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EIGENVALUES AND POSTIVE SOLUTION FOR A SECOND-ORDER THREE-POINT BOUNDARY VALUE PROBLEM*

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ABSTRACT: In this paper, we discuss the following second-order three-point boundary value probelm

$$u''(t) = f(t,u,u',u''),\ 0 \le t \le 1$$

$$u'(0) = \alpha u(0),\ u(\eta) = \eta u(1),$$

where $\alpha \in (0,\infty), \eta \in (0,1); f:[0,1]\times[0,\infty)^3 \to [0,\infty)$ is continuous, in which the second-order derivative may occur nonlinearly and $f(t,0,0,0) \not\equiv 0$, $t \in [0, 1]$. We show that under the appropriate growth cnditinos on the inhomogeneous term and by the using Schauder fixed point theorem the problem has at least one positive soulution. As an application of the main result, we also study the existence of positive eigenvalues for this problem. The emphasis in this work is the highest-order derivative occurs nolinearly in our problem.

Keywords: positive solution; three-point boundary value problem; fixed point theorem; eigenvalue

AMS Subject Classification: 34B10, 34B15

1. INTRODUCTION

In this paper we study the existence of positive solution of the following second-order three-point boundary value problem (BVP):

$$u''(t) = f(t,u,u',u''),\ 0 \le t \le 1 \tag{1.1}$$

$$u'(0) = \alpha u(0),\ u(\eta) = \eta u(1), \tag{1.2}$$

where $\alpha \in (0,\infty), \eta \in (0,1); f:[0,1]\times[0,\infty)^3 \to [0,\infty)$ is continuous, in which the second-order derivative may occur nonlinearly and $f(t,0,0,0) \not\equiv 0$, $t \in [0, 1]$.

The three-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. In the past few years, there has been much attention focused on questions of positive solutions of

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three-point boundary value problems for nonlinear ordinary differential equations. The main approach is rerformulate the problem to an operator equation of the form $u = Tu,$ where T is a suitable oprator and then by applying the Leray-Schauder continuation theorem, nonlinear alternative of Leray-Schauder, coincidence degree theory, Krasnoselskii's fixed point theorem, or Leggett-Williams fixed point theorem, for example, see [2-12, 14-17] and references therein. The aim of the present paper is to establish a simple criterion for the existence of positive solution to the BVP (1.1), (1.2) by Schauder fixed point theorem. The emphasis in this work is the highest -order derivative occurs nonlinearly.

The organization of this paper is as follows. In Section 2, we present some preliminary results that will be used to prove our main results. In Section 3, we discuss the existence of positive solution for the BVP (1.1), (1.2), the existence of positive eigenvalues for this problem and the give two examples to illustrate our results.

2. PRELIMINARIES

In this section, we shall give some preliminary results. Consider the three-point BVP:

$$u'(t) = h(t),\ 0 \le t \le 1, \tag{2.1}$$

$$u'(0) = \alpha u(0),\ u(\eta) = \eta u(1), \tag{2.2}$$

where $\eta \in (0, 1)$.

Lemma 2.1 *For any* $\alpha \in \mathbf{R}$, $h \in C[0,1]$, *the three-point BVP* (2.1), (2.2) *has a unique solution*

$$u(t) = \int_0^t (t-s)h(s)ds + (1+\alpha t)\left(\int_0^\eta sh(s)ds + \frac{\eta}{1-\eta}\int_\eta^1 (1-s)h(s)ds\right) \tag{2.3}$$

Proof. For $t \in [0,1]$, integrate (2.1) from 0 to t we get

$$u'(t) = \int_0^t h(s)ds + B$$

For $t \in [0,1]$, integrate from 0 to t yields

$$u(t) = \int_0^t \left(\int_0^x h(s)ds\right)dx + Bt + A,$$

i.e.

