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FIXED POINT THEOREMS OF MATKOWSKI TYPE ON COMPLETE GAUGE SPACES

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ABSTRACT: A fixed point theory for contractive maps of Matkowski type on complete uniform spaces is presented.

1. INTRODUCTION

This paper presents new fixed point results for generalized contractive maps first on complete metric spaces and then more generally on complete gauge spaces (i.e. complete uniforn spaces). Our results complement those in [1, 2, 5]. We note that the proofs in [1, 2, 5] are indirect in nature (i.e. one argues by contradiction that the sequence considered is Cauchy). However in this paper we supply a direct proof so in addition we obtain new applicable estimates.

2. FIXED POINT THEORY IN METRIC SPACES

This section presents fixed point results for generalized contractions of Matkowski type [4] on complete metric spaces. We begin with a global result.

Theorem 2.1: Let (X, d) be a complete metric space and $F : X \to X$. Suppose there is a nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$ such that for $x, y \in X$ we have

$$d(Fx, Fy) \leq \psi(\max\{d(x, F x), d(y, F y),$$
$$\frac{1}{2}[d(x, F y) + d(y, Fx)]\}).$$

Then there exists a unique $x \in X$ with $x = F x$.

Proof: First notice that $\psi(t) < t$ for $t > 0$. To see this suppose there exists $t_0 > 0$ with $t_0 \leq \psi(t_0)$. Then since ψ is nondecreasing we see that $t_0 \leq \psi^n(t_0)$ for each $n \in (1, 2, ...\}$, a contradiction. Note also that $\psi(0) = 0$.

To show uniqueness suppose there exists $x, y \in X$ with $x = F x$, $y = F y$ and $x \neq y$. Then

$$d(x, y) = d(Fx, Fy) \leq \psi(\max\{d(x, y), 0, 0, \frac{1}{2}[d(x, y) + d(y, x)]\}) = \psi(d(x, y)),$$

a contradiction. It remains to show existence. Let $x_0 \in X$ and let $x_n = F\, x_{n-1}$ for $n \in \{1, 2, ...\}$. We first show

$$d(x_n, x_{n+1}) \le \psi(d(x_{n-1}, x_n) \text{ for } n \in \{1, 2, ...\}. \tag{2.1}$$

To see (2.1) notice

$$\begin{aligned} d(x_n, x_{n+1}) &\le \psi(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad \tfrac{1}{2}\left[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)\right]\}) \\ &\le \psi(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad \tfrac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]\}). \end{aligned}$$

Let

$$\eta_n = \psi\left(\max\left\{d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right), \frac{1}{2}\left[d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right]\right\}\right).$$

If $\eta_n = d(x_{n-1}, x_n)$ then (2.1) holds. If $\eta_n = d(x_n, x_{n+1})$ then $d(x_n, x_{n+1}) = 0$ since if not

$$d(x_n, x_{n+1}) \le \psi(x_n, x_{n+1}) < d(x_n, x_{n+1}),$$

a contradiction. Thus $d(x_n, x_{x+1}) = 0$ and (2.1) is immediate. Finally suppose $\eta_n = \frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]$. If $\eta_n = 0$ then $d(x_n, x_{n+1}) = 0$ and (2.1) is immediate. If $\eta_n \ne 0$ we have

$$\begin{aligned} d(x_n, x_{n+1}) &\le \psi\left(\frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]\right) \\ &< \frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right], \end{aligned}$$

so

$$\frac{1}{2} d\left(x_n, x_{n+1}\right) < \frac{1}{2} d\left(x_{n-1}, x_n\right),$$

and as a result

$$\eta_n = \frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]$$

$$< \frac{1}{2}d(x_{n-1}, x_n) + \frac{1}{2}d(x_{n-1}, x_n) = d(x_{n-1}, x_n),$$

which contradicts the definition of η_n. In all cases (2.1) is true. Thus

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \text{ for } n \in \{1, 2, ...\},$$

and since $\lim_{n\to\infty}\psi^n(a) = 0$ for $a > 0$ we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Let $\epsilon > 0$ be fixed. Choose $n \in \{1, 2, ...\}$ so that

$$d(x_n, x_{n+1}) < \epsilon - \psi(\epsilon). \tag{2.2}$$

Now (2.1) and (2.2) imply

$$\begin{aligned} d(x_n, x_{n+2}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \leq [\epsilon - \psi(\epsilon)] + \psi(d(x_n, x_{n+1})) \\ &\leq [\epsilon - \psi(\epsilon) + \psi(\epsilon)) \leq \{\epsilon - \psi(\epsilon)] + \psi(\epsilon) = \epsilon, \end{aligned}$$

so $x_{n+2} \in \overline{B(x_n, \epsilon)} = \{x \in X : d(x, x_n) \leq \epsilon\}$. We now claim that

$$x_{n+k} \in \overline{B(x_n, \varepsilon)} \text{ for } k \in \{1, 2, ...\}. \tag{2.3}$$

