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A CLASS OF SECOND ORDER DIFFERENCE EQUATIONS WITH DELAYS AND IMPULSES

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ABSTRACT: This paper is devoted to the investigation of the oscillation of a class of second-order nonlinear impulsive delay difference equations. Some interesting results are obtained by using analysis technique and impulsive difference inequality, and some examples which illustrate that impulsive perturbations play a very important role in giving rise to oscillations of equations are also included.

Keywords and Phrases: Difference equation; Oscillation; Impulsive; Delay

2000 Mathematics Subject Classification: 39A11.

1. INTRODUCTION

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of ordinary differential equations. On the other hand, the theory of impulsive differential equations has attracted the interest of many researchers in the past 20 years since they provide a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Such processes are often investigated in various fields of science and technology such as physics, ecology, optimal control, etc. Recently, the corresponding theory for impulsive differential equations has been studied by several authors. (see[1-5] and the references therein). Only a few papers are impulsive difference equations [6, 7].

In this paper, we consider the following impulsive delay difference equation:

$$\begin{cases} \Delta(a_{n-1}(\Delta_\alpha x(n-1)+\Delta x(n-1))) + f(n, x(n-l)) = g(x(n)), \alpha > 0,\ n \neq n_k,\ k \in N, \\ a_{n_k}\Delta_\alpha x(n_k) = b_k(a_{n_k-1}(\Delta_\alpha x(n_k-1))), \\ a_{n_k}\Delta x(n_k) = b_k(a_{n_k-1}(\Delta x(n_k-1))), \end{cases} \tag{1.1}$$

where $\Delta x(n) = x(n+1) - x(n)$, $\Delta_\alpha x(n) = x(n+1) - \alpha x(n)$, $l \in N$, N is the natural number set, $0 \leq n_0 < n_1 < \ldots < n_k, < \ldots$, and $lim_{k\to\infty} n_k = \infty$.

Throughout this paper, we assume that the following conditions hold:

(c1) $uf(n, u) > 0(u \neq 0)$ and there exists a nonnegative sequence $\{p_n\}$ such that

$$\frac{f(n,u)}{u} \geq p_n;$$

(c2) $\{b_k\}_{k_0}^{\infty}$ is a positive sequence;

(c3) $\{a_n\}_{n_0}^{\infty}$ is a positive sequence;

(c4) $vg(v) \leq 0(v \neq 0)$.

For convince, we let

$$N[n_1, n_2] = \{n | n \in N, n_1 \leq n \leq n_2\},$$
$$N[n_1, n_2) = \{n | n \in N, n_1 \leq n < n_2\},$$
$$N[n_1, \infty) = \{n | n \in N, n_1 \leq n < \infty\}$$

and

$$S(n) = a_{n-1}(\Delta_\alpha x(n-1) + \Delta x(n-1)).$$

By a solution of Eq. (1.1), we mean a real valued sequence $\{x(n)\}$ defined on $N[n_0 - l, \infty)$ which satisfied Eq. (1.1) for $n \geq n_0$. It is obvious that Eq. (1.1) has a unique solution $\{x(n)\}_{n_0-l}^{\infty}$, under the initial conditions

$$x_i = y_i;\ i = n_0 - l, ..., n_0, \tag{1.2}$$

in which $y_i(i = n_0 - l, ..., n_0)$ are given real constants.

A solution of Eq. (1.1) is said to be nonoscillatory if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

