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STABILITY OF GENERALIZED ADDITIVE EQUATIONS ON BANACH SPACES AND GROUPS

Valeriĭ Faĭziev & John M. Rassias

ABSTRACT: In 1941 D.H. Hyers solved the well-known Ulam stability problem for linear mappings. In 1951 D.G. Bourgin was the second author to treat the Ulam problem for general additive mappings. In 1982-2004 J.M. Rassias established the Hyers-Ulam stability for the Ulam problem of linear and nonlinear mappings. In 1983 F. Skof was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 2005 V.A. Faiziev and P.K. Sahoo established on groups the stability of a Jensen type functional equation introduced by J.M. Rassias and M.J. Rassias in 2003. In this paper we investigate the Ulam stability of generalized functional additive mappings on Banach Spaces and groups. In this paper we consider also the problem of stability for functional equations of the form $f(x^n y^n) = nf(x) + nf(y)$, $n \in \mathbb{N}$ on groups.

Keywords and Phrases: Additive mapping, Ulam stability, Banach Space, group, stability of functional equation, wreath product of groups.

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1. INTRODUCTION

In 1940,1960 and 1964 S. M. Ulam ([26]) proposed the Ulam stability problem:

"When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In 1941 D. H. Hyers ([16]) solved this problem for linear mappings. In 1951 D. G. Bourgin ([2]) was the second author to treat the Ulam problem for general additive mappings. In 1978, according to P.M. Gruber ([15]), this kind of stability problems is of particular interest in probability theory and in the case of functional equations of different types. In 1980, I. Fenyö ([13]) established the stability of the Ulam problem for other mappings. Other interesting stability results have been achieved also by the following authors: P. W. Cholewa ([3]), St. Czerwik ([4]), and H. Drljevic ([5]). In 1982-2004 J. M. Rassias ([18]-[22], [24]) and in 2003 J.M. Rassias and M. J. Rassias ([23]) established the Ulam stability for different mappings. In 1999 P.Gavruta ([14]) answered a question of J.M.Rassias ([20]) concerning the stability of the Cauchy equation. See also papers of C. Badea [1] and S. M. Jung [17].

In 2005 V. A. Faiziev and P. K. Sahoo ([6]) established on groups the stability of a Jensen type functional equation introduced by J. M. Rassias and M. J. Rassias in 2003. In this paper we investigate the Ulam stability of generalized functional additive mappings Banach Spaces and groups.

Throughout this paper, let X be a real normed space and Y be a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations.

Definition 1.1: A mapping $A : X \to Y$ is called *additive* if A satisfies the functional equation

$$A(x_1 + x_2) = A(x_1) + A(x_2), \tag{1.1}$$

for all $x_1, x_2 \in X$.

Definition 1.2: A mapping $A : X \to Y$ is called *generalized additive* if A satisfies the functional equation

$$A(a_1x_1 + a_2x_2) = a_1A(x_1) + a_2A(x_2), \tag{1.2}$$

for a fixed pair of real numbers $a_1, a_2 \in \mathbb{R}$ and all $x_1, x_2 \in X$.

Throughout this paper we denote $m = a_1 + a_2$.

Definition 1.3: A mapping $f : X \to Y$ is called *approximately odd* if f satisfies the functional inequality

$$\|f(x) + f(-x)\| \leq \theta, \tag{1.3}$$

for some fixed $\theta \geq 0$ and all $x \in X$.

Theorem 1.4: (Hyers [16]). If a mapping $f : X \to Y$ satisfies the approximately additive inequality

$$\|f(x_1 + x_2) - f(x_1) - f(x_2)\| \leq \delta,$$

for some fixed $\delta > 0$ and all $x_1, x_2 \in X$, then there exists a unique additive mapping $A : X \to Y$, satisfying the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x),$$

equation (1.1) and inequality

$$\|f(x) - A(x)\| \leq \delta$$

for some fixed $\delta > 0$ and all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

2. STABILITY OF GENERALIZED ADDITIVE EQUATION (1.2)

Employing the above Theorem 1.4 (Hyers' direct method [16]), we prove the following Theorem 2.1 on Banach Spaces.

Theorem 2.1: If a mapping $f : X \to Y$ satisfies the approximately generalized additive inequality

$$\|f(a_1x_1 + a_2x_2) - a_1f(x_1) - a_2f(x_2)\| \leq \delta, \tag{2.1}$$

for some fixed $\delta > 0$, and for a fixed pair of real numbers $a_1, a_2 \in \mathbb{R}$ such that $m = a_1 + a_2 \neq \pm 1$ and $a_i \neq 0$; $i = 1, 2$ then inequalities

$$\left\|f(0)\right\| \leq \frac{1}{|1-m|}\delta, \tag{2.2}$$

$$\left\|f(-x)+f(x)\right\| \leq \frac{2}{\left|1-|m|\right|}\delta, \tag{2.3}$$

$$\left\|f(x)-a_1 f\left(a_1^{-1}x\right)\right\| \leq \delta+|a_2|\left\|f(0)\right\| \leq \frac{|1-m|+|a_2|}{\left|1-|m|\right|}\delta, \tag{2.4}$$

hold for all $x \in X$, and there exists a unique generalized additive mapping $A : X \to Y$, satisfying formula

