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MULTIPLE POSITIVE SOLUTIONS FOR A CLASS OF ONHOMOGENEOUS ELLIPTIC EQUATIONS ON PERIODIC DOMAINS

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ABSTRACT: In this article, we consider the following problem

$$-\Delta u + u = f(x, u) + h(x) \text{ in } \Omega,\ u > \text{ in } \Omega,\ u \in H_0^1(\Omega), \tag{*}$$

where $0 \leq f(x, u) \leq a_0 u + b_0 u^{p-1}$ for all $x \in \Omega$, $u \geq 0$ with $a_0 \in [0, 1)$, $b_0 > 0$, $2 < p < (2N/(N-2))$, if $N \geq 3$, $2 < p < \infty$ if $N = 2$ and Ω is a smooth periodic domain in $\mathbb{R}^N$. We prove that (*) has at least two positive solutions if

$$\|h\|_{H^{-1}(\Omega)} < C_p S^{p/2(p-2)}$$

and $h \geq 0$, $h \not\equiv 0$ in Ω, where S is the best Sobolev constants in Ω and

$$C_p = b_0^{-1/(p-2)} (p-2)(p-1)^{-(p-1)/(p-2)} (1 - a_0)^{(p-1)/(p-2)}.$$

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1. INTRODUCTION

In this paper, we shall study the multiplicity of solutions of semilinear elliptic equations

$$\begin{cases} -\Delta u + u = f(x,u) + h(x) \text{ in } \Omega, \\ u \in H_0^1(\Omega),\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth domain in $\mathbb{R}^N$, $h \in H^{-1}(\Omega)$, $N \geq 2$ and $2 < p < 2^*$, $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = \infty$ for $N = 2$, and the nonlinear function $f(x, t)$ satisfies the following assumptions:

(f1) $f(x, t) \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ and $\lim_{t \to 0} \frac{\partial f}{\partial t}(x, t) = \frac{\partial f}{\partial t}(x, 0)$ uniformly in $x \in \Omega$;

(f2) there exist $a_0 \in [0, 1)$ and $b_0 > 0$ such that

$$0 < f(x, t) \le a_0 t + b_0 t^{p-1} \text{ for } x \in \Omega,\ t > 0,$$

where $2 < p < 2^*$;

($f3$) there exists a constant $\theta > 2$ such that

$$0 < \theta F(x, t) \le f(x, t)t, \text{ for all } x \in \Omega,\ t \ge 0$$

where $F(x,t) = \int_0^t f(x,s)ds$;

($f4$) thre exists $\overline{f} : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|x| \to \infty} f(x,t) = \overline{f}(t)$ uniformly for bounded $t > 0$, $f(x,t) \ge \overline{f}(t)$, for all $x \in \Omega$, $t \ge 0$, and $\lim_{t \to \infty} \frac{f(x,t)}{t} = \infty$ uniformly in $x \in \Omega$;

($f5$) $f(x, \cdot) \in C^2(0, +\infty)$ and $\frac{\partial^2}{\partial t^2} f(x,t) \ge 0$ for all $x \in \Omega$, $t \ge 0$;

($f5$)* $f(x, t)/t$ is strictly increasing in t uniformly in $x \in \Omega$ in the following sense:

$$\inf_{t \in [r_1, r_2],\, x \in \Omega} \frac{\partial}{\partial t}\left(\frac{f(x,t)}{t}\right) > 0 \text{ for all } 0 < r_1 < r_2.$$

Let

$$S = \inf\{\textstyle\int_\Omega (|\nabla u|^2 + u^2)dx : u \in H_0^1(\Omega), \int_\Omega |u|^p dx = 1\},$$
$$C_p = b_0^{-1/(p-2)}(p-2)(p-1)^{-(p-1)/(p-2)}(1-a_0)^{(p-1)/(p-2)}.$$

We also assume that:

($h1$) $\|h\|_{H^{-1}(\Omega)} \le C_p S^{p/2(p-2)}$.

($h2$) $\|h\|_{H^{-1}(\Omega)} < C_p S^{p/2(p-2)}$.

If Ω is bounded, see Bahri-Berestycki [2], Bahri-Lions [3], Tanaka [16] and the references therein for similar problems. If Ω is the whole space $\mathbb{R}^N$, Adachi-Tanaka [1], Cao-Zhou [6], Hirano [10], Jeanjean [12] and Zhu [18] have showed the existence of at least two positive solutions of (1.1) under some suitable conditions. For Ω is an infinite strip, Hsu [11] has studied the multiplicity of positive solutions of similar problems. The main aim of this paper is to study (1.1) on the general periodic domains (see Definition 2.1). In this paper, we used the fact that the concentration compactness principle holds on periodic domains and some ideas of Adachi-Tanaka [1] and Cao-Zhou [6], to prove our main results.

Now, we state main results in the following:

Theorem 1.1: Let Ω be a smooth domain in $\mathbb{R}^N$. If $(f1)$ $(f2)$, $(h1)$ hold, $h(x) \geq 0$ and $h(x) \not\equiv 0$ in Ω, then problem (1.1) has at least one positive solution.