$$u(t) = \int_0^t (t-s)h(s)ds + Bt + A.$$

So,

$$u(0) = A,$$

$$u'(0)=Bu'(0) = B,$$

$$u(1) = \int_0^1 (1-s)h(s)ds + B + A,$$

$$u(\eta) = \int_0^\eta (\eta - s)h(s)ds + \eta B + A.$$

Combining with (2.2) we conclude that

$$A = \frac{1}{1-\eta}\left(\eta\int_0^1 (1-s)h(s)ds - \int_0^\eta (\eta-s)h(s)ds\right)$$

$$= \frac{1}{1-\eta}\left((\eta-1)\int_0^\eta sh(s)ds + \eta\int_\eta^1 (1-s)h(s)ds\right)$$

$$= \int_0^\eta sh(s)ds + \frac{\eta}{1-\eta}\int_\eta^1 (1-s)h(s)ds,$$

$$B = \alpha\int_0^\eta sh(s)ds + \frac{\alpha\eta}{1-\eta}\int_\eta^1 (1-s)h(s)ds.$$

Therefore, the three-point BVP (2.1), (2.2) has a unique solution

$$u(t) = \int_o^t (t-s)h(s)ds + (1+\alpha t)\left(\int_0^\eta sh(s)ds + \frac{\eta}{1-\eta}\int_\eta^1 (1-s)h(s)ds\right).$$

This completes the proof.

From (2.3) we have

Lemma 2.2 *Let* $\alpha \in (0, \infty)$, $h \in C^+[0, 1]$. *Then the unique solution* $u(t)$ *of the BVP* (2.1), (2.2) *is nonnegative on* [0,. 1] *and if* $h(t) \not\equiv 0$, *then* $u(t) > 0$, $t \in [0, 1]$**.**

Define an integral operator $T : E \to E$ *by*

$$Tu(t) = (1+\alpha t)\left(\int_0^\eta sf(s,u,u',u'')ds + \frac{\eta}{1-\eta}\int_\eta^1 (1-s)f(s,u,u'u'')ds\right)$$

$$+\int_0^t (t-s)f(s,u,u'.u'')ds,\ \ t\in[0,1]. \tag{2.4}$$

It is easy to see that the BVP(1.1), (1.2) has a solution $u = u(t)$ if and only if u is a fixed point of the operator T defined by (2.4).

Theorem 2.1 *Let E be Banach space and $B \subset E$ be a bounded closed conver subset, $T : E \to E$ be a completely continuous operator such that*

$$T(B) \subset B.$$

Then T has a fixed point in B.

3. MAIN RESULTS

In this section, we study the existence of positive solution for the BVP (1.1), (1.2). We obtain the following existence results.

Theorem 3.1 *Assume that* $\alpha \in (0,\infty), f : [0,1] \times [0,\infty)^3 \to [0,\infty)$ *is continuous and* $f(t,0,0,0) \not\equiv 0,\ t \in [0,1]$. *If there exist constant* $M > 0$ such that

$$(2 + \alpha\eta)\, L \le 2\mathrm{M}$$

where

$$L = max\, \{f(t,u,v,w)\, |t \in [0,1], u, v, w \in [0, M]\}.$$

Then the BVP (1.1), (1.2) *has at least one positive solution.*

Proof Let $E = C^2[0,1]$ be a Banach space with norm $\|u\| = \max\,\{|u|_0, |u'|_0, |u''|_0\}$, where $|u|_0 = \max_{0 \le t \le 1}\ |u(t)|$, $B = \{u(t)|u(t)C^2[0,1],\ u(t), u'(t), u''(t) \in [0, M]\}$, then B is a bounded closed convex subset of E. It is easy to see that the operator T defined by (2.4) is completely continuous. Now we prove

$$T(B) \subset B. \tag{3.1}$$

In fact, for any $u \in B$, from (2.4) we have

$$\begin{aligned}
0 \le Tu(t) &= (1+\alpha t)\left(\int_0^\eta sf(s,u,u',u'')ds + \frac{\eta}{1-\eta}\int_\eta^1 (1-s)f(s,u,u',u'')ds\right) \\
&\quad + \int_0^t (t-s)f(s,u,u',u'')ds \\
&\le (1+\alpha)\left(\int_0^\eta sLds + \frac{\eta}{1-\eta}\int_\eta^1 (1-s)Lds\right) + \int_0^1 (1-s)Lds \\
&= (1+\alpha)\,.\,\frac{1}{2}\eta L + \frac{1}{2}L \\
&= \frac{1}{2}(1+\eta+\alpha\eta)L \\
&< \frac{1}{2}(2+\alpha\eta)L \le M,
\end{aligned}$$