$$A(x) = \begin{cases} \lim_{n\to\infty} m^{-n} f\left(m^n x\right) & \text{if } |m| > 1 \\ \lim_{n\to\infty} m^{n} f\left(m^{-n} x\right) & \text{if } |m| < 1 \end{cases}, \tag{2.5}$$

equation (1.2) and inequality

$$\left\|f(x)-A(x)\right\| \leq \frac{\delta}{\left|1-|m|\right|} \tag{2.6}$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof: Substituting $x_1 = x_2 = 0$ in (2.1), one gets inequality (2.2). We assume $|m| > 1$. If we replace $x_1 = x_2 = x$ in (2.1), we find inequality

$$\|f(mx) - mf(x)\| \leq \delta, \tag{2.7}$$

holds for all $x \in X$, and $|m| > 1$. Thus from this inequality, inequality (1.3) and triangle inequality, we get

$$|m|\,\|f(-x)+f(x)\| \le \|-[f(mx)-mf(x)]\| + \|-[f(-mx)-mf(-x)]\| + \|f(-mx)+f(mx)]\| \le \delta+\delta+\theta = 2\delta+\theta,$$

or

$$\|f(-x)+f(x)\| \le \frac{2}{|m|}\delta + \frac{1}{|m|} = \theta, \text{ for } m \neq 0$$

or

$$\|f(-x)+f(x)\| \le \theta = \frac{2}{|m|-1}\delta, \text{ for } |m| > 1.$$

We assume $|m| < 1$. If we replace $x_1 = x_2 = \dfrac{x}{m}$ in (2.1), we obtain inequality

$$|f(x) - mf(m^{-1}x)\| \le \delta, \tag{2.8}$$

holds for all $x \in X$, and $|m| < 1$. Denote $\mu = m^{-1} \neq 0$, $|\mu| > 1$. Similarly, one gets

$$|\mu|\,\|f(-x)+f(x)\| \le \|-[f(\mu x)-\mu f(x)]\| + \|-[f(-\mu x)-\mu f(-x)]\| + \|f(-\mu x)+f(\mu x)\| \le \delta|\mu| + \delta|\mu| + \theta = 2\delta|\mu| + \theta,$$

or

$$\|f(-x)+f(x)\| \le \frac{2|\mu|}{|\mu|} + \frac{1}{|\mu|}\theta = \theta, \ \mu \neq 0,$$

or

$$\|f(-x)+f(x)\| \le \theta = \frac{2|\mu|}{|\mu|-1}\delta = \frac{2}{1-|m|}\delta, \text{ for } |m| < 1.$$

Therefore the proof of the above inequality (2.3) is complete.

Assume $a_1 \neq 0$. Substituting $x_1 = a_1^{-1}x$, $x_2 = 0$ in (2.1) and employing the triangle inequality and inequality (2.2), one finds inequality (2.4).

Assuming $|m| > 1$, we get from inequality (2.7) and the triangle inequality the basic inequality

$$\left\| f\left(x\right) - m^{-1} f\left(mx\right) \right\| \leq \frac{1}{|m|}\delta \text{ for } m \neq 0, \tag{2.9}$$

and thus the general inequality

$$\begin{aligned} \| f(x) - m^{-n} f(m^n x)\| &\leq \| f(x) - m^{-1} f(mx)\| + m^{-1}\| f(mx) - m^{-1} f(m^2 x)\| \\ &\quad + \dots + m^{-(n-1)}\| f(m^{n-1}x) - m^{-1} f(m^n x)\| \\ &\leq \frac{1}{|m|}\delta\left(1 + \frac{1}{|m|} + \dots + \frac{1}{|m|^{n-1}}\right) \\ &= \frac{1 - |m|^{-n}}{|m| - 1}\delta, \end{aligned} \tag{2.10}$$

for all $x \in X$, and $|m| > 1$.

Similarly, assuming $|m| < 1$, we find from inequality (2.8) and the triangle inequality the other general inequality

$$\begin{aligned} \| f(x) - m^n f(m^{-n} x)\| &\leq \| f(x) - mf(m^{-1}x)\| + m\| f(m^{-1}x) - mf(m^{-2}x)\| \\ &\quad + \dots + |m|^{n-1}\| f(m^{-(n-1)}x) - mf(m^{-n}x)\| \\ &\leq \delta(1 + |m| + \dots + |m|^{n-1}) \\ &= \frac{1 - |m|^n}{1 - |m|}\delta, \end{aligned} \tag{2.11}$$

for all $x \in X$, and $|m| < 1$.

Let us denote,

$$f_n\left(x\right) = \begin{cases} m^{-n} f\left(m^n x\right) & \text{if } |m| > 1 \\ m^n f\left(m^{-n} x\right) & \text{if } |m| < 1 \end{cases},$$

for all $x \in X$.

Therefore from inequalities (2.10)-(2.11), we obtain the general inequality

$$\begin{cases} \left\| f(x) - f_n(x) \right\| \le \dfrac{1}{\left|1-|m|\right|}\delta\left(1-|m|^{-n}\right) & \text{if } |m|>1 \\ \left\| f(x) - f_n(x) \right\| \le \dfrac{1}{\left|1-|m|\right|}\delta\left(1-|m|^{n}\right) & \text{if } |m|<1 \end{cases}, \tag{2.12}$$

for all $x \in X$.