Theorem 1.2: Let Ω be a periodic domain in $\mathbb{R}^N$ and $h(x) \geq 0$, $h(x) \not\equiv 0$, $h(x) \in L^2(\Omega) \cap L^q(\Omega)$ ($q < N/2$ if $N \geq 4$, $q = 2$ if $N = 2, 3$) hold. If $(f1) - (f5)$ and $(h1)^*$ hold, then problem (1.1) has at least two solutions, one of which is a local minimizer of $I(u)$.

Theorem 1.3: Let Ω be a periodic domain in $\mathbb{R}^N$. If $(f1) - (f4)$ and $(f5)^*$ hold, then there exists a constant $M > 0$ such that if $h(x) \geq 0$, $h(x) \not\equiv 0$ and $\|h\|_{H^{-1}(\Omega)} \leq M$, then problem (1.1) has at least two solutions, one of which is a local minimizer of $I(u)$.

2. PRELIMINARIES

In this section, we shall give some notations and some known results. Let $H_0^1(\Omega)$ be the Sobolev space of the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|$, where

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 + u^2\right) dx\right)^{1/2}.$$

Throughout this paper, we denote <, > the usual scalar product in $H_0^1(\Omega)$, the universal constants by C and C_i ($i = 1, 2, \ldots$) unless some special statement is given, and set $\|u\|_p = \left(\int_\Omega |u|^p dx\right)^{1/p}$ for $1 \leq p < \infty$ and $u \in L^p(\Omega)$, $\|u\|_\infty = \sup_{x\in\Omega} |u(x)|$ for $u \in L^\infty(\Omega)$, and denote $\frac{\partial}{\partial t} f(x,t)$ and $\frac{\partial^2}{\partial t^2} f(x,t)$ by $f'(x, t)$ and $f''(x, t)$, respectively, in what follows.

Definition 2.1: Let Ω be a domain, there is a partition $\{\Omega_n\}$ of Ω and points $\{y_n\}_{n=1}^\infty$ in $\mathbb{R}^N$ satisfying the following conditions:

(i) $\{y_n\}_{n=1}^\infty$ forms a sub-group of R^N;

(ii) Ω_0 is bounded;

(iii) $\Omega_n = y_n + \Omega_0$.

Then Ω is called a periodic domain.

Now, we give some typical examples of periodic domains:

Example 2.2: $\Omega = \mathbb{R}^N$ is the whole space.

Example 2.3: $\Omega = O \times \mathbb{R}^n$ is an infinite strip in $\mathbb{R}^N$, where O is a bounded domain in $\mathbb{R}^m$, $m + n + N$, $m \geq 1$ and $n \leq 1$.

Example 2.4: $\Omega = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^n : r_1 < |y| < r_2\}$ is an infinite hole strip, where $m + n = N$, $m \geq 1$, $n \geq 1$ and $r_2 > r_1 > 0$.

We seek solutions of (1.1) as critical points of the functional I associated with (1.1) and given by

$$I(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right)dx - \int_\Omega F(x, u^+)dx - \int_\Omega hudx.$$

where $F(x,u) = \int_0^u f(x,t)dt$ for all $x \in \Omega$ and $u \in \mathbb{R}$.

Since we look for only positive solutions, we may assume without loss of generality that

$$f(x, t) = 0 \text{ for all } x \in \Omega \text{ and } t \leq 0.$$

Then $I(u)$ belongs to $C^1(H_0^1(\Omega), \mathbb{R})$ under assumptions $(f1) - (f2)$. Moreover, we have

Lemma 2.5: Assume $(f1)$ and $(f2)$, $h \geq 0$ and suppose that $u \in H_0^1(\Omega)$ is a critical point of $I(u)$. Then $u(x)$ is a nonnegative solution of (1.1). Moreover, if $u \not\equiv 0$ or $h \not\equiv 0$, then u is a positive solution of (1.1).

Proof: Suppose that $I'(u) = 0$, then for all $\psi \in H_0^1(\Omega)$, $< I'(u), \psi >= 0$. Thus u is a weak solution of

$$-\Delta u + u = f(x, u^+) + h(x) \text{ in } \Omega,$$

and by $(f2)$ and $h \geq 0$, $f(x, u^+) + h(x)$ is nonnegative. Then by the maximum principle we have that u is nonnegative. If $u \not\equiv 0$ or $h \not\equiv 0$, we can see that $f(x, u^+) + h(x) \geq 0$ and $f(x, u^+) + h(x) \not\equiv 0$, then u is a positive solution of (1.1).

Let us now introduce the equation at infinity associated with equation (1.1) in a periodic domain Ω.

$$\begin{cases} -\Delta u + u = \overline{f}(u) \text{ in } \Omega, \\ u \in H_0^1(\Omega), u > 0 \text{ in } \Omega, \end{cases} \tag{2.1}$$

and its associated energy functional I^∞ defined by

$$I^\infty(u) = \int_\Omega \left[\frac{1}{2}\left(|\nabla u|^2 + |u|^2 \right) - \lambda \overline{F}\left(u^+\right) \right] dx, u \in H_0^1(\Omega),$$

where $\overline{F}(u) = \int_0^u \overline{f}(t)dt$.

