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THE GENERALIZED CESÀRO OPERATOR ON THE UNIT BALL IN C^n

Stevo Stević

ABSTRACT: We define the generalized Cesàro operator $C_{\zeta_0}^{\gamma}$ on the space of holomorphic functions on the unit ball $B \subset C^n$ as follows

$$C_{\zeta_0}^{\gamma}(f)(z) = (\gamma+1)\int_0^1 f(tz)\frac{(1-t)^{\gamma}}{(1-\langle tz,\zeta_0\rangle)^{\gamma+1}}\,dt,$$

where $\Re\gamma > -1$ and $\zeta_0 \in \partial B$

The boundedness of the operator on some spaces of holomorphic functions on the unit ball is considered.

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1. INTRODUCTION

Let U be the unit disc in the complex plane C, $dm(z) = rdr\frac{d\theta}{\pi}$ the normalized Lebesgue area measure on U and $H(U)$ the space of all analytic functions in U.

For each complex γ with $\Re\gamma > -1$ and k nonnegative integer let A_k^{γ} be defined as the kth coefficient in the expression

$$\frac{1}{(1-x)^{\gamma+1}} = \sum_{k=0}^{\infty} A_k^{\gamma} x^k,$$

so that $A_k^{\gamma} = \frac{(\gamma+1)...(\gamma+k)}{k!}$

For an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on U, the generalized Cesàro operator is defined by

$$C^{\gamma}(f)(z)=\sum_{n=0}^{\infty}\left(\frac{1}{A_n^{\gamma+1}}\sum_{k=0}^{n}A_{n-k}^{\gamma}a_k\right)z^n. \tag{1}$$

These operators were introduced in [12] on Hardy spaces and have been subsequently studied and proved bounded on all Hardy spaces in [17]. The boundedness of the operator on other spaces of analytic functions were considered in [15], [16] and [17]. For $\gamma = 0$ we obtain the classical Cesàro operator $C^0 = C$, which was investigated, for example, in [1, 2, 7, 8, 9, 10, 11]. Adjoint of Cesàro operator was investigated in [8, 9, 17].

The integral form of C^{γ} is (see [12])

$$C^{\gamma}(f)(z)=\frac{\gamma+1}{z^{\gamma+1}}\int_0^z f(\zeta)\frac{(z-\zeta)^{\gamma}}{(1-\zeta)^{\gamma+1}}\,d\zeta$$

or, taking simply as a path the segment joining 0 and z,

$$C^{\gamma}(f)(z)=(\gamma+1)\int_0^1 f(tz)\frac{(1-t)^{\gamma}}{(1-tz)^{\gamma+1}}\,dt.$$

Closely related operators on the polydisc were investigated in [13, 14].

Let $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)$ be points in complex vector space $\mathbf{C}^n$. For a holomorphic function f we denote

$$\nabla f=\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right).$$

The aim of this note is to define a generalized Cesàro operator on the unit ball $B\subset \boldsymbol{C}^n$ and to prove its boundedness on some spaces of analytic functions on B. The class of all analytic functions on B will be denoted by $H(B)$.

We define the generalized Cesàro operator on $H(B)$ as follows

$$C_{\zeta_0}^{\gamma}(f)(z)=(\gamma+1)\int_0^1 f(tz)\frac{(1-t)^{\gamma}}{(1-\langle tz,\zeta_0\rangle)^{\gamma+1}}\,dt,\ f\in H(B),\ z\in B, \tag{2}$$

where γ is a complex number such that $\Re g > -1$, ζ_0 is a fixed point which lies on the boundary ∂B of the unit ball B and

$$\langle z,w\rangle = z_1w_1+\cdots+z_nw_n.$$

The Hardy space $H^p(B)$ $(0 < p < \infty)$ is defined on B by

$$H^p(B) = \{f \mid f \in H(B) \text{ and } \|f\|_{H^p(B)} < \infty\},$$

where

$$\|f\|_{H^p(B)}^p = \sup_{0 \le r < 1} \int_{\partial B} |f(r\zeta)|^p \, d\sigma(\zeta),$$

where $d\sigma$ is the normalized surface measure on ∂B.

The weighted Bergman space $\mathcal{A}_\alpha^p(B), \alpha > -1, p > 0,$ is the space of all analytic functions f on B for which

$$\|f\|_{\mathcal{A}_\alpha^p(B)} = \left[\int_B |f(z)|^p \, (1 - |z|^2)^\alpha \, dV(z)\right]^{1/p}$$

$$= \left[2n \int_0^1 M_p^p(f, r)(1 - r^2)^\alpha r^{2n-1} dr\right]^{1/p} < \infty,$$

where $M_p(f, r) = \left(\int_{\partial B} |f(r\zeta)|^p \, d\sigma(\zeta)\right)^{1/p}$ and dV is the normalized Lebesgue's measure on the unit ball B.

Let $a > 0$. The a-Bloch space $\mathcal{B}^a = \mathcal{B}^a(B)$ is the space of all analytic functions f on B such that

$$b_a(f) = \sup_{z \in B} (1 - |z|^2)^a \, |\nabla f(z)| < \infty.$$

It is clear that $\mathcal{B}^a$ is a normed space, modulo constant functions and $\mathcal{B}^{az} \subset \mathcal{B}^{az}$ for $a_1 < a_2$.