Following the proof of our theorems ([18]-[24]), we easily prove that sequence $\{f_n(x)\}$ is a Cauchy sequence and thus convergent, because Y is a complete space. In fact, assuming $|m| > 1$, and $k \ge n > 0$ and setting $h = m^n x$, one gets from (2.10), that

$$\begin{aligned} \| f_k(x) - f_n(x)\| &= \|m^{-k} f(m^k x) - m^{-n} f(m^n x)\| \\ &= |m|^{-n} \|f(h) - m^{-(k-n)} f(m^{k-n} h)\| \\ &\le |m|^{-n} \frac{1-|m|^{-(k-n)}}{|m|-1}\delta \\ &= \frac{|m|^{-n} - |m|^{-k}}{|m|-1}\delta \\ &\le \frac{|m|^{-n}}{|m|-1}\delta \to 0, \text{ as } n \to \infty, \end{aligned} \tag{2.13}$$

holds for all $x \in X$. In addition, assuming $|m| < 1$, and $k \ge n > 0$ and setting $\ell = m^{-n}x$, one gets from (2.11) that

$$\begin{aligned} \|f_k(x) - f_n(x)\| &= \|m^k f(m^{-k}x) - m^n f(m^{-n}x)\| \\ &= |m|^n \|f(\ell) - m^{k-n} f(m^{-(k-n)}\ell)\| \\ &\le |m|^n \frac{1-|m|^{k-n}}{1-|m|}\delta \\ &= \frac{|m|^n - |m|^k}{1-|m|}\delta \end{aligned} \tag{2.14}$$

$$\leq \frac{|m|^n}{1-|m|}\delta \to 0, \text{ as } n \to \infty,$$

holds for all $x \in X$.

Therefore formula (2.5) exists. Thus from inequalities (2.1) and (2.12), one proves that there exists a generalized additive mapping $A : X \to Y$, satisfying inequality (2.6).

Let us denote,

$$A_n(x) = \begin{cases} m^{-n} A(m^n x) & \text{if } |m| > 1 \\ m^n A(m^{-n} x) & \text{if } |m| < 1 \end{cases},$$

for all $x \in X$.

Following the proof of (2.12), we easily prove that

$$A_n^{(x)} = A(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$.

This formula is important for the proof of *the uniqueness* of a generalized additive mapping $A : X \to Y$.

The rest of the proof of this theorem is omitted as analogous to the proof of our theorems ([18]-[24]). Similarly, we prove the last assertion of this theorem 2.1 that if, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$, then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$, completing the proof of this theorem.

Note that a similar stability result for the generalized additive mapping is also considered in [1], and [17].

3. AUXILIARY RESULTS

Note if in the equation (1.1) we take $a_1 = n$, $a_2 = m \in \mathbb{N}$, and $x_1 = x$, $x_2 = y \in X$ then this equation can be rewritten as

$$f(nx + my) = nf(x) + mf(y)$$

and can be considered on any abelian group G. This m is different from the above employed $m = a_1 + a_2$ on Banach Spaces. If we rewrite the latter equation in the form

$$f(x^n y^m) = nf(x) + mf(y),$$

then it can be considered on any group G. In the following part of the paper we consider a particular case of the last equation. Namely, we will consider the equation

$$f(x^n y^n) = nf(x) + nf(y).$$

Suppose that G is an arbitrary group and E is an arbitrary real Banach space.

Definition 3.1: We will say that a mapping $f : G \to E$ is a *n-additive mapping* if for any $x, y \in G$ we have

$$f(x^n y^n) - nf(x) - nf(y) = 0. \tag{3.1}$$

We denote the set of all n-additive mappings from G to E by $Ad^n(G; E)$.

If $n = 1$, then 1-additive mapping is simply additive one. In what follows we will assume that $n \geq 2$.

Definition 3.2: We will say that a mapping $f : G \to E$ is an *n-quasiadditive mapping* if there is $c > 0$ such that for any $x, y \in G$ we have

$$\| f(x^n y^n) - nf(x) - nf(y)\| \leq c. \tag{3.2}$$

It is clear that the set of n–quasiadditive mappings is a linear real space. Denote it by $KA^n(G; E)$. From (3.2) we obtain

$$\|f(x^n) - nf(x) - nf(1)\| \leq c,$$

therefore

$$\|f(x^n) - nf(x)\| \leq d = c + n\|f(1)\|, \tag{3.3}$$

From (3.3) it follows that

$$\|f((x^n)^n) - nf(x^n)\| \leq d,$$

or

$$\|f(x^{n^2}) - nf(x^n)\| \leq d,$$

Now using (3.3) we get

$$\|f(x^{n^2}) - n^2 f(x)\| \leq d + dn.$$

It follows that

$$\|f((x^n)^{n^2}) - n^2 f(x^n)\| \leq d + dn.$$

Again using (3.3) we obtain

$$\|f(x^{n^3}) - n^3 f(x)\| \leq d + dn + dn^2.$$

Continuing this way we get

$$\|f(x^{n^k}) - n^k f(x)\| \leq d(1 + n + n^2 + \ldots + n^{k-1}) \leq dn^k.$$

It follows that

$$\left\| \frac{1}{n^k} f\left(x^{n^k}\right) - f\left(x\right) \right\| \leq d\left(\frac{1}{n^k} + \frac{n}{n^k} + \frac{n^2}{n^k} + \ldots + \frac{n^{k-1}}{n^k}\right) \leq d. \tag{3.4}$$