If (f1) – (f3) hold, using results of Chen-Tzeng [7, Remark 5], we know that (2.1) has a ground state $\overline{w}(x) > 0$ in Ω such that

$$S^\infty = I^\infty\left(\overline{w}\right) = \sup_{t\geq 0} I^\infty\left(t\overline{w}\right). \tag{2.2}$$

Let us recall that a sequence $\{u_n\} \subset H_0^1(\Omega)$ is called a $(PS)_c$-sequence for $I(u)$ if $I'(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. We say also I satisfies $(PS)_c$-condition if any $(PS)_c$-sequence possesses a strongly convergent sequence in $H_0^1(\Omega)$. We need the following concentration compactness principle which provides a precise description of a behavior of $(PS)_c$-sequence for I.

Lemma 2.6: Assume (f1) – (f4) hold. Let $\{u_n\}$ be a $(PS)_c$-sequence of I. Then there exists a subsequence (still denoted by $\{u_n\}$) for which the following holds: there exist an integer $l \geq 0$, sequence $\{x_n^i\} \subseteq \Omega$, a solution $\overline{u}$ of (1.1) and solutions u^i, $1 \leq i \leq l$ of (2.1), such that, for some subsequence $\{u_n\}$, as $n \to \infty$, we have

$$u_n \rightharpoonup \overline{u} \text{ weakly in } H_0^1(\Omega),$$

$$I\left(u_n\right) \to I\left(\overline{u}\right) + \sum_{i=1}^{l} I^\infty(u^i),$$

$$u_n - \left(\overline{u} + \sum_{i=1}^{l} u^i\left(x - x_k^i\right) \right) \to 0 \text{ strongly in } H_0^1(\Omega)$$

$$\left|x_n^i\right| \to \infty, \left|x_n^i - x_n^j\right| \to \infty \text{ for } 1 \leq i \neq j \leq l,$$

where we agree that in the case $l = 0$ the above holds without u^i, x_n^i.

Proof: Lemma 2.6 can be derived directly from the arguments in Bahri-Lions [4] (or [13], [14], [15], [17]). Thus we omit the proof here.

We also quote an asymptotic behavior of the solution of (1.1) at infinity in Hsu [11].

Lemma 2.7: Let Ω be a $C^{1,1}$ unbounded domain in $\mathbb{R}^N$. If (f1), (f2), (h1) hold and $u \in H_0^1(\Omega)$ is a weak solution of (1.1) in Ω, then $u \in L^\infty(\Omega)$ and $\lim_{|x|\to\infty} u(x) = 0$.

Proof: See Hsu [11] for the proof.

3. THE PROOFS OF THE MAIN RESULTS

In this section, we give the proofs of our main results. Repeating the same arguments explored by Cao-Zhou [6], we can deduce Theorem 1.1.

We define

$$J(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + (1-a_0)u^2\right)dx - \frac{b_0}{p}\int_\Omega |u|^p\, dx - \int_\Omega hudx,$$

$$\Lambda = \{u \in H_0^1(\Omega) : \langle J'(u), u\rangle = 0\},$$

$$\Lambda^+ = \left\{u \in \Lambda : \|u\|^2 - a_0\|u\|_2^2 - b_0(p-1)\|u\|_p^p > 0\right\}.$$

First, we quote the following theorem for the special case $f(x, u) = a_0 u + b_0 u^{p-1}$ in Cao-Zhou [6].

Theorem 3.1: Let Ω be a smooth domain in $\mathbb{R}^N$ and h satisfy (h1). Then

$$c_0 = \inf_{\Lambda} J = \inf_{\Lambda^+} J$$

is achieved at a point $w_0 \in \Lambda^+$ which is a critical point for J. Moreover, if $h(x) \geq 0$ and $h(x) \not\equiv 0$, then w_0 is a positive solution of the following equation.

$$\begin{cases} -\Delta u + u = a_0 u + b_0 u^{p-1} + h(x) \text{ in } \Omega, \\ u \in H_0^1(\Omega), u > 0 \text{ in } \Omega. \end{cases} \tag{3.1}$$

Proof: Modifying the proof Cao-Zhou [6, Theorem 2.1]. Here we omit it.

Proof of Theorem 1.1: It follows from Theorem 3.1 that under conditions of Theorem 1.1, (3.1) has a positive solution w_0. By (f2), w_0 is a weak supersolution of

(1.1). On the other hand, 0 is a subsolution of (1.1). Thus, by the standard barrier method, (1.1) has a solution u_0 such that $0 \le u_0 \le w_0$ (see [9]). Since $h(x) \ge 0$ and $h(x) \not\equiv 0$, by the maximum principle, we have that $u_0 > 0$. Thus we complete the proof of Theorem 1.1.