In this note we prove the following results.

Theorem 1. *The operator* $\mathcal{C}_{\zeta_0}^{\gamma}$ *is bounded on* $H^p(B)$ *if* $p \in (0,1]$.

Theorem 2. *The operator* $\mathcal{C}_{\zeta_0}^{\gamma}$ *is bounded on* $\mathcal{A}_\alpha^p(B)$ *if* $\alpha > -1$ *and* $p \in (0,1]$.

Theorem 3. *The operator* $\mathcal{C}_{\zeta_0}^{\gamma}$ *is bounded on* $B^a(B)$ *if* $a \in (1,\infty)$.

2. AUXILIARY RESULTS

In order to prove our main results, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 1. ([3])*If $p \in (0,\infty)$ and $f \in H^p(B)$, then there is a constant* **C** *depending only on p and n such that*

$$\int_{\partial B} \sup_{0\le r<1} |f(r\zeta)|^p \, d\sigma(\zeta) \le C \| f \|^p_{H^p(B)}.$$

Lemma 2. ([6, 1.4.10]) Let

$$I_c(z) = \int_{\partial B} \frac{d\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+c}}$$

Then for $c > 0$ the following relationship holds

$$I_c(z) \asymp (1-|z|)^{-c.}$$

The above means that there are finite positive constants C and C' such that

$$C(1-|z|)^{-c} \le I_c(z) \le C'(1-|z|)^{-c}.$$

Lemma 3. ([4, p.29]) *Let $0 < p < q \le \infty$, $p \le \lambda$, $f \in H^p(B)$ and* $1 = n\left(\frac{1}{p}-\frac{1}{q}\right)$.

Then there is a constant C independent of f such that

$$\int_0^1 M_q^\lambda(f,r)(1-r)^{\lambda s-1} dr \le C \| f \|^p_{H^p(B)},$$

for all $f \in H(B)$.

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Let $p \in (0, 1]$. Without loss of generality we may assume that γ is a real number greater then -1. Let $f \in H^p(B)$, $0 \le r < 1$, $t_k = 1 - 2^{-k}$, $k = 0, 1, \ldots$ and

$F_k(r\zeta) = \sup_{t_{k-1}<t<t_k} | f(tr\zeta)/(1-\langle tr\zeta,\zeta_0\rangle)^{\gamma+1} |, \zeta \in \partial B.$

Then

$$\frac{M_p^p(C_{\zeta_0}^\gamma, r)}{(\gamma+1)^p} = \int_{\partial B} \left| \int_0^1 \frac{f(tr\zeta)(1-t)^\gamma}{(1-\langle tr\zeta,\zeta_0\rangle)^{\gamma+1}} dt \right|^p d\sigma(\zeta)$$

$$\leq \int_{\partial B}\left(\int_0^1\left|\frac{f(tr\zeta)(1-t)^{\gamma}}{(1-\langle tr\zeta,\zeta_0\rangle)^{\gamma+1}}\right|dt\right)^p d\sigma(\zeta)$$

$$=\int_{\partial B}\left(\sum_{k=1}^{\infty}\int_{t_{k-1}}^{t_k}\left|\frac{f(tr\zeta)}{(1-\langle tr\zeta,\zeta_0\rangle)^{\gamma+1}}\right|(1-t)^{\gamma}dt\right)^p d\sigma(\zeta)$$

$$\leq C\int_{\partial B}\left(\sum_{k=1}^{\infty}F_k(r\zeta)2^{-k(\gamma+1)}\right)^p d\sigma(\zeta)$$

$$\leq C\sum_{k=1}^{\infty}2^{-k(\gamma+1)p}\int_{\partial B}F_k^p(r\zeta)d\sigma(\zeta)$$

$$\leq C\sum_{k=1}^{\infty}2^{-k(\gamma+1)p}\int_{\partial B}\sup_{0<t<t_k}\left|\frac{f(tr\zeta)}{(1-\langle tr\zeta,\zeta_0\rangle)^{\gamma+1}}\right|^p d\sigma(\zeta)$$

$$\leq C\sum_{k=1}^{\infty}2^{-k(\gamma+1)p}M_p^p\left(\frac{f}{(1-\langle\cdot,\ \zeta_0\rangle)^{\gamma+1}}t_k r\right)$$

$$\leq C\sum_{k=1}^{\infty}\int_{t_k}^{t_{k+1}}(1-t)^{p(\gamma+1)-1}M_p^p\left(\frac{f}{(1-\langle\cdot,\ \zeta_0\rangle)^{\gamma+1}}tr\right)dt$$

$$\leq C\int_0^1(1-t)^{p(\gamma+1)-1}M_p^p\left(\frac{f}{(1-\langle\cdot,\zeta_0\rangle)^{\gamma+1}},tr\right)dt$$