Now for any $m \in \mathbb{N}$ we have

$$\left\| \frac{1}{n^k} f\left(x^{n^{k+m}}\right) - f\left(x^{n^m}\right) \right\| \leq d. \tag{3.5}$$

It follows that

$$\left\| \frac{1}{n^{k+m}} f\left(x^{n^{k+m}}\right) - \frac{1}{n^m} f\left(x^{n^m}\right) \right\| \leq \frac{1}{n^m} d. \tag{3.6}$$

From the latter it follows that the sequence

$$\left\{ \frac{1}{n^k} f\left(x^{n^k}\right) \middle| k \in \mathbb{N} \right\}$$

is a Cauchy sequence. Since the space E is complete, the above sequence has a limit and we denote it by $\varphi_m(x)$. Thus

$$\varphi_n\left(x\right) = \lim_{k\to\infty} \frac{1}{n^k} f\left(x^{n^k}\right). \tag{3.7}$$

From (3.4) it follows that

$$\|\varphi_n(x) - f(x)\| \leq d, \forall x \in G. \tag{3.8}$$

Lemma 3.3: For any $x \in G$ and any $q \in \mathbb{N}$ we have

$$\varphi_n(x^{n^q}) = n^q \varphi_n(x). \tag{3.9}$$

Proof:

$$\varphi_n(x^{n^q}) = \lim_{k\to\infty} \frac{1}{n^k} f\left(\left(x^{n^q}\right)^{n^k}\right)$$

$$= \lim_{k\to\infty} \frac{n^q}{n^{k+q}} f\left(x^{n^{k+q}}\right)$$

$$= n^q \varphi_n(x).$$

Lemma 3.4: Let $f \in KA^n(G; E)$ such that

$$\|f(x^n y^n) - nf(x) - nf(y)\| \le c \quad \forall x, y \in G.$$

Then $\varphi_n \in KA^n(G; E)$.

Proof: From (3.3) it follows that

$$\|f(x^n y^n) - f(x^n) - f(y^n)\| \le c + 2d.$$

Now using (3.8) we have

$$\| \varphi_n(x^n y^n) - \varphi_n(x^n) - \varphi_n(y^n)\| \le c + 2d + 3d.$$

Therefore, from (4.9) we obtain

$$\|\varphi_n(x^n y^n) - n\varphi_n(x) - n\varphi_n(y)\| \le c + 5d.$$

This completes the proof of the lemma.

Definition 3.5: By $n-(G; E)$-*pseudoadditive mapping* we will mean an $n-(G; E)$-quasiadditive mapping f such that $f(x^{n^k}) = n^k f(x)$ for any $x \in G$ and any $k \in \mathbb{N}$.

Remark 3.6: If $f \in PA^n(G; E)$, then:

1. $f(x^{-n^k}) = -n^k f(x)$ for any $x \in G$ and $k \in \mathbb{N}$;
2. if $y \in G$ is an element of finite order, then $f(y) = 0$;
3. if f is a bounded mapping on G, then $f \equiv 0$.

Proof: Let for some $c > 0$ the following relation hold

$$\|f(x^n y^n) - nf(x) - nf(y)\| \le c.$$

Then we have

$$\|f(1) - nf(y) - nf(y^{-1})\| \le c,$$

$$\|nf(y) + nf(y^{-1})\| \le c_2 = c + \|f(1)\|, \text{ for any } y \in G,$$

It follows that for any $k \in N$ we have

$$\|nf(y^{n^k}) + nf((y^{n^k})^{-1})\| \le c_2,$$

$$\|nf(y^{n^k}) + nf((y^{-1})^{n^k})\| \le c_2,$$

$$n^{k+1}\|f(y) + f((y^{-1}))\| \le c_2.$$

The last inequality is equivalent to the next one $\|f(y) + f(y^{-1})\| \leq \frac{c_2}{n^{k+1}}$ for all $y \in G$ and all $k \in \mathbb{N}$. The latter implies $f(y^{-1}) = -f(y)$. Thus for any $k \in \mathbb{N}$ we have $f(y^{-n^k}) = f((y^{n^k})^{-1}) = -f(y^{n^k}) = -n^k f(y)$. Hence, the assertion 1 is established.

Now let us verify 2: Let x be an element of order q. Then there exist k and p such that $k > p$ and the following relation holds:

$$x^{n^k} = x^{n^p}$$

It follows that

$$f(x^{n^k}) = f(x^{n^p})$$

and

$$n^k f(x) = n^p f(x)$$

$$(n^k - n^p)f(x) = 0.$$

and we see that $f(x) = 0$.

The assertion 3 is obvious.

We denote by $B(G; E)$ the space of all bounded mappings on a group G that take values in E.