Next, we look for another positive solution of (1.1) by mountain pass type argument. We introduce the following auxiliary problem:

$$\begin{cases} -\Delta u + u = f(x,u) + \lambda h(x) \text{ in } \Omega, \\ u \in H_0^1(\Omega), u > 0 \text{ in } \Omega, \lambda > 0. \end{cases} \tag{3.2$_\lambda$}$$

If $(h1)^*$ holds, then there exists $\lambda_0 > 1$ such that

$$\|\lambda_0 h\|_{H^{-1}(\Omega)} = \lambda_0 \|h\|_{H^{-1}(\Omega)} < C_p S^p / 2(p-2),$$

namely, $(h1)^*$ still holds for $h_{\lambda_0} = \lambda_0 h$. Therefore, it follows from Theorem 1.1 that $(3.2)_{\lambda_0}$ has a positive solution u_{λ_0} which is a supersolution of (1.1). As in the proof of Theorem 1.1, we can find a minimal solution u_0 of on the interval $[0, u_{\lambda_0}]$, that is $0 < u_0 < u_{\lambda_0}$ (see [9] for the definition of minimal solution). Now, we shall prove that u_0 is also a local minimizer of $I(u)$. Similarly as in Cao-Zhou [6], we have

Lemma 3.2: Assume $(f1)$ – $(f5)$ hold, the minimization problem

$$\inf\left\{\int_\Omega \left(|\nabla v|^2 + (1 - f'(x,0))v^2\right)dx : v \in H_0^1(\Omega), \int_\Omega (f'(x,u_0) - f'(x,0))v^2 dx = 1\right\}.$$

can be achieved by some $u_0 > 0$ Furthermore, $\mu > 1$.

Proof: By $(f1)$ and $(f2)$, we have that $f'(x, 0) \le a_0 \in [0, 1)$,

$$\int_\Omega \left(|\nabla v|^2 + (1 - f'(x,0))v^2\right)dx \ge (1 - a_0)\|v\|^2.$$

Indeed, by the definition of μ we know that $0 < \mu < +\infty$. Let $\{v_n\} \subset H_0^1(\Omega)$ be a minimizing sequence of μ that is

$$\int_\Omega (f'(x,u_0) - f'(x,0))v_n^2 dx = 1 \text{ and } \int_\Omega \left(|\nabla v_n|^2 + (1 - f'(x,0))v_n^2\right)dx \to \mu \text{ as } n \to \infty.$$

This implies that $\{v_n\}$ is bounded in $H_0^1(\Omega)$, then there is a subsequence, still denoted by $\{v_n\}$ and some $v_0 \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v_0 \text{ weakly in } H_0^1(\Omega),$$

$$v_n \to v_0 \text{ almost everywhere in } \Omega,$$

$$v_n \to v_0 \text{ strongly in } L^s_{loc}(\Omega) \text{ for } 2 \le s < 2^*.$$

Thus,

$$\int_\Omega \left(|\nabla v_0|^2 + (1 - f'(x,0))v_0^2 \right) dx \le \liminf_{n\to\infty} \int_\Omega \left(|\nabla v_n|^2 + (1 - f'(x,0))v_n^2 \right) dx = \mu.$$

To prove that v_0 achieves, μ it suffices to show that

$$\int_\Omega \left(f'(x,u_0) - f'(x,0) \right) v_0^2 dx = 1.$$

By Lemma 2.7 and ($f1$), we have $f'(x, u_0) \to f'(x, 0)$ as $|x| \to \infty$, it follows that there exists a constant $C > 0$ such that

$$|f'(x, u_0) - f'(x, 0)| \le C \text{ for all } x \in \Omega.$$

Furthermore, for any $\varepsilon > 0$, there exists $R > 0$ such that for $x \in \Omega$ and $|x| \ge R$, $|f'(x, u_0) - f'(x, 0)| < \varepsilon$. Let $B_R = \{x \in \mathbb{R}^N : |x| < R\}$, then we have

$$\begin{aligned}
&\left| \int_\Omega (f'(x,u_0) - f'(x,0)) |v_n - v_0|^2 dx \right| \\
&\le \int_{B_R\cap\Omega} |f'(x,u_0) - f'(x,0)| |v_n - v_0|^2 dx + \int_{\Omega\setminus B_R} |f'(x,u_0) - f'(x,0)| |v_n - v_0|^2 dx \\
&\le C\int_{B_R\cap\Omega} |v_n - v_0|^2 dx + \varepsilon \int_{\Omega\setminus B_R} |v_n - v_0|^2 dx.
\end{aligned}$$

It follows from the Sobolev embedding theorem that there exists n_1, such that for $n \ge n_1$,

$$\int_{B_R\cap\Omega} |v_n - v_0|^2 dx < \varepsilon.$$

Since $\{v_n\}$ is bounded in $H_0^1(\Omega)$, this implies that there exists a constant $C_1 > 0$ such that

$$\int_{\Omega\setminus B_R} |v_n - v_0|^2 dx \le C_1.$$

Therefore, we conclude that for $n \ge n_1$,

$$\left| \int_\Omega f'(x,u_0) |v_n - v_0|^2 dx \right| \le C\varepsilon + C_1\varepsilon.$$

