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POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS WITH THE SCALAR *p*-LAPLACIAN

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ABSTRACT: In this paper we establish the existence of at least one nontrivial positive solution under complete resonance at $+\infty$. At 0^+ we allow only partial interaction with the spectrum of the negative scalar p-Laplacian with Dirichlet boundary conditions (nonuniform nonresonance).

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1. INTRODUCTION

In this paper we study the existence of positive solutions for the following nonlinear scalar boundary value problem

$$\left\{\begin{array}{l} -(|x'(t)|^{p-2}x'(t))' \in \partial j(t,x(t)) \ a.e. \ on \ T=[0,b], \\ x(0)=x(b)=0, 1<p<\infty \end{array}\right\} \tag{1.1}$$

Here the potential function $j : T \times \mathbb{R} \to \mathbb{R}$