which means

$$0 \le Tu(t) \le M, \ \forall t \in [0,1]. \tag{3.2}$$

In addition,

$$0 \le (Tu)'(t) = \alpha\left(\int_0^\eta sf(s,u(s),u'(s))ds + \frac{\eta}{1-\eta}\int_\eta^1 (1-1)f(s,u(s),u'(s))ds\right)$$

$$+\int_0^t f(s,u(s),u',(s))ds$$

$$\le \alpha\left(\int_0^\eta sLds + \frac{\eta}{1-\eta}\int_\eta^1 (1-s)Lds\right) + \int_0^1 Lds$$

$$= \frac{1}{2}\alpha\eta L + L$$

$$= \frac{1}{2}(2+\alpha\eta)L \le M,$$

which means

$$0 \le (Tu)'(t) \le M, \ \forall t \in [0,1]. \tag{3.3}$$

Further,

$$0 \le (Tu)''(t) = f(t,u,u',u'') \le L < L < \frac{1}{2}(2+\alpha\eta)L \le M, \ \forall t \in [0,1]. \tag{3.4}$$

Therefore, from (3.2), (3.3) and (3.4) we get

$$Tu \in B,$$

which impiles (3.1) holds. Thus, by Schauder fixed theorem, T has a fixed point $u^* \in B$. From $f(t,0,0,0) \not\equiv 0,\ t \in [0,1]$ we know u^* is a positive solution of the BVP (1.1), (1.2).

Example 3.1 Consider the three-point BVP

$$u''(t) = \frac{1}{2}(1-t^2)(1+\sqrt{u}) + \frac{1}{9}(1-t)u' + \frac{2t}{81}(u'')^2, \quad 0 \le t \le 1, \tag{3.5}$$

$$u'(0) = 2u(0), \quad u(\frac{1}{2}) = \frac{1}{2}u(1). \tag{3.6}$$

Set $f(t,u,v,w) = \frac{1}{2}(1-t^2)(1+\sqrt{u}) + \frac{1}{9}(1-t)v + \frac{2t}{81}w^2,\ M = 9.$ A direct computation shows $L = 5$ and $(2 + \alpha\eta)\ L = 15 < 18 = 2M$, by Theorem 3.1, the BVP (3.5), (3.6) has at least one positive solution.

As an application of Theorem 3.1 we consider the following eigenvalue problem:

$$u''(t) = \lambda f(t,u,u',u''),\ 0 \le t \le 1, \tag{3.7}$$

$$u'(0) = \alpha u(0), u(\eta) = \eta u(1). \tag{3.8}$$

We obtain the following existence result.

Theorem 3.2 *Assume that* $\alpha \in (0,\infty), \eta \in (0,1)$ *are constants,* $\lambda > 0$ *is parameter,* $f: [0,1] \times [0,\infty)^3 \to [0,\infty)$ *is continuous and* $f(t,0,0,0) \not\equiv 0$, $t \in [0,1]$. *Then for each* $\lambda \in (0, \frac{2M}{(2+\alpha\eta)L}]$, *where* $M > 0$ *and* $L = \max\{f(t,u,v,w) \mid t \in [0,1], u,v,w \in [0, M]\}$, *the three point BVP* (3.7), (3.8) *has at least one positive solution.*

Example 3.2 Consider the three-point BVP

$$u''(t) = \lambda\left[1 + \frac{1}{7}t^2(2+3\sqrt{u}) + \frac{1}{32}(1+t)u' + \frac{t}{16}u''\right],\ 0 \le t \le 1, \tag{3.9}$$

$$u'(0) = 3u(0),\ u(\frac{2}{5}) = \frac{2}{5}u(1). \tag{3.10}$$

Set $f(t,u,v,w) = 1 + \frac{1}{7}t^2(2+3+\sqrt{u}) + \frac{1}{32}(1+t)v + \frac{t}{16}w,\ M = 16.$ A direct computation shows $L = 5$ and $(2+\alpha\eta)L = 16,$ by Theorem 3.2, for each $\lambda \in (0,2],$ the BVP (3.9), (3.10) has at least one positive solution.

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