Take $\varepsilon \to 0$, we obtain that

$$\int_\Omega \left(f'(x,u_0)-f'(x,0)\right)v_0^2dx=1.$$

Hence

$$\int_\Omega \left(\left|\nabla v_0\right|^2+\left(1-f'(x,0)\right)v_0^2\right)dx \ge \mu.$$

This implies that v_0 achieves μ. Clearly, $|v_0|$ also achieves μ. Hence we may assume that $v_0 \ge 0$ in Ω and v_0 satisfies

$$-\Delta v_0 + (1-f'(x,0))v_0 = \mu(f'(x,u_0)-f'(x,0))v_0. \tag{3.3}$$

By the maximum principle for weak solutions (see Gilbarg-Trudinger [8, Theorem 8.19]), $(f1)$ and $(f5)^*$, we deduce that $v_0 > 0$ in Ω.

We shall now prove that $\mu > 1$. By the definition of u_{λ_0} and u_0, we obtain

$$\begin{aligned} -\Delta(u_{\lambda_0}-u_0)+(u_{\lambda_0}-u_0) &= f(x,u_{\lambda_0})-f(x,u_0)+(\lambda_0-1)h(x) \\ &> f'(x,u_0)(u_{\lambda_0}-u_0)+(\lambda_0-1)h(x) \qquad (3.4) \\ &\quad (\text{by } f''(x,u)\ge 0 \text{ for } x\in\Omega) \end{aligned}$$

Multiplying (3.4) by v_0 and integrating it on Ω, we get

$$\begin{aligned} &\int_\Omega \nabla\left(u_{\lambda_0}-u_0\right)\nabla v_0dx+\int_\Omega\left(u_{\lambda_0}-u_0\right)v_0dx \\ \ge\ &\int_\Omega f'(x,u_0)\left(u_{\lambda_0}-u_0\right)v_0dx+\int_\Omega(\lambda_0-1)h(x)v_0dx \\ >\ &\int_\Omega f'(x,u_0)\left(u_{\lambda_0}-u_0\right)v_0dx. \end{aligned} \tag{3.5}$$

By (3.3), we have

$$\begin{aligned} &\int_\Omega \nabla\left(u_{\lambda_0}-u_0\right)\nabla v_0dx+\int_\Omega\left(u_{\lambda_0}-u_0\right)v_0dx \\ &=\mu\int_\Omega\left(f'(x,u_0)-f'(x,0)\right)\left(u_{\lambda_0}-u_0\right)v_0dx+\int_\Omega f'(x,0)\left(u_{\lambda_0}-u_0\right)v_0dx. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6) we deduce that

$$\mu\int_\Omega\left(f'(x,u_0)-f'(x,0)\right)\left(u_{\lambda_0}-u_0\right)v_0dx > \int_\Omega\left(f'(x,u_0)-f'(x,0)\right)\left(u_{\lambda_0}-u_0\right)v_0dx,$$

which implies that $\mu > 1$.

By the fact that $\mu > 1$, we have

$$\int_\Omega\left(\left|\nabla v\right|^2+\left(1-f'(x,0)\right)v^2\right)dx \ge \mu\int_\Omega\left(f'(x,u_0)-f'(x,0)\right)v^2dx \tag{3.7}$$

for all $v \in H_0^1(\Omega)$.

Lemma 3.3: If $(f1)-(f5)$ and $(h1)^*$ hold, then u_0 is a local minimizer of I, that is, there exists an $\varepsilon_0 > 0$ such that

$$I(u_0 + v) > I(u_0) \text{ for all } v \in H_0^1(\Omega),\ \|v\| \le \varepsilon_0. \tag{3.8}$$