Certainly (2.3) is true for $k = 1$ and $k = 2$. Suppose $x_{n+m} \in \overline{B(x_n, \epsilon)}$ for $m \in \{1, 2, ..., p\}$; here $p \in \{1, 2, ...\}$. We will now show that $x_{n+p+1} \in \overline{B(x_n, \epsilon)}$ so then (2.3) will follow from the principle of induction. Along the way to prove $x_{n+p+1} \in \overline{B(x_n, \epsilon)}$ we will need to show

$$d(x_{n+1}, x_{n+j}) \in \psi(\epsilon) \text{ for } j \in \{1, 2, ..., p\}. \tag{2.4}$$

Certainly (2.4) is true for $j = 1$ and $j = 2$ since

$$d(x_{n+1}, x_{n+2}) \in \psi(d(x_n, x_{n+1})) \leq \psi(\epsilon - \psi(\epsilon)) \leq \psi(\epsilon).$$

Suppose (2.4) is true for a fixed $j \in \{2, ..., p - 1\}$. Then

$$d(x_n, x_{n+j+1}) \in \psi(\max\{d(x_n, x_{n+j}), d(x_n, x_{n+1}), d(x_{n+j}, x_{n+j+1}),$$

$$\frac{1}{2}\left[d(x_n, x_{n+j+1}) + d(x_{n+j}, x_{n+1})\right]\Big\}\Big)$$

$$\leq \ \psi\left(\max\left\{\epsilon, \epsilon - \psi(\epsilon), \psi^j(\epsilon), \frac{1}{2}\left[\epsilon + \psi(\epsilon)\right]\right\}\right)$$

since $x_{n+j} \in \overline{B(x_n, \epsilon)}$ (note $j \in \{2, ..., p-1\}$), $d(x_n, x_{n+1}) < \epsilon - \psi\{\epsilon)$, $d(x_{n+j}, x_{n+j+1}) \leq \psi^j(d(x_n, x_{n+1})) \leq \psi^j(\epsilon)$, $x_{n+j+1} \in \overline{B(x_n, \epsilon)}$ since $j + 1 \in \{3, ..., p\}$ and $d(x_{n+j}, x_{n+1}) \leq \psi(\epsilon)$ since (2.4) is assumed true for this j. Consequently $d(x_{n+1}, x_{n+j+1}) \in \psi(\epsilon)$, so by induction (2.4) is true. Now

$$\begin{aligned} d(x_{n+p+1}, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p+1}) \\ &\leq [\epsilon - \psi(\epsilon)] + \psi(\max\{d(x_n, x_{n+p}), d(x_n, x_{n+1}), \\ &\quad d\left(x_{n+p}, x_{n+p+1}\right), \frac{1}{2}\left[d(x_n, x_{n+p+1}) + d(x_{n+p}, x_{n+1})\right]\Big\}\Big) \\ &\leq [\epsilon - \psi(\epsilon) + \psi(\max\{\epsilon, \epsilon - \psi(\epsilon), \psi^p(d(x_n, x_{n+1})), \\ &\quad \frac{1}{2}\left[d(x_n, x_{n+p+1}) + \psi(\epsilon)\right]\Big\}\Big) \end{aligned}$$

since $x_{n+p} \in \overline{B(x_n, \epsilon)}$ and $d(x_{n+p}, x_{n+1}) \leq \psi(\epsilon)$ from (2.4). Now

$$\psi^p(d(x_n + x_{n+1})) \leq \psi^p(\epsilon - \psi(\epsilon)) \leq \psi^p(\epsilon) \leq \epsilon$$

so

$$d\left(x_{n+p+1}, x_n\right) \leq \left[\epsilon - \psi(\epsilon)\right] + \psi\left(\max\left\{\varepsilon, \frac{1}{2}\left[d\left(x_n, x_{n+p+1}\right) + \psi(\epsilon)\right]\right\}\right).$$

Let

$$\tau_p = \max\left\{\epsilon, \frac{1}{2}\left[d\left(x_n, x_{n+p+1}\right) + \psi(\epsilon)\right]\right\}.$$