2. SOME LEMMAS

Lemma 2.1: Assume that

$$\begin{cases} \Delta m(n) \leq l_n m(n) + q_n, n \neq n_k, k \in N, \\ m(n_k + 1) \leq b_k m(n_k) + e_k, \end{cases}$$

where $\{l_n\}$ and $\{q_n\}$ are two real valued sequences and $l_n > -1$, e_k, b_k are constants and $b_k \geq 0$. Then

$$m(n) \leq m(n_0) \prod_{n_0<n_k<n} b_k \prod_{n_0<i<n,\, i\neq n_k,\, k\in N} (1+l_i) + \sum_{n_0<n_k<n} e_k \prod_{n_k<n_j<n} b_j \prod_{n_k<i<n,\, i\neq n_j,\, j\in N} (1+l_i)$$

$$+ \sum_{i=n_0,\, i\neq n_k}^{n-1} \prod_{i<n_k<n} b_k \prod_{i<s<n,\, s\neq n_k} (1+l_s) q_i,\ n \geq n_0. \tag{2.1}$$

Proof: this lemma is a discrete version of Theorem 1.4.1 in [8] and Lemma 2.2 in [9]. The proof can be followed from mathematical induction and direct analysis:

If $n \in N[n_0, n_1]$, $m(n) \leq m(n_0) \prod_{n_0<i<n} (1+l_i) + \sum_{i=n_0}^{n-1} \prod_{i<s<n} (1+l_s) q_i$, obviously, for $n \in N[n_0, n_1]$, (2.1) holds. We might assume for $n \in N[n_0, n_p]$, (2.1) also holds, thus, for $n \in N(n_p, n_{p+1}]$, we get

$$\begin{aligned} m(n) &\leq m(n_p+1) \prod_{n_p<i<n} (1+l_i) + \sum_{i=n_p}^{n-1} \prod_{i<s<n} (1+l_s) q_i \\ &\leq \left(b_p m(n_p) + e_p\right) \prod_{n_p<i<n} (1+l_i) + \sum_{i=n_p}^{n-1} \prod_{i<s<n} (1+l_s) q_i, \end{aligned}$$

from the induction hypothesis, the above inequality turns into

$$m(n) \leq \Bigg\{ b_p \Bigg[m(n_0) \prod_{n_0<n_k<n_p} b_k \prod_{n_0<i<n_p,\, i\neq n_k,\, k\in N} (1+l_i) + \sum_{n_0<n_k<n_p} e_k \prod_{n_k<n_j<n_p} b_j \prod_{n_k<i<n_p,\, i\neq n_j,\, j\in N} (1+l_i)$$

$$+ \sum_{i=n_0,\, i\neq n_k}^{n_p-1} \prod_{i<n_k<n_p} b_k \prod_{i<s<n_p,\, s\neq n_k} (1+l_s) q_i \Bigg] + e_p \Bigg\} \prod_{n_p<i<n} (1+l_i) + \sum_{i=n_p}^{n-1} \prod_{i<s<n} (1+l_s) q_i,$$

which on simplification gives the estimate (2.1) for $n \in N[n_0, n_{p+1}]$. This completes the proof.

Lemma 2.2: Let $x(n)$ be a solution of Eq. (1.1). Suppose that there exists some $N^* \geq n_0$ such that $x(n) > 0$ for $n \geq N^*$, and the following conditions holds:

$(h1)(c1) - (c4)$;

$(h2)$ for all sufficiently large $n_j \geq n_1$ the following inequality hold:

$$\lim_{n\to\infty}\sum_{m=0}^{n-n_j}\frac{2^m}{(\alpha-1)^{(m+1)}a_{n_j+m}}\prod_{n_j\leq n_k\leq n_j+m} b_k = +\infty.$$

Then

$$\Delta_\alpha x(n_{k-1}) + \Delta x(n_{k-1}) \geq 0,\ \Delta_\alpha x(n) + \Delta x(n) \geq 0,\ n \in N[n_k, n_{k+1})(n_k - l > N^*).$$