In particular, we can find a suitable $\kappa > 0$ such that

$$I(u_0 + v) > I(u_0) + \kappa \text{ for } \|v\| = \varepsilon_0. \tag{3.9}$$

Proof: By (3.7), for every $v \in H_0^1(\Omega)$, we have

$$\begin{aligned}
I(u_0+v) &= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|v\|^2 + \int_\Omega \left(\nabla u_0 \nabla v + u_0 v\right) dx \\
&\quad - \int_\Omega F\left(x, u_0 + v\right) dx - \int_\Omega h u_0 dx - \int_\Omega hv dx \\
&= I\left(u_0\right) + \frac{1}{2}\int_\Omega \left(\left|\nabla v\right|^2 + \left(1 - f'\left(x,0\right)\right)v^2\right) dx \\
&\quad - \int_\Omega \left(F\left(x,u_0+v\right) - F\left(x,u_0\right) - f\left(x,u_0\right)v - \frac{1}{2}f'\left(x,0\right)v^2\right) dx \\
&= I\left(u_0\right) + \frac{1}{2}\left(1-\frac{1}{\mu}\right)\int_\Omega \left(\left|\nabla v\right|^2 + \left(1 - f'\left(x,0\right)\right)v^2\right) dx \\
&\quad + \frac{1}{2\mu}\int_\Omega \left(\left|\nabla v\right|^2 + \left(1 - f'\left(x,0\right)\right)v^2\right) dx \\
&\quad - \int_\Omega \left(F\left(x,u_0+v\right) - F\left(x,u_0\right) - f\left(x,u_0\right)v - \frac{1}{2}f'\left(x,0\right)v^2\right) dx \\
&\ge I\left(u_0\right) + \frac{\mu-1}{2\mu}\left(1-a_0\right)\|v\|^2 \\
&\quad - \int_\Omega \left(F(x,u_0+v) - F(x,u_0) - f(x,u_0)v - \frac{1}{2}f'(x,u_0)v^2\right) dx \\
&= I\left(u_0\right) + \frac{\mu-1}{2\mu}\left(1-a_0\right)\|v\|^2
\end{aligned}$$

$$-\int_\Omega \int_0^v \left(f(x,u_0+s)-f(x,u_0)-f'(x,u_0)s\right) dsdx$$

From Lemma 2.7, we deduce that $u_0 \in L^\infty(\Omega)$ and $\lim_{|x|\to\infty} u_0(x) = 0$. Thus, by ($f$1) and ($f$2), we obtain

$$\lim_{s\to 0} \frac{f(x,u_0+s)-f(x,u_0)-f'(x,u_0)s}{s} = 0,$$

and

$$0 \le \limsup_{s\to\infty} \frac{f(x,u_0+s)-f(x,u_0)-f'(x,u_0)s}{s^{p-1}} \le b_0.$$

This implies that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$f(x,u_0+s)-f(x,u_0)-f'(x,u_0)s \le \varepsilon s + C_\varepsilon s^{p-1}, \text{ for } x \in \Omega,\ s \ge 0. \qquad (3.10)$$

Therefore, by (3.10) and the Sobolev inequality, we have

$$\begin{aligned} &\int_\Omega \int_0^v \left(f(x,u_0+s)-f(x,u_0)-f'(x,u_0)s\right) dsdx \\ \le\ &\int_\Omega \int_0^{|v|} \left(f(x,u_0+s)-f(x,u_0)-f'(x,u_0)s\right) dsdx \\ \le\ &\frac{\varepsilon}{2}\|v\|^2 + C_\varepsilon \|v\|^p. \end{aligned}$$

Thus, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$I(u_0+v) \ge I(u_0) + \left(\frac{\mu-1}{2\mu} - \frac{\varepsilon}{2}\right)\|v\|^2 - C_\varepsilon \|v\|^p.$$

Taking $\varepsilon = \varepsilon_0 > 0$ small enough, we have

$$I(u_0+v) \ge I(u_0) + \frac{\mu-1}{4\mu}\|v\|^2 \text{ for } \|v\| \le \varepsilon_0,$$

for which we deduce (3.8) and (3.9) for suitable $\kappa > 0$.

For $\rho > 0$, we denote $B_\rho = \{u \in H_0^1(\Omega) : \|u\| < \rho\}$.

Lemma 3.4: Assume that (f1) – (f4) hold. Then there exists a constant $M > 0$ such that if $h(x)\ \Omega\ 0$, $h(x) \not\equiv 0$, $\|h\|_{H^{-1}(\Omega)} \le M$, then we have

(i) there exists a constant $\rho_0 > 0$ such that

$$I(u) \ge 0 \text{ for all u} \in \partial B_{\rho_0},$$

(ii) there exists $\bar{u}_0 \in B_{\rho_0}$ is a local minimizer of I, that is

$$I(\bar{u}_0) = \inf_{u \in B_{\rho_0}} I(u) < 0.$$

Moreover, $\bar{u}_0$ is a positive solution of (1.1).

Proof: See Adachi-Tanaka [1, Lemma 2.1, Propostion 2.2].

Let $\bar{w}$ be a ground state solution of (2.1), then we have $S^\infty = I^\infty(\bar{w})$ = $\sup_{t\ge 0} I^\infty(t\bar{w})$.

Lemma 3.5: If (f1) – (f5) hold, then

(i) there exists $t_0 > 0$ such that $I(u_0 + t\bar{w}) < I(u_0)$ for $t \ge t_0$;

(ii) $\sup_{t\ge 0} I(u_0 + t\bar{w}) < I(u_0) + S^\infty$.