If $\tau_p = \frac{1}{2}\left[d\left(x_n, x_{n+p+1}\right) + \psi(\epsilon)\right]$ (note $\tau_p > 0$) then

$$d(x_{n+p+1}, x_n) < [\epsilon - \psi(\epsilon)] + \frac{1}{2}[d(x_n, x_{n+p+1}) + \psi(\epsilon)]$$

so

$$\frac{1}{2}d(x_{n+p+1}, x_n) < [\epsilon - \psi(\epsilon)] + \frac{1}{2}\psi(\epsilon),$$

and consequently

$$\tau_p = \frac{1}{2}\left[d(x_n, x_{n+p+1}) + \psi(\epsilon)\right] < \left\{[\epsilon - \psi(\epsilon)] + \frac{1}{2}\psi(\epsilon)\right\} + \frac{1}{2}\psi(\epsilon) = \epsilon,$$

and this contractícts the definition of τ_p. Thus $\tau_p = \epsilon$ so

$$d(x_{n+p+1}, x_n) \le [\epsilon - \psi(\epsilon)] + \psi(\epsilon) = \epsilon,$$

so $x_{n+p+1} \in \overline{B(x_n, \epsilon)}$. Thus (2.3) is true i.e. $d(x_m, x_n) \le \epsilon$ for all $m \ge n$.

Consequently $\{x_n\}$ is a Cauchy sequence. Since X is complete there exists a $x \in X$ with $\lim_{n\to\infty} x_n = x$.

Suppose $d(x, F\,x) = a > 0$. Choose $N \in \{1, 2, ...\}$ with $d(x, x_n) < \frac{a}{2}$ for $n \ge N$. Now for $n \ge N$ we have

$$\begin{aligned} d(x, F(x)) &\le d(x, x_{n+1}) + d(F\,x_n, F\,x) \\ &\le d(x, x_{n+1}) + \psi(\max\{d(x, x_n), d(x, F\,x), d(x_n, x_{n+1}), \\ &\quad \frac{1}{2}\left[d(x_n, F\,x) + d(x, x_{n+1})\right]\}) \\ &\le d(x, x_{n+1}) + \psi(d(x, F\,x)) \end{aligned}$$

since $d(x_n, x) < \frac{a}{2} \le a = d(x, F\,x)$,

$$d(x_n, x_{n+1}) \le d(x_n, x) + d(x, x_{n+1}) < \frac{a}{2} + \frac{a}{2} = a,$$

and

$$\frac{1}{2}\left[d\left(x_n, F\,x\right)+d\left(x, x_{n+1}\right)\right] \;<\; \frac{1}{2}\left[d(x_n,x)+d(x,F\,x)+\frac{a}{2}\right]$$

$$\leq \; \frac{1}{2}\left[\frac{a}{2}+a+\frac{a}{2}\right]=a.$$

As a result for $n \geq N$ we have $d(x, F\,x) \leq d(x, x_{n+1}) + \psi(d(x, F\,x))$ so letting $n \to \infty$ yields $d(x, F\,x) \leq \psi(d(x, F\,x))$ which is a contradiction. Thus $d(x, F\,x) = 0$.

Remark 2.1: Note if $\psi(0.\ \infty) \to [0, \infty)$ is a continuous function (or upper semicontinuous from the right) with $\psi(t) < t$ for $t > 0$ then $\lim_{n\to\infty}\psi^n(t) = 0$ for $t > 0$ since for fixed $t > 0$ if $a_n = \psi^n(t)$ then $a_n = \psi(a_{n-1}) \leq a_{n-1}$ so $a_n \downarrow \beta$ say, and now note that $\beta = \psi(\beta)$ (or $\beta \leq \psi(\beta)$) so $\beta = 0$.

Remark 2.2: It is possible also to obtain common fixed point results using the ideas in Theorem 2.1 with those in [5].

Next we present a local result.

Theorem 2.2: Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ with $F:\overline{B(x_0, r)} \to X$ where $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$. Suppose there is a nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty}\psi^n(t) = 0$ for each $t > 0$ such that for $x,\ y \in \overline{B(x_0, r)}$ we have

$$d(Fx, Fy) \leq \psi(\max\{d(x, y), d(x, F\,x), d(y, F\,y),$$

$$\frac{1}{2}[d(x,F\,y)+d(y,Fx)]\}).$$

Also suppose

$$d(x_0, F\,x_0) < r - \psi(r). \tag{2.5}$$

Then there exists a unique $x \in \overline{B(x_0, r)}$ with $x = F\,x$.