Now, choose $a>1$ such that $1-\dfrac{p(\gamma+1)}{n}<1/a<1.$ By the Hölder inequality we obtain

$$M_p^p\left(\frac{f}{(1-\langle z,\zeta_0\rangle)^{\gamma+1}},tr\right)\ \leq\left(\int_{\partial B}|f(tr\zeta)|^{ap}\ d\sigma(\zeta)\right)^{1/a}\left(\int_{\partial B}|1-\langle rt\zeta,\zeta_0\rangle|^{-\frac{(\gamma+1)pa}{a-1}}\ d\sigma(\zeta)\right)^{\frac{a-1}{a}}$$

$$\leq \left(\int_{\partial B} |f(tr\zeta)|^{ap}\, d\sigma(\zeta)\right)^{1/a} (1-tr)^{n(a-1)/a-(\gamma+1)p}$$

Hence

$$M_p^p(C_{\zeta_0}^{\gamma}, r) \leq C\int_0^1 M_{ap}^p(f_r, t)(1-t)^{n\frac{a-1}{a}-1}\, dt$$

$$\leq C\, \| f_r \|_p^p = CM_p^p(f, r), \tag{3}$$

as desired.

Remark 1. From the proof of Theorem 1 we see that we have proved a stronger result, that is, that $M_p(C_{\zeta_0}^{\gamma}, r)$ is dominated by a constant multiple of $M_p(f, r)$.

Proof of Theorem 2. Multiplying inequality (3) by $(1-r^2)^{\alpha} r^{2n-1}$ and then integrating obtained inequality in r from 0 to 1 we obtain the result.

Given $0 < p, q < \infty$, and positive Borel measure μ, on $r \in (0, 1)$, the weighted space $\mathcal{A}_{\mu}^{p,q}(B)$ consists of those functions f analytic on B for which

$$\| f \|_{\mathcal{A}_{\mu}^{p,q}(B)} = \left[\int_0^1 \left(\int_{\partial B} |f(r\zeta)|^p\, d\sigma(\zeta)\right)^{\frac{q}{p}} d\mu(r)\right]^{1/q} < \infty.$$

Similar to Theorem 2 we can prove the following result:

Corollary 1. *The generalized Cesàro operator is bounded on* $\mathcal{A}_{\mu}^{p,q}(B)$ *for* $p,q >$ 0. *Moreover, there is a constant C independent of f, such that*

$$\| C_{\zeta_0}^{\gamma}(f) \|_{\mathcal{A}_{\mu}^{p,q}(B)} \leq C\, \| f \|_{\mathcal{A}_{\mu}^{p,q}(B)}.$$

Before proving Theorem 3 we need another auxiliary result which is incorporated in the following lemma.

Lemma 4. *Let* $\gamma > -1$, $a > 0$ *and* $r \in (0, 1)$. *Then*

$$\int_0^1 \frac{(1-t)^{\gamma}}{(1-tr)^{a+\gamma+1}}\, dt \leq \frac{a+\gamma+1}{a(\gamma+1)} \frac{1}{(1-r)^a}.$$

Proof. We have

$$\int_0^1 \frac{(1-t)^\gamma}{(1-tr)^{a+\gamma+1}} = \int_0^r \frac{(1-t)^\gamma}{(1-tr)^{a+\gamma+1}} dt + \int_r^1 \frac{(1-t)^\gamma}{(1-tr)^{a+\gamma+1}}$$

$$\leq \int_0^r \frac{1}{(1-t)^{a+1}} dt + \int_r^1 \frac{(1-t)^\gamma}{(1-r)^{a+\gamma+1}} dt$$

$$= \frac{1}{a} \frac{1}{(1-r)^a} - \frac{1}{a} + \frac{1}{\gamma+1} \frac{1}{(1-r)^a}$$

$$< \frac{a+\gamma+1}{a(\gamma+1)} \frac{1}{(1-r)^a},$$

as desired.

Proof of Theorem 3. Let $a > 1$. Then it is well known that

$$\| f \|_{B^a} \asymp \| f \|'_{B^a} = \sup_{z \in B} (1-|z|)^{a-1} |f(z)|.$$

We have

$$|C_{\zeta_0}^\gamma f(z)| \leq |\gamma+1| \int_0^1 \frac{|f(tz)|}{|1-\langle tz, \zeta_0 \rangle|^{\gamma+1}} (1-t)^\gamma dt$$

$$\leq |\gamma+1| \int_0^1 \frac{|f(tz)|(1-t|z|)^{a-1}}{(1-t|z|)^{a+\gamma}} (1-t)^\gamma dt$$

$$\leq |\gamma+1| \| f \|'_{B^a} \int_0^1 \frac{(1-t)^\gamma}{(1-t|z|)^{a+\gamma}} dt$$

$$\leq \frac{|a+\gamma|}{|a-1|} \| f \|'_{B^a} \frac{1}{(1-|z|)^{a-1}} \quad \text{(by Lemma 4)},$$

from which the result follows.

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Stevo Stević
Mathematical Institute of the Serbian Academy of Science
Knez Mihailova 35/I, 11000 Beograd, Serbia
E-mail: *sstevic@ptt.yu; sstevo@matf.bg.ac.yu*