Proof: Firstly, we show that

$$\Delta_\alpha x(n_k - 1) + \Delta x(n_k - 1) \geq 0$$

for any $n_k \geq N^*$. Otherwise, there exists some j such that

$$\Delta_\alpha x(n_j - 1) + \Delta x(n_j - 1) < 0$$

for *nj ¡* 1 > *N£*, from *Eq:*(1*:*1), (*c*2) and (*c*3), we get

$$\begin{aligned} a_{n_j}(\Delta_\alpha x(n_j) + \Delta x(n_j)) &= b_j(a_{n_j-1}\Delta_\alpha x(n_j-1)) + b_j(a_{n_j-1}\Delta x(n_j-1)) \\ &= b_j a_{n_j-1}(\Delta_\alpha x(n_j-1) + x(n_j-1)) < 0. \end{aligned}$$

Let

$$a_{n_j-1}(\Delta_\alpha x(n_j - 1) + \Delta x(n_j - 1)) = -\beta\ (\beta > 0),$$

From Eq. (1.1), for $n \in N(n_{j+i-1}, n_{j+i})$, $i = 1, 2, \ldots$, we have

$$\Delta S(n) = \Delta(a_{n-1}(\Delta_\alpha x(n-1) + \Delta x(n-1))) = -f(n, x(n-l)) + g(x(n)) \leq -p_n(x(n-l)).$$

Hence, $S(n)$ is monotonically decreasing in $N(n_{j+i-1}, n_{j+i})$. So,

$$\begin{aligned} a_{n_{j+1}-1}(\Delta_\alpha x(n_{j+1}-1) + \Delta x(n_{j+1}-1)) &\leq a_{n_j}(\Delta_\alpha x(n_j) + \Delta x(n_j)) \\ &= -b_j\beta < 0 \end{aligned}$$

and

$$\begin{aligned} a_{n_{j+2}-1}(\Delta_\alpha x(n_{j+2}-1) + \Delta x(n_{j+2}-1)) &\leq a_{n_{j+1}}(\Delta_\alpha x(n_{j+1}) + \Delta x(n_{j+1})) \\ &= b_{j+1}(a_{n_{j+1}-1}\Delta_\alpha x(n_{j+1}-1)) + b_{j+1}(a_{n_{j+1}-1}\Delta x(n_{j+1}-1)) \\ &= b_{j+1}a_{n_{j+1}-1}(\Delta_\alpha x(n_{j+1}-1) + \Delta x(n_{j+1}-1)) \leq -b_{j+1}b_j\beta \\ &< 0. \end{aligned}$$

By induction, we obtain

$$a_n\left(\Delta_\alpha x(n) + \Delta x(n)\right) \leq -\beta\prod_{n_j\leq n_k\leq n} b_k < 0,$$

that is

$$\Delta_\alpha x(n) + \Delta x(n) \le -\frac{\beta}{a_n} \prod_{n_j \le n_k \le n} b_k.$$

Now we go on the following calculation

$$x(n_j+1) - \frac{(\alpha+1)}{2} x(n_j) = x(n_j+1) - \frac{(\alpha+1) b_j}{2} \frac{a_{n_j-1}}{a_{n_j}} x(n_j-1) \le -\frac{\beta}{2a_{n_j}} b_j,$$

$$x(n_j+2) - \frac{(\alpha+1)^2}{4} x(n_j) \le \frac{(\alpha+1)\beta}{2\cdot 2a_{n_j}} b_j - \frac{\beta}{2a_{n_j+1}} \prod_{n_j \le n_k \le n_j+1} b_k,$$

$$x(n_j+3) - \frac{(\alpha-1)^3}{2^3} x(n_j) \le -\frac{(\alpha+1)^2 \beta}{2^2 \cdot 2a_{n_j}} b_j - \frac{(\alpha-1)\beta}{2\cdot 2a_{n_j+1}} \prod_{n_j \le n_k \le n_j+1} b_k - \frac{\beta}{2a_{n_j+2}} \prod_{n_j \le n_k \le n_j+2} b_k,$$