Proof: (i) By (f3) and (f5), for all $x \in \Omega$, $t_1, t_2 \ge 0$,

$$\begin{cases} f(x, t_1 + t_2) \ge f(x, t_1) + f(x, t_2), \\ f(x, t_1 + t_2) \not\equiv f(x, t_1) + f(x, t_2). \end{cases} \tag{3.11}$$

By (3.11) and (f4), we have

$$\begin{aligned} I(u_0 + t\bar{w}) &= \frac{1}{2}\int_\Omega \left(|\nabla u_0|^2 + u_0^2\right)dx + \frac{t^2}{2}\int_\Omega \left(|\nabla \bar{w}|^2 + \bar{w}^2\right)dx + t\int_\Omega \left(\nabla_{u_0} \nabla \bar{w} + u_0\bar{w}\right)dx \\ &\quad - \int_\omega F(x, u_0 + t\bar{w})dx - \int_\Omega hu_0 dx - t\int_\Omega h\bar{w}dx \\ &= I(u_0) + I^\infty(t\bar{w}) - \int_\Omega \left(F(x, u_0 + t\bar{w}) - F(x, u_0) - \bar{F}(t\bar{w}) - tf(x, u_0)\bar{w}\right)dx \end{aligned}$$

$$= I(u_0) + I^\infty(t\overline{w}) - \int_\Omega \int_0^{t\overline{w}} \left(f(x, u_0 + s) - f(x, u_0) - f(x, s) \right) ds dx$$

$$\leq I(u_0) + I^\infty(t\overline{w}). \tag{3.12}$$

By (f4), so $I^\infty(t\overline{w}) \to -\infty$ as $t \to \infty$. Hence (i) holds.

(ii) From (i), we know that

$$\sup_{t \geq 0} I(u_0 + t\overline{w}) = \sup_{t \leq t_0} I(u_0 + t\overline{w}).$$

By the continuity of $I(u_0 + t\overline{w})$ as a function of $t \geq 0$ and $I(0) = u_0$, we can find some $t_1 \in (0, t_0)$ such that

$$\sup_{0 \leq t \leq t_1} I(u_0 + t\overline{w}) < I(u_0) + S^\infty.$$

Thus, we only need to show that

$$\sup_{t_1 \leq t \leq t_0} I(u_0 + t\overline{w}) < I(u_0) + S^\infty.$$

To this end, by (3.11) and (3.12), we have

$$\sup_{t_1 \leq t \leq t_0} I(u_0 + t\overline{w}) \leq I(u_0) + S^\infty - \inf_{t_1 \leq t \leq t_0} \int_\Omega \int_0^{t\overline{w}} \left(f(x, u_0 + s) \right)$$

$$-f(x, u_0) - f(x, s) ds dx < I(u_0) + S^\infty$$

Therefore (ii) holds.

Remark 3.6: We replace u_0 and (*f*5) by $\overline{u}_0$ and (*f*5)*, respectively. From (*f*5)*, we can easily deduce that (3.11) holds. Therefore, repeating the same argument in Lemma 3.5, we have

(i) there exists $t_0 > 0$ such that $I(\overline{u}_0 + t\overline{w}) < I(\overline{u}_0)$ for $t \geq t_0$;

(ii) $\sup_{t \geq 0} I(\overline{u}_0 + t\overline{w}) < I(\overline{u}_0) + S^\infty$.

Proof of Theorem 1.2: We shall use the Mountain Pass Lemma without the (PS) condition in Brezis-Nirenberg [5] to obtain the existence of the second positive solution. For this purpose, fixing t_0 large enough such that (i) in Lemma 3.5 holds $\|t_0\overline{w}\| > \varepsilon_0$ for ε_0 chosen in Lemma 3.3.

Let u_0 be the minimum solution. Set

$$\Gamma = \{p \in C([0,1], H_0^1(\Omega)) : p(0) = u_0,\ p(1) = u_0 + t_0\overline{w}\},$$

$$c = \inf_{p\in\Gamma} \max_{s\in[0,1]} I(p(s)).$$

By Lemma 3.3 and Lemma 3.5, we have

$$\kappa + I(u_0) < c < I(u_0) + S^{\infty}. \tag{3.13}$$

Applying the Mountain Pass Lemma of Brezis-Nirenberg [5], there exists a $(PS)_c$-sequence $\{u_n\}$ such that

$$I(u_n) \to c,$$
$$I'(u_n) \to 0 \text{ strongly in } H^{-1}(\Omega).$$

Thus, by Lemma 2.6, there exist a subsequence (still denoted by $\{u_n\}$), an integer $l \geq 0$, sequence $\{x_n^i\}$ in Ω, $1 \leq i \leq l$, a solution $\overline{u}$ of (1.1) solutions u^i of (2.1) such that

$$c = I\left(\overline{u}\right) + \sum_{i=0}^{l} I^{\infty}\left(u^i\right).$$

We shall show that $\overline{u}$ is a solution different from u_0. In fact, by (3.13), we have

$$c = I(\overline{u}) \geq I(u_0) + \kappa > I\left(u_0\right) \text{ if } l = 0;\ I\left(u_0\right) + S^{\infty} > c \geq I\left(\overline{u}\right) + S^{\infty} \text{ if } l \geq 1.$$

This implies that $\overline{u} \not\equiv u_0$. Applying the maximum principle again, we have $\overline{u} > 0$ in Ω. Hence we have completed the proof of Theorem 1.2.