Proof: Let $x_1 = F\,x_0$. Then from (2.5) we have

$$d(x_1, x_0) = d(F\,x_0, x_0) < r - \psi(r),$$

so $x_1 \in B(x_0, r)$. Let $x_2 = F\,x_1$ (this is possible since $x_1 \in B(x_0, r)$). For $n \in \{3, 4, ...\}$ we let $x_n = F\,x_{n-1}$. This is possible if we show $x_{n-1} \in \overline{B(x_0, r)}$ for $n \in \{3, 4, ...\}$. To show this we will in fact establish more i.e. we will show

$$\begin{cases} d\left(x_n, x_{n+1}\right) \le \psi\left(d\left(x_{n-1}, x_n\right)\right) \text{ for } n \in \{1, 2, ...\} \\ \text{and } x_i \in \overline{B(x_0, r)} \text{ for } i \in \{0, 1, ..., n\}. \end{cases} \tag{2.6}$$

Note (essentially the same ideas as in (2.1)) that

$$d(x_1, x_2) = d(F\,x_0, F\,x_1) \le \psi(d(x_0, x_1)),$$

and also note

$$\begin{aligned} d(x_0, x_2) &\le d(x_0, x_1) + d(x_1, x_2) \le d(x_0, x_1) + \psi(d(x_0, x_1)) \\ &< [r - y(r)] + \psi(r) = r, \end{aligned}$$

so $x_2 \in \overline{B(x_0, r)}$. Similarly (see the ideas in (2.1)) we have

$$d(x_2, x_3) = d(F\,x_1, F\,x_2) \le \psi(d(x_1, x_2)).$$

Now suppose there exists a $k \in \{2, 3, ...\}$ with

$$d(x_m, x_{m+1}) \le \psi(d(x_{m-1}, x_m)) \text{ for } m \in \{1, 2, ..., k\}$$

and $x_m \in \overline{B(x_0, r)}$ for $m \in \{1, 2, ..., k\}$. We now show $x_{k+1} \in \overline{B(x_0, r)}$ and $d(x_{k+1}, x_{k+2}) \le \psi(d(x_k, x_{k+1}))$. Essentially the same reasoning as in (2.4) yields

$$d(x_1, x_k) \le \psi(r); \tag{2.7}$$

note $d(x_1, x_2) \le \psi(d(x_0, x_1) \le \psi(r)$ and if we assume $d(x_1, x_j) \le \psi(r)$ for $j \in \{3, ..., k-1\}$ then

$$\begin{aligned} d(x_1, x_{j+1}) &\le \psi(\max\{d(x_0, x_j), d(x_0, x_1), d(x_j, x_{j+1}), \\ &\qquad \frac{1}{2}\left[d\left(x_0, x_{j+1}\right) + d\left(x_j, x_1\right)\right]\}) \\ &\le \psi\left(\max\left\{r, r - \psi(r), \psi^j\left(d\left(x_0, x_1\right)\right), \frac{1}{2}\left[r + \psi(r)\right]\right\}\right) \\ &\le \psi(\max\{r, \psi^j(r)\}) = \psi(r), \end{aligned}$$

so (2.7) is true. Now following the proof in Theorem 2.1 we obtain

$$\begin{aligned} d(x_1, x_{j+1}) &\le d(x_0, x_1) + d(F\,x_0, F\,x_k) \\ &\le [r - \psi(r)] + \psi(\max\{d(x_0, x_k), d(x_0, x_1), d(x_k, x_{k+1}), \\ &\quad \frac{1}{2}\left[d\left(x_0, x_{k+1}\right) + d\left(x_k, x_1\right)\right]\}) \\ &\le [r - \psi(r)] + \psi(\max\{r, r - \psi(r), \psi^k(r), \\ &\quad \frac{1}{2}\left[d\left(x_0, x_{k+1}\right) + \psi(r)\right]\}) \end{aligned}$$

so

$$d\left(x_0, x_{k+1}\right) \le \left[r - \psi(r)\right] + \psi\left(\max\left\{r, \frac{1}{2}\left[d\left(x_0, x_{k+1}\right) + \psi(r)\right]\right\}\right).$$