Theorem 3.7: For an arbitrary group G the following decomposition holds

$$KA^n(G; E) = PA^n(G; E) \oplus B(G; E).$$

Proof: It is clear that $PA^n(G; E)$ and $B(G; E)$ are subspaces of $KA^n(G; E)$, and $PA^n(G; E) \cap B(G;E) = \{0\}$. Hence the subspace of $KA^n(G; E)$ generated by $PA^n(G; E)$ and $B(G; E)$ is their direct sum. That is $PA^n(G; E) \oplus B(G; E) \subseteq KA^n(G; E)$. Let us verify that $KA^n(G; E) \subseteq PA^n(G; E) \oplus B(G; E)$. Indeed, if $f \in KA^n(G; E)$, then the function φ_n defined by the formula (3.7) is an element of $PA^n(G; E)$. Now from the relation (3.8) it follows that

$$\|\varphi_n(x) - f(x)\| \leq d, \forall x \in G.$$

So, $\varphi_n - f \in B(G; E)$.

Definition 3.8: A *quasicharacter* of a semigroup S is a real-valued function f on S such that the set $\{f(xy) - f(x) - f(y) \mid x, y \in S\}$ is bounded.

Definition 3.9: By a *pseudocharacter* of a semigroup S we mean its quasicharacter f that satisfies $f(x^n) = nf(x)$ for all $x \in S$ and all $n \in \mathbb{N}$.

The set of all (S; E)-quasiadditive mappings is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers), which will be denoted by $KAM(S; E)$. The subspace of $KAM(S; E)$ consisting of (S; E)-pseudoadditive mappings will be denoted by $PAM(S; E)$ and the subspace consisting of additive mappings from S to E will be denoted by $Hom(S; E)$. We say that an (S; E)-pseudoadditive mapping φ of the semigroup S is *nontrivial* if $\varphi \notin Hom(S; E)$.

The space of quasicharacters will be denoted by $KX(S)$, the space of pseudocharacters will be denoted by $PX(S)$, and the space of real additive characters on S will be denoted by $X(S)$.

4. STABILITY

Suppose that G is a group and E is a real Banach space.

Definition 4.1: We shall say that the equation (3.1) *is stable* for a pair (G; E) if for any $f : G \to E$ satisfying functional inequality

$$\|f(x^n y^n) - nf(x) - nf(y)\| \le c \qquad \forall x, y \in G$$

for some $c > 0$ there is a solution j of the functional equation (3.1) such that the mapping $j(x) - f(x)$ belongs to $B(G; E)$.

It is clear that the equation (3.1) is stable on G if and only if $PA^n(G; E) = Ad^n(G; E)$.

Theorem 4.2: Let E_1, E_2 be Banach spaces over reals. Then the equation (3.1) is stable for pair (G; E_1) if and only if it is stable for pair (G; E_2).

Proof: Let E be a Banach space and $\mathbb{R}$ be the set of reals. Suppose that the equation (3.1) is stable for the pair (G; E). Suppose that (3.1) is not stable for the pair (G, $\mathbb{R}$), then there is a nontrivial real-valued n-pseudoadditive function f on G. Now let $e \in E$ and $\|e\| = 1$. Consider the mapping $\varphi : G \to E$ given by the formula $\varphi(x) = f(x) \cdot e$. It is clear that φ is a nontrivial n-pseudoadditive E-valued mapping, and we obtain a contradiction.

Now suppose that the equation (3.1) is stable for the pair (G, $\mathbb{R}$), that is, $PA^n(G; \mathbb{R}) = Ad^n(G, \mathbb{R})$. Denote by E^* the space of linear bounded functionals on E endowed by a functional norm topology. It is clear that for any $\psi \in PA^n(G, H)$ and any $\lambda \in H^*$ the function $\lambda \circ \psi$ belongs to the space $PA^n(G, \mathbb{R})$. Indeed, let for some $c > 0$ and any $x; y \in G$ we have $\|\psi(x^n y^n) - n\psi(x) - n\psi(y)\| \le c$. Hence

$$|\lambda \circ \psi(x^n y^n) - \lambda \circ \psi(x^n) - \lambda \circ \psi(y^n)| = |\lambda(\psi(x^n y^n) - \psi(x^n) - \psi(y^n))| \le c\|\lambda\|.$$

Obviously, $\lambda \circ \psi(x^{n^k}) = n^k\lambda \circ \psi(x)$ for any $x \in G$ and for any $n \in \mathbb{N}$. Hence the mapping $\lambda \circ \psi$ belongs to the space $PA^n(G, \mathbb{R})$. Let $f : G \to H$ be a nontrivial n-pseudoadditive mapping. Then there are $x, y \in G$ such that $f(x^ny^n) - nf(x) - nf(y) \neq 0$. Hahn-Banach Theorem implies that there is a $\ell \in H^*$ such that $\ell(f(x^ny^n)- nf(x)- nf(y)) \neq 0$, and we see that $\ell \circ f$ is a nontrivial n-pseudoadditive real-valued mapping on G. This contradiction proves the theorem.

In what follows the space $KA^n(G; \mathbb{R})$ will be denoted by $KX^n(G)$, the space $PA^n(G; \mathbb{R})$ will be denoted by $PX^n(G)$, the space $A^n(G, \mathbb{R})$ will be denoted by $X^n(G)$.

Corollary 4.3: The equation (3.1) over a group G is stable if and only if $PX^n(G) = X^n(G)$.