Proof of Theorem 1.3: By Lemma 3.4, we have already shown the existence of one positive solution $\overline{u}_0$ as a minimizer of $I(u)$ in B_{ρ_0}. Here, we assume that

$$I\left(\overline{u}_0\right) = \inf\{I(u) : u \in H_0^1(\Omega), I'(u) = 0\}. \tag{3.14}$$

If not, clearly $I(u)$ has a critical point different from $\overline{u}_0$ and (1.1) has at least two positive solutions. Under (3.14), by Lemma 2.6, $(PS)_c$-condition holds for $c \in \left(-\infty, I\left(\bar{u}_0\right) + S^{\infty}\right)$.

Set

$$\Gamma = \left\{ p \in C\left(\left[0,1\right], H_0^1\left(\Omega\right)\right) : p\left(0\right) = \bar{u}_0,\, p\left(1\right) = \bar{u}_0 + t_0 \bar{w} \right\},$$

$$c = \inf_{p \in \Gamma} \max_{s \in [0,1]} I\left(p\left(s\right)\right).$$

where t_0 is defined as in Remark 3.6(i). By Lemma 2.6, Lemma 3.4, Remark 3.6 and applying the Mountain Pass Lemma of Brezis-Nirenberg [5], we can use the same argument in the proof of Theorem 1.2 to obtain the conclusion of Theorem 1.3.

REFERENCES

[1] S. Adachi and K. Tanaka, Existence of Positive Solutions for a Class of Nonhomogeneous Elliptic Equations in $\mathbb{R}^N$, Nonlinear Anal. **48** (2002), 685-705.

[2] A. Bahri and H. Berestycki, A Perturbation Method in Critical Point Theory and Application, *Trans. Amer. Math. Soc.*, **297** (1987), 1-32.

[3] A. Bahri and P.L. Lions, Morse Index of Some Min-max Critical Points, I, Application to Multiplicity Results, *Comm. Pure Appl. Math.*, **41** (1988), 1027-1037.

[4] A. Bahri and P.L. Lions, On the existence of a positive solution of semilinear elliptic equations in unbounded domains, Ann. I. H. P. Analyse non lineaire, 14 (1997), 365–413.

[5] H. Brezis and L. Nirenberg, Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents, *Comm. Pure Appl. Math.*, **36** (1983), 437-477.

[6] D.M. Cao and H.S. Zhou, Multiple Positive Solutions of Nonhomogeneous Semilinear Elliptic Equations in , *Proc. Royal Soc. Edinb., Ser.*, **126A** (1996), 443-463.

[7] C.N. Chen and S.Y. Tzeng, Some Properties of Palais-Smale Sequences with Applications to Elliptic Boundary-value Problem, *Electron J. Diff. Eqns.*, **1999(17)** (1999), 1-29.

[8] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, New York, 1983.

[9] J. Graham-Eagle, Monotone Method for Semilinear Elliptic Equations in Unbounded Domains, *J. Math. Anal. Appl.*, **137** (1989), 122-131.

[10] N. Hirano, Existence of Entire Positive Solutions for Nonhomogeneous Elliptic Equations, *Nonlinear Anal.*, **29** (1997), 889-901.

[11] T.S. Hsu, Eigenvalue Problems and Bifurcation of Nonhomogeneous Semilinear Elliptic Equations on Unbounded Cylinder Domains, Dyna. of Cont. Disc. and Impu. Syst. Ser. A: Math. Anal., **14** (2007), 99-122.

[12] L. Jeanjean, Two Positive Solutions for a Class of Nonhomogeneous Elliptic Equations, *Diff. Integral Eqns.* **10** (1997), 609-624.

[13] W.C. Lien, S.Y. Tzeng and H.C. Wang, Existence of Solutions of Semilinear Elliptic Problems on Unbounded Domains, *Diff. and Int. Equ.*, **6** (1993), 1281-1298.

[14] P.L. Lions, The Concentration-compactness Principle in the Calculus of Variations, The Locally Compact Case, Ann. Inst. H. Poincaré, Analyse nonlinéaire **1** (1984), 109-145 and 223–283.

[15] P.L. Lions, On Positive Solutions of Semilinear Elliptic Equations in Unbounded Domains, In Nonlinear Diffusion Equations and Their Equilibrium States, eds. W.M. Ni, L.A. Peletier and J. Serrin (New York/ Berlin: Springer, 1988).

[16] K. Tanaka, Morse Indices at Critical Points Related to the Symmetric Mountain Pass Theorem and Applications, *Comm. Partial Diff. Eqns.*, **14** (1989), 99-128.

[17] H.C. Wang, Palais-Smale Approaches to Semilinear Elliptic Equations in Unbounded Domains, *Electron. J. Diff. Eqns. Monograph*, **6** (2004), (142 pages).

[18] X.P. Zhu, A Perturbation Result on Positive Entire Solutions of a Semilinear Elliptic Equation, *J. Diff. Eqns.*, **92** (1991), 163-178.

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