x^2 - 35x^3 - 15rx^2 + 78r^2x - 30r^3x \\ - 210r^2x^2 + 105r^2x^3 + 30r^3x^2 + 105r^4x^2 - 105r^4x^3 - 15r^5x^2 + 35r^6x^3 + 35 \end{pmatrix}}{25(r-1)(r+1)^3(rx-x+1)^6}$$

5. NUMERICAL PLOTS

In the next section, the pressure distribution in the bearing is determined for various values of the parameter λ_1, λ_3 and clearance ratio r.

Fig. 2 indicates the variation of the pressure with respect to x when r is fixed, $\lambda_3 = 0$ and λ_1 is varied. It is seen that the pressure increases with increasing λ_1, which mean higher load capacity for the bearing.

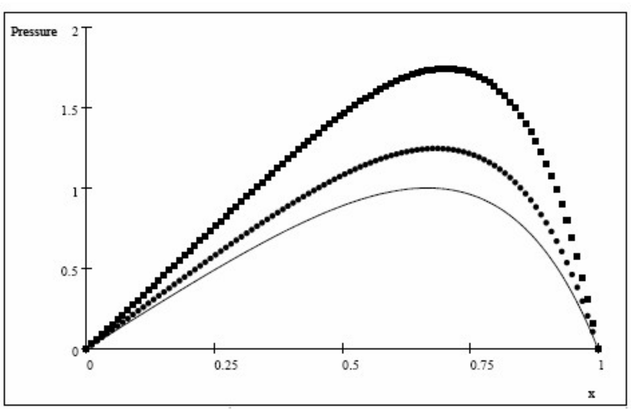

Figure 2: $r = 0.5, \lambda_3 = 0(-\lambda_1 = 0, \cdots, \lambda_1 = 0.1, \cdots, \lambda_1 = 0.2, \cdots, \lambda_1 = 0.3)$

Fig. 3 indicates the variation of the pressure with respect to x when r is fixed, $\lambda_1 = 0$ and λ_3 is varied. It is seen again that the pressure increases with increasing λ_3, which mean higher load capacity for the bearing.

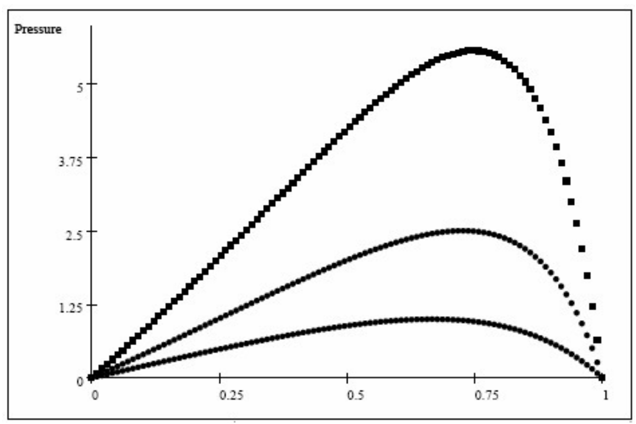

Figure 3: $r = 0.5, \lambda_1 = 0(-\lambda_3 = 0, \cdots, \lambda_3 = 0.1, \cdots, \lambda_3 = 0.2, \cdots, \lambda_3 = 0.3)$

In Fig. 4, for $\lambda_1 = \lambda_3 = 0.1$, the dimensionless length versus dimensionless pressure is plotted for different clearance ratios r. It is seen that pressure build up for lower clearance ratios.

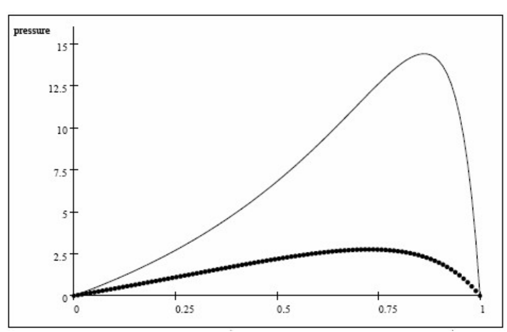

Figure 4: $\lambda_1 = \lambda_3 = 0(-r = 0.3, \cdots, r = 0.5, \cdots, r = 0.7)$

Therefore we conclude that the maximum load carrying capacity of the bearing depends on parameter λ_1, λ_3 of the lubricant and clearance ratio *r*. The present analysis suggests that the load capacity of a bearing lubricated with second and third grade fluid can be obtained after giving an appropriate design to the bearings.

6. CONCLUSION

In this paper the homotopy analysis method is successfully applied to give an explicit analytical solution of the slider bearing with non-Newtonian lubricants. The velocity profile and pressure distribution in the inclined shaped slider bearing are calculated using homotopy analysis method. In this study we do not need the so called small parameter assumption at all, which is necessary in the perturbation method. That is the homotopy analysis method is independent of any small or large quantities. The findings of the present study provide useful information for engineers in designing and application of bearing systems. Thus from the above discussion we conclude that the analytical method used in this paper is to be useful for the analysis of lubrication theory and also for solving nonlinear problems with strong nonlinearity and with no small or large parameter. The success of the homotopy analysis method for considered problems verifies once again that it is indeed in a useful analytic tool for non-linear problems in science and engineering, although further improvements are necessary.

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