Due to the previous theorem we may simply say that equation (3.1) is stable or not stable.

Now our goal is to show that, in general, the equation (3.1) is not stable on a group. First we recall some facts from the paper [8].

Let F be a free group over group alphabet X such that $|X| \geq 2$. Recall that a word $v = x_{i_i}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_n}^{\varepsilon_n}$ ($\varepsilon_i \in \{1, -1\}$, $x_{i_j} \in X$) is reduced if $x_{i_k}^{\varepsilon_k} \neq x_{i_{k+1}}^{-\varepsilon_{k+1}}$ $k = 1, 2, \dots, n-1$.

By the length of v we mean the number n which we denote by $|v|$. Let $v = x_{i_i}^{\varepsilon_{i_1}} x_{i_2}^{\varepsilon_{i_2}} \dots x_{i_{n-1}}^{\varepsilon_{i_{n-1}}} x_{n}^{\varepsilon_{i_n}}$ be a reduced word. We recall (see [8]) that the set of "beginnings" $\mathcal{B}(v)$ and the set of "ends" $\mathcal{E}(v)$ of the word v is defined as follows: if $n \leq 1$, then $\mathcal{B}(v) = \mathcal{E}(v) = \{\wedge\}$, where $\wedge$ is empty word. If $n \geq 2$, then

$$\mathcal{B}(v) = \left\{\wedge, x_{i_i}^{\varepsilon_{i_1}}, x_{i_i}^{\varepsilon_{i_2}} x_{i_2}^{\varepsilon_{i_2}}, \dots, x_{i_1}^{\varepsilon_{i_1}} x_{i_2}^{\varepsilon_{i_2}} \dots, x_{i_{n-1}}^{\varepsilon_{i_n}}\right\}$$

$$\mathcal{E}(v) = \left\{x_{i_2}^{\varepsilon_{i_2}} \dots x_{i_n}^{\varepsilon_{i_n}}, \dots, x_{i_{n-1}}^{\varepsilon_{i_{n-1}}} x_{i_n}^{\varepsilon_{i_n}}, x_{n}^{\varepsilon_{i_n}}, \wedge\right\}$$

Denote by E the set of words w such that $\mathcal{B}(w) \cap \mathcal{E}(w) = \emptyset$ and w is not conjugate to w^{-1} in F. In [8] for any $w \in E$ was constructed a pseudocharacter e_w.

Lemma 4.4: The system of pseudocharacters few $\{e_w; w \in E\}$ has the following properties:

(1) $|e_w(uv) - e_w(u) - e_w(v)| \leq 15$ for any u, v from F and any w from E;

(2) if $|w_1| < |w_2|$, then $e_{w_2}(\mathrm{w}_1) = 0$;

(3) if $|w_1| = |w_2|$ and $w_1 \neq w_2$, then $e_{w_2}(w_1) = 0$;

(4) $e_w(w) = 1$ for each $w \in E$;

(5) $e_w(w_1) = -1$ if w_1 is conjugate to w^{-1}.

Proof: See [8].

From this lemma we get the following one

Lemma 4.5: If $w \in E$ and $w = uv$, such that $|u| \geq 1$, $|v| \geq 1$, then for any $n \in \mathbb{N}$ we have:

1. $u^n v^n \in E$;
2. $e_w(u^n v^n) - e_w(u^n) - e_w(v^n) = 1$.

Remark 4.6: In general, equation (3.1) is not stable on a group G.

Proof: It is clear that for any $n \in N$ we have the following including $PX(G) \subseteq PX^n(G)$. Let φ be a nontrivial pseudocharacter of G. Suppose that there is $j \in X^n(G)$ such that the mapping $\varphi - j$ is bounded. Then there is $c > 0$ such that $|\varphi(x) - j(x)| \leq c$ for any $x \in G$. Hence for any $k \in \mathbb{N}$ we have $c \geq |\varphi(x^{n^k}) - j(x^{n^k})| = n^k|\varphi(x) - j(x)|$ and we see that the latter is possible if $\varphi(x) = j(x)$. So, $\varphi \in PX(G) \cap X^n(G)$.

Now let F be a free group over a group alphabet X such that $|X| \geq 2$.

From the previous lemma it follows that for any $w \in E$ we get $e_w \notin X^n(F)$.

Note that in papers ([8]-[10]) full description of the spaces of pseudocharacters on free groups, free products of groups as well as some extensions of free groups was given. In the papers [11] and [12] an application of pseudocharacters e_w, $w \in E$ to a problem of expressibility in group theory was given.

Definition 4.7: For $n \in N$ the group G is said to be n-abelian if for any x, y the following relation holds:

$$(xy)^n = x^n y^n.$$

It is easy to check that if a group G is n-abelian, then for any $k \in N$ we have

$$(xy)^{n^k} = x^{n^k} y^{n^k}.$$

Theorem 4.8: If G is n-abelian group, then $PX^n(G) = X(G)$. Therefore the equation (4.1) is stable on G.