Let

$$\tau_k = \max\left\{r, \frac{1}{2}\left[d\left(x_0, x_{k+1}\right) + \psi(r)\right]\right\}$$

and as in Theorem 2.1 we have $\tau_k = r$ so

$$d(x_0, x_{k+1}) \le [r - \psi(r)] + \psi(r) = r$$

which yields $x_{k+1} \in \overline{B(x_0, r)}$. Also (see the ideas in (2.1)) we have

$$d(x_{k+1}, x_{k+2}) = d(F\,x_k, F\,x_{k+1}) \le \psi(d(x_k, x_{k+1})).$$

Then by induction $x_n \in \overline{B(x_0, r)}$ for $n \in \{1, 2, ...\}$ and

$$d(x_n, x_{n+1}) \le \psi(d(x_{n-1}, x_n)) \text{ for } n \in \{1, 2, ...\}.$$

In particular

$$d(x_n, x_{n+1}) \le \psi^n(d(x_0, x_1)) \text{ for } n \in \{1, 2, ...\},$$

so $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $\epsilon > 0$ and $\epsilon > r$ be fixed. Choose $n \in \{1, 2, ...\}$ so that

$$d(x_n, x_{n+1}) < \epsilon - \psi(\epsilon).$$

Essentially the same argument as in Theorem 2.1 guarantees that $x_{n+k} \in \overline{B(x_n, \epsilon)}$ for $k \in \{1, 2, ...\}$ so $d(x_m, x_n) \le \epsilon$ for all $m \ge n$. Thus $\{x_n\}$ is a Cauchy sequence and the rest of the proof follows as in Theorem 2.1.

3. FIXED POINT THEORY IN GAUGE SPACES

Let $E = (E, \{d_\alpha\}_{\alpha\in\Lambda})$ (here Λ is a directed set) be a gauge space endowed with a complete gauge structure $\{d_\alpha : \alpha \in \Lambda\}$ (see Dugundji [3 pp. 198, 308]). For $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0, \infty)^\Lambda$ and $x_0 \in X$, we define the pseudo-ball centered at x_0 of radius r by

$$\overline{B(x_0, r)} = \{y \in E : d_\alpha(x_0, y) \le r_\alpha \text{ for all } \alpha \in \Lambda\}.$$

Theorem 3.1: Let E be a complete gauge space and $F : E \to E$. Suppose for each $\alpha \in \Lambda$ there is a nondecreasing function $y_\alpha : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \psi_\alpha^n(t) = 0$ for each $t > 0$ such that for $x, y \in E$ we have

$$d_\alpha(Fx, Fy) \le \psi(\max\{d_\alpha(x, y), d_\alpha(x, F x), d_\alpha(y, F y),$$

$$\frac{1}{2}\left[d_\alpha(x, Fy) + d_\alpha(y, Fx)\right]\}).$$

Then there exists a unique $x \in E$ with $x = F x$.

Proof: Let $x_0 \in E$ and $x_n = F x_{n-1}$ for $n \in \{1, 2, ...\}$. Fix $\alpha \in \Lambda$. Essentially the same reasoning as in Theorem 2.1 guarantees that $d_\alpha(x_n, x_{n-1}) \le \psi\alpha(d\alpha(x_{n-1}, x_n))$ for $n \in \{1, 2, ...\}$ and $\{x_n\}$ is a Cauchy sequence with respect to d_α. Now since we can do this argument for each $\alpha \in \Lambda$ we have that $\{x_n\}$ is Cauchy. Thus there exists a $x \in E$ with $x_n \to x$. Fix $\alpha \in \Lambda$. Essentially the same reasoning as in Theorem 2.1 guarantees that $d_\alpha(x, F x) = 0$. We can do this argument for each $\alpha \in \Lambda$ so $d_\alpha(x, F x) = 0$ for each $\alpha \in \Lambda$ and so $x = F x$.

Similarly following the ideas in Theorem 2.2 and Theorem 3.1 we obtain

Theorem 3.2: Let E be a complete gauge space, $x_0 \in E$, $r = \{r_\alpha\}_{\alpha\in\Lambda} \in (0, \infty)^\Lambda$ and $F : \overline{B(x_0, r)} \to E$. Suppose for each $\alpha \in \Lambda$ there is a nondecreasing function ψ_α: $[0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \psi_\alpha^n(t) = 0$ for each $t > 0$ such that for $x, y \in \overline{B(x_0, r)}$ we have

$$d_\alpha(Fx, F y) \le \psi(\max\{d_\alpha(x, y), d_\alpha(x, F x), d_\alpha(y, F y),$$

$$\frac{1}{2}\left[d_\alpha(x, Fy) + d_\alpha(y, Fx)\right]\}).$$

Also assume for each $\alpha \in \Lambda$ we have

$$d_\alpha(x_0, F x_0) < r_\alpha - \psi_\alpha(r_\alpha).$$

Then there exists a unique $x \in \overline{B(x_0, r)}$ with $x = F\,x$.

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