By induction, we obtain

$$x(n+1) \le \left(\frac{\alpha+1}{2}\right)^{n-n_j+1} b_j \frac{a_{n_j-1}}{a_{n_j}} x(n_j-1) - \left(\frac{\alpha+1}{2}\right)^{n-n_j} \frac{\beta}{2a_{n_j}} b_j$$

$$-\left(\frac{\alpha+1}{2}\right)^{n-n_j-1} \frac{\beta}{2a_{n_j+1}} \prod_{n_j \le n_k \le n_j+1} b_k - \left(\frac{\alpha+1}{2}\right)^{n-n_j-2} \frac{\beta}{2a_{n_j+2}} \prod_{n_j \le n_k \le n_n+2} b_k$$

$$-\ldots - \frac{\alpha-1}{2} \frac{\beta}{2a_{n-1}} \prod_{n_j \le n_k \le n-1} b_k - \frac{\beta}{2a_n} \prod_{n_j \le n_k \le n} b_k, \tag{2.2}$$

in view of $x(n) > 0$, it follows from ($h2$) that the right side of (2.2) converges to $-\infty$, however, the left side of (2.2) is eventually positive, which is a contradiction.

Therefore

$$\Delta_\alpha x(n_k - 1) + \Delta x(n_k - 1) \ge 0,\ n_k - 1 \ge N^*.$$

By ($c2$), for $\forall n_k \ge N^*$, $a_{n_k}(\Delta_\alpha x(n_k) + \Delta x(n_k)) = b_k a_{n_k-1}(\Delta_\alpha x(n_k-1) + \Delta x(n_k-1)) \ge 0$. Because $S(n)$ is monotonically decreasing in $N(n_{j+i-1}, n_{j+i}]$, we get $S(n) \ge 0$ for $n \in N(n_{j+i-1}, n_{j+i}]$, which implies

$$\Delta_\alpha x(n) + \Delta x(n) \geq 0.$$

This completes the proof.

Remark 2.3: Suppose that $x(n)$ is eventually negative, if $(h1)$ and $(h2)$ hold true, then we get $\Delta_\alpha x(n_k - 1) + \Delta x(n_k - 1) \leq 0$, $\Delta_\alpha x(n) + \Delta x(n) \leq 0$, $n \in N[n_k, n_{k+1})(n_k - l > N^*)$.

3. OSCILLATION CRITERIA

Theorem 3.1: Suppose that condition $(h1)$ and $(h2)$ hold, and for all sufficiently large n_j,

$$\sum_{i=n_j+1,\, i\neq n_k}^{n} p_i \prod_{n_j<n_k\leq i} \frac{\alpha+1}{2b_k} \to +\infty\left(n\to\infty\right) \tag{3.1}$$

holds. Then every solution of Eq. (1.1) is oscillatory.

Proof: If Eq. (1.1) has a nonoscillatory solution $x(n)$, Without loss of generality, we might assume that $x(n) > 0(n > n_0)$. From lemma 2.2, get $\Delta_\alpha x(n) + \Delta x(n) \geq 0$, $n \in N[n_k, n_{k+1}](n_1 > n_0 + l)$, $k = 1, 2, \ldots$.

Let

$$w(n) = \frac{a_{n-1}\left(\Delta_\alpha x(n-1) + \Delta x(n-1)\right)}{x(n-l)}.$$

Then

$$w(n_k) \geq 0(k = 1, 2, \ldots),\ w(n) \geq 0(n \geq n_0).$$

Using Eq. (1.1) and $(c1)$, we get

$$\begin{aligned}\Delta w(n) &= \frac{a_n\left(\Delta_\alpha x(n) + \Delta x(n)\right)}{x(n+1-l)} - \frac{a_{n-1}\left(\Delta_\alpha x(n-1) + \Delta x(n-1)\right)}{x(n-l)} \\ &= \frac{\Delta\left(a_{n-1}\left(\Delta_\alpha x(n-1) + \Delta x(n-1)\right)\right)}{x(n-1)} - \frac{a_n\left(\Delta_\alpha x(n) + \Delta x(n)\right)}{x(n-l)} \\ &\quad + \frac{a_n\left(\Delta_\alpha x(n) + \Delta x(n)\right)}{x(n+1-l)}\end{aligned}$$

$$= \frac{-f\left(n, x(n-l) + g\left(x(n)\right)\right)}{x(n-l)} - \frac{a_n\left(\Delta_\alpha x(n) + \Delta x(n)\Delta x(n-l)\right)}{x(n-l)x(n+1-l)} \leq -p_n,$$

$$w(n_k + 1) = \frac{a_{n_k}\left(\Delta_\alpha x(n_k) + \Delta_x(n_k)\right)}{x(n_k+1-l)}$$

$$= \frac{b_k a_{n_k-1}\left(\Delta_\alpha x(n_k-1) + \Delta x(n_k-1)\right)}{x(n_k+1-l)} \leq \frac{2b_k}{(\alpha+1)} w(n_k),$$

It follows from the above inequalities that $w(n)$ satisfies the following difference inequalities

$$\begin{cases} \Delta w(n) \leq -p_n, n \neq n_k, k \in N, \\ w(n_k+1) \leq \dfrac{2b_k}{(\alpha+1)} w(n_k). \end{cases}$$

Applying Lemma 2.1, we have

$$w(n+1) \leq w(n_j) \prod_{n_j < n_k \leq n} \frac{2b_k}{(\alpha+1)} - \sum_{i=n_j+1, i\neq n_k}^{n} p_i \prod_{i<n_k\leq n} \frac{2b_k}{(\alpha+1)}$$

$$\prod_{n_j<n_k\leq n} \frac{2b_k}{\alpha+1}\left(w(n_j) - \sum_{i=n_j+1, i\neq n_k}^{n} p_i \prod_{n_i<n_k\leq i} \frac{\alpha+1}{2b_k}\right), n \geq n_j. \tag{3.2}$$

By (3.1), (3.2) and $w(n) > 0$, we can draw a contradiction as $n \to \infty$. Hence, every solution of Eq. (1.1) is oscillatory. This completes the proof.

Corollary 3.2: Assume that ($h1$) and ($h2$) hold and there exists a positive integer k_0 such that $\alpha + 1 \geq 2b_k$ for $k \geq k_0$. If

$$\sum_{n\neq n_k, k\in N} p_n = +\infty, \tag{3.3}$$

then every solution of Eq. (1.1) is oscillatory.

Proof: Without loss of generality. Let $k_0 = 1$, it follows from $\alpha + 1 \geq 2b_k$ that

$$\sum_{i=n_j+1,\, i\neq n_k,\, k\in N}^{n} p_i \prod_{n_j<n_k\leq i} \frac{\alpha+1}{2b_k} \geq \sum_{i=n_j+1,\, i\neq n_k,\, k\in N}^{n} p_i. \tag{3.4}$$

Let $n \to \infty$, applying (3.3) and (3.4), we get (3.1). According to Theorem 3.1 we obtain that every solution of Eq. (1.1) is oscillatory.

Corollary 3.3: Assume that ($h1$) and ($h2$) hold and there exist a positive integer k_0 and a positive constant λ such that $\frac{\alpha+1}{2} \geq \left(\frac{n_{k+1}}{n_k}\right)^{\lambda} b_k$ for $k \geq k_0$. If

$$\sum_{n\neq n_k,\, k\in N} n^{\lambda} p_n = +\infty, \tag{3.5}$$

then, every solution of Eq. (1.1) is oscillatory.

Proof: Without loss of generality. Let $k_0 = 1$, it follows from $\frac{\alpha+1}{2} \geq \left(\frac{n_{k+1}}{n_k}\right)^{\lambda} b_k$

$$\begin{aligned}\sum_{i=n_0,\, i\neq n_k,\, k\in N}^{n} p_i \prod_{n_0\leq n_k\leq i} \frac{\alpha+1}{2b_k} &= \sum_{i=n_0}^{n_1-1} p_i + \frac{\alpha+1}{2b_1}\sum_{i=n_1+1}^{n_2-1} p_i + \cdots + \frac{(\alpha+1)^j}{2b_1 2b_2 2b_3 \cdots 2b_j} \\ &\quad \times \sum_{i=n_j+1,\, i\neq n_k,\, k\in N[j+1,\infty)}^{n} p_i \geq \frac{\alpha+1}{2b_1}\sum_{s=n_1+1}^{n_2-1} p_s \\ &\quad + \cdots + \frac{(\alpha+1)^j}{2b_1\, 2b_2\, 2b_3 \cdots 2b_j} \sum_{s=n_j+1,\, i\neq n_k,\, k\in N[j+1,\infty)}^{n} p_s \\ &\geq \frac{1}{n_1^{\lambda}}\left[\sum_{s=n_1+1}^{n_2-1} n_2^{\lambda} p_s + \cdots + \sum_{s=n_j+1}^{n} n_{j+1}^{\lambda} p_s\right] \\ &\geq \frac{1}{n_1^{\lambda}} \sum_{s=n_1+1,\, s\neq n_k,\, k\in N}^{n} s^{\lambda} p_s,\ n\in N(n_j, n_{j+1}).\end{aligned}$$

Let $n \to \infty$, applying (3.5), we get (3.1). According to Theorem 3.1, we get every solution of Eq. (1.1) is oscillatory.

Remark 3.4: Using the similar method, we can discuss oscillation criteria for the following advanced and mixed difference equations with impulsive effect:

$$\begin{cases} \Delta\left(a_{n-1}\left(\Delta_\alpha x(n-1)+\Delta x(n-1)\right)\right)+f\left(n, x(n+l)\right)=0, \alpha>0, n \neq n_k, k \in N, \\ a_{n_k}\Delta_\alpha x(n_k)=b_k\left(a_{n_k-1}\left(\Delta_\alpha x(n_k-1)\right)\right), \\ a_{n_k}\Delta x(n_k)=b_k\left(a_{n_k-1}\left(\Delta x(n_k-1)\right)\right), \end{cases}$$

$$\begin{cases} \Delta\left(a_{n-1}\left(\Delta_\alpha x(n-1)+\Delta x(n-1)\right)\right)+f\left(n, x(n+l)\right)=g\left(x(n)\right), \alpha>0, n \neq n_k, k \in N, \\ a_{n_k}\Delta_\alpha x(n_k)=b_k\left(a_{n_k-1}\left(\Delta_\alpha x(n_k-1)\right)\right), \\ a_{n_k}\Delta x(n_k)=b_k\left(a_{n_k-1}\left(\Delta x(n_k-1)\right)\right), \end{cases}$$

$$\begin{cases} \Delta\left(a_{n-1}\left(\Delta_\alpha x(n-1)+\Delta x(n-1)\right)\right)+f\left(n, x(n), x(n+l)\right)=0, \alpha>0, n \neq n_k, k \in N, \\ a_{n_k}\Delta_\alpha x(n_k)=b_k\left(a_{n_k-1}\left(\Delta_\alpha x(n_k-1)\right)\right), \\ a_{n_k}\Delta x(n_k)=b_k\left(a_{n_k-1}\left(\Delta x(n_k-1)\right)\right), \end{cases}$$

$$\begin{cases} \Delta\left(a_{n-1}\left(\Delta_\alpha x(n-1)+\Delta x(n-1)\right)\right)+f\left(n, x(n), x(n+l)\right)=g\left(x(n)\right), \alpha>0, n \neq n_k, k \in N, \\ a_{n_k}\Delta_\alpha x(n_k)=b_k\left(a_{n_k-1}\left(\Delta_\alpha x(n_k-1)\right)\right), \\ a_{n_k}\Delta x(n_k)=b_k\left(a_{n_k-1}\left(\Delta x(n_k-1)\right)\right), \end{cases}$$

$$\begin{cases} \Delta\left(a_{n-1}\left(\Delta_\alpha x(n-1)+\Delta x(n-1)\right)\right)+f\left(n, x(n-l), x(n+l)\right)=g\left(x(n)\right), \alpha>0, n \neq n_k, k \in N, \\ a_{n_k}\Delta_\alpha x(n_k)=b_k\left(a_{n_k-1}\left(\Delta_\alpha x(n_k-1)\right)\right), \\ a_{n_k}\Delta x(n_k)=b_k\left(a_{n_k-1}\left(\Delta x(n_k-1)\right)\right), \end{cases}$$

4. EXAMPLES

Example 1: Consider impulsive delay difference equation:

$$\begin{cases} \Delta\left(\frac{1}{n-1}\left(\Delta_{\frac{1}{2}}x(n-1)+\Delta x(n-1)\right)\right)+\frac{\ln\left[\left(\frac{n}{n-1}\right)^{\frac{3}{2(n-1)}}\frac{n^{\frac{7n-3}{3n}}}{(n+1)^{\frac{2}{n}}}\right]-\frac{1}{e^n}}{\ln(n-l)}x(n-l) \\ =-\frac{1}{e^n}\operatorname{sgn}(x(n)), n\neq 2k, k\in N, \\ \frac{1}{2k}\Delta_{\frac{1}{2}}x(2k)=\frac{3k-1}{4k}\left(\frac{1}{2k-1}\Delta_{\frac{1}{2}}x(2k-1)\right), \\ \frac{1}{2k}\Delta x(2k)=\frac{3k-1}{4k}\left(\frac{1}{2k-1}\Delta x(2k-1)\right), \end{cases} \tag{4.1}$$

in which $a_n=\frac{1}{n}, \alpha=\frac{1}{2}, p_n=\frac{\ln\left[\left(\frac{n}{n-1}\right)^{\frac{3}{2(n-1)}}\frac{n^{\frac{7n-3}{3n}}}{(n+1)^{\frac{2}{n}}}\right]-\frac{1}{e^n}}{\ln(n-l)}, b_k=\frac{3k-1}{4k}$, applying Corollary 3.2, we get every solution of Eq. (4.1) is oscillatory. But the delay difference equation

$$\Delta\left(\frac{1}{n-1}\left(\Delta_{\frac{1}{2}}x(n-1)+\Delta x(n-1)\right)\right)+\frac{\ln\left[\left(\frac{n}{n-1}\right)^{\frac{3}{2(n-1)}}\frac{n^{\frac{7n-3}{3n}}}{(n+1)^{\frac{2}{n}}}\right]-\frac{1}{e^n}}{\ln(n-l)}x(n-l)$$

$$=-\frac{1}{e^n}\operatorname{sgn}(x(n)), n\neq 2k, k\in N,$$

has a nonoscillatory solution $x(n) = \ln n$.

Example 2: Consider impulsive delay difference equation:

$$\begin{cases} \Delta\left(\frac{1}{e^n}\left(\Delta_2 x(n-1)+\Delta x(n-1)\right)\right)+n^{3s-1}x(n-l) \\ =-\exp(x(n))\operatorname{sgn}(x(n)),\, n\neq 2k,\, k\in N, \\ \frac{1}{e^{2k}}\Delta_2 x(2k)=\frac{3k}{2k+3}\left(\frac{1}{e^{(2k-1)}}\left(\Delta_2 x(2k-1)\right)\right), \\ \frac{1}{e^{2k}}\Delta x(2k)=\frac{3k}{2k+3}\left(\frac{1}{e^{(2k-1)}}\left(\Delta x(2k-1)\right)\right), \end{cases} \tag{4.2}$$

in which $s(>1)$ is integer, $a_n=\frac{1}{e^n}, \alpha=2, \gamma=1, p_n=n^{3s-1}, b_k=\frac{3k}{2k+3}$, applying Corollary 3.3, we derive that every solution of Eq. (4.2) is oscillatory.

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