Proof: For any $x, y \in G$ and $k \in N$ we have

$$|\varphi(x^{n^k} y^{n^k}) - \varphi(x^{n^k}) - \varphi(y^{n^k})| \leq \delta,$$

$$|\varphi((xy)^{n^k}) - \varphi(x^{n^k}) - \varphi(y^{n^k})| \leq \delta,$$

$$n^k|\varphi(xy) - \varphi(x) - \varphi(y)| \leq \delta.$$

From the latter relation it follows that

$$\varphi(xy) - \varphi(x) - \varphi(y) = 0.$$

Therefore $PX^n(G) = X(G) \subseteq X^n(G)$.

Corollary 4.9: Let G be an arbitrary group and x, y are elements of G such that $xy = yx$. Then for any $f \in PX^n(G)$ we have

$$f(xy) = f(x) + f(y).$$

Proof: It is clear that subgroup of G generated by elements x and y is abelian group. Obviously, any abelian group A is n-abelian group for any $n \in N$. So, we can apply Theorem 4.8.

5. METABELIAN GROUPS

Definition 5.1: We will say that a group G is *metabelian* if for any x, y, z the following relation

$$[[x, y], z] = 1$$

holds. Here $[a, b] = a^{-1}b^{-1}ab$.

For example any group $UT(3, K)$ is metabelian for any commutative field K.

In this section G denotes a metabelian group.

The following lemma can be easily proved by direct calculation.

Lemma 5.2: For any $x, y, z \in G$ and any $m \in N$ we have

$$(xy)^m = x^m y^m [y, x]^{\frac{m(m-1)}{2}}. \tag{5.1}$$

Theorem 5.3: $PX^n(G) = X(G)$

Proof: From (5.1) it follows $x^m y^m = (xy)^m [x, y]^{\frac{m(m-1)}{2}}$. Therefore for any $k \in N$ and any $f \in PX^n(G)$ we get

$$f\left(x^{n^k}y^{n^k}\right)=f\left((xy)^{n^k}[x,y]^{\frac{n^k(n^k-1)}{2}}\right)=f\left((xy)^{n^k}\right)+f\left([x,y]^{\frac{n^k(n^k-1)}{2}}\right)$$

Now taking into account that the group generated by $[y, x]$ is abelian we get

$$\begin{aligned}\delta &\geq |f(x^{nk}y^{nk})-f(x^{nk})-f(y^{nk})|\\ &=\left|f\left((xy)^{n^k}[x,y]^{\frac{n^k(n^k-1)}{2}}\right)-f\left(x^{n^k}\right)-f\left(y^{n^k}\right)\right|\\ &=\left|f\left((xy)^{n^k}\right)+f\left([x,y]^{\frac{n^k(n^k-1)}{2}}\right)-f\left(x^{n^k}\right)-f\left(y^{n^k}\right)\right|\\ &=\left|n^kf(xy)+\frac{n^k(n^k-1)}{2}f([x,y])-n^kf(x)-n^kf(y)\right|\\ &=n^k\left|f(xy)+\frac{n^k-1}{2}f([x,y])-f(x)-f(y)\right|.\end{aligned}$$

It follows that for any $x, y \in G$ the following relation holds

$$\left|f(xy)-f(x)-f(y)+\frac{n^k-1}{2}f([x,y])\right|\leq\frac{1}{n^k}\delta.$$

The latter is possible only if

$$f(xy)-f(x)-f(y)=0,\; f([x,y])=0.$$

So, $f \in X(G)$. The proof is complete.

6. WREATH PRODUCT

First recall the notion of *wreath product* of groups.

Let A and B be arbitrary groups. For each $b \in B$ denote by $A(b)$ a group that is isomorphic to A under isomorphism $a \to a(b)$. Denote by

$H = A^{(B)} = \Pi_{b\in B} A(b)$ the direct product of groups $A(b)$. It is clear that if $a_1(b_1)a_2(b_2)$ ... $a_k(b_k)$ is an element of H, then for any $b \in B$, the mapping

$$b^* : a_1(b_1)a_2(b_2) \ldots a_k(b_k) \to a_1(b_1b)a_2(b_2b) \ldots a_k(b_kb)$$

is an automorphism of H and $b \to b^*$ is an embedding of B into $Aut\ H$. Hence, we can form a semidirect product $G = B \cdot D$. This group is called *the wreath product* of the groups A and B, and will be denoted by $G = A \wr B$. Let A be an arbitrary group and B be a cyclic group of order n with generator b. Consider the group $G = A \wr B$. Denote by D *diagonal* subgroup of G, i.e., subgroup consisting of elements of the form $a(1)a(b) \ldots a(b^{n-1})$, $a \in A$. It is clear that D is isomprphic to A under isomorphism $a \to a(1)a(b) \ldots a(b^{n-1})$. For any $i \in N$ denote by C_i a cyclic group of order n with generator c_i. For any group A consider the following sequence of groups G_k.

$$G_0 = A,\ G_1 = G_0 \wr C_1,\ \ldots.,\ G_{k+1} = G_k \wr C_{k+1},\ \ldots \tag{6.1}$$

Let $\sigma_i : G_i \to G_{i+1}$ be a embedding G_i into G_{i+1} identifying G_i with diagonal of G_{i+1}. Then we have the following sequence

$$A = G_0 \xrightarrow{\sigma_0} G_1 \xrightarrow{\sigma_1} \ldots \xrightarrow{\sigma_{k-1}} G_k \xrightarrow{\sigma_k} G_{k+1} \xrightarrow{\sigma_{k+1}} \ldots \tag{6.2}$$

Let G be direct limit of this sequence.

Lemma 6.1: For any $g \in G$ the equation $x^n = g$ has a solution. Let $f \in PX(G)$ and $f|_{G_0} \equiv 0$, then $f \equiv 0$ on G.

Proof: Denote by D_k the diagonal subgroup of $G_k = G_{k-1} \wr C_k$. Then we have

$$G_0 = D_1,\ G_1 = D_2,\ \ldots,\ G_i = D_{i+1},\ \ldots$$

It is clear that if $g \in G_k$, then g can be presented in the form

$$g = g\left(c_k^0\right) g\left(c_k^1\right) \ldots g\left(c_k^{n-1}\right).$$

Now if we set $v = c_k g\left(c_k^0\right)$ we get the equality $\left(c_k g\left(c_k^0\right)\right)^n = g$.

7. THEOREM OF EMBEDDING

Let A be an arbitrary group and let G be direct limit of the sequence (6.2). And let p be a prime number greater than n. Let K be direct product $K = \prod_{i=1}^{\infty} Z_p(i)$, where $Z_p(i)$ is a cyclic group of order p with generator c_i. Now consider the group $Q = G \wr K$. By the theorem 3.2 from [7] we have $PX(Q) = X(G)$.

Theorem 7.1: $PX^n(Q) = PX(Q) = X(G)$.

Proof: Let $\widehat{G}$ be subgroup of Q generated by $G(c)$, $c \in K$. From above we know that the equation $x^n = g$ has a solution for any $g \in \widehat{G}$. The same is true for any $b \in K$. It follows that $PX^n\left(\widehat{G}\right) = PX\left(\widehat{G}\right)$. For any $b \in K$ and any $v \in \widehat{G}$ there are c and u such that $c^n = b$ and $u^n = v$. It implies that $(u^b)^n = v^b$. Here we use the following notation $x^b = b^{-1}xb$. Then for $f \in PX^n(Q)$ such that

$$|f(x^n y^n) - f(x^n) - f(y^n)| \leq \delta, \forall x, y \in Q$$

we have

$$|f(c^n u^n) - f(c^n) - f(u^n)| \leq \delta,$$

$$|f(c^n u^n) - f(u^n)| \leq \delta,$$

$$|f(bv) - f(v)| \leq \delta,$$

$$|f(b_v^{\ b}) - f(v^b)| \leq \delta. \tag{7.1}$$

Similarly

$$|f(vb) - f(v)| \leq,$$

therefore

$$|f(bv^b) - f(v)| \leq \delta, \tag{7.2}$$

From (7.1), (7.2) it follows

$$|f(v^b) - f(v)| \leq 2\delta,$$

The last relation is true for any v, so for any $k \in N$ we have

$$|f((v^k)^b) - f(v^k)| \leq 2\delta,$$

$$k|f(v^b) - f(v)| \leq 2\delta.$$

The latter relation is possible only if

$$f(v^b) = f(v) \tag{7.3}$$

Denote by $\psi = f\big|_{\widehat{G}}$. Then we have $\psi \in PX(\widehat{G}, K)$. Now continue ψ onto Q by the rule:

$$\psi(bv) = \psi(v). \tag{7.4}$$

It is easily verified that ψ becomes now an element of the space $KX(Q)$. Now consider the function $\varphi = f - \psi$ on Q. It is clear that $\varphi\big|_{K\cup\hat{G}} \equiv 0$. The functions f and ψ are elements of $KX^n(Q)$, so $\varphi \in KX^n(Q)$. Therefore there is an $\delta_1 > 0$ such that $|\varphi(x^n y^n) - \varphi(x^n) - \varphi(y^n)| \leq \delta_1$ for all $x, y \in Q$. Now we have

$$|\varphi(c^n u^n) - \varphi(c^n) - \varphi(u^n)| \leq \delta_1,$$

$$|\varphi(c^n u^n) - \varphi(u^n)| \leq \delta_1,$$

$$|\varphi(c^n u^n)| \leq \delta_1.$$

So, we see that φ is a bounded function on Q. It follows that

$$f = \psi + \varphi \in KX(Q) \cap PX^n(Q) = PX(Q).$$

Now from the relation $PX(Q) = X(G)$ follows that $PX^n(Q) = X(G)$. The proof of the theorem is complete.

Corollary 7.2: Any group A can be embedded into a group Q such that the equation (3.1) is stable over Q.

Proof: Let $Q = G \wr K$ be the group from the above theorem. The group A is a subgroup of G. We will identify the group G with subgroup $G(1)$ of $H = \prod_{c\in K} G(c) \subset Q$, where $1 \in K$. Hence, we can assume that A is a subgroup of Q. From the theorem 7.1 we have $PX^n(Q) = X(G)$. So, the equation (3.1) is stable over Q.

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Valeriĭ Faĭziev
Tver State Agricultural Academy
Tver Sakharovo, Russia
E-mail: vfaiz@tvcom.ru

John M. Rassias
National and Capodistrian University of Athens
4, Agamemnonos Str.
Aghia Paraskevi, Attikis 15342
Athens, Greece
E-mail: jrassias@primedu.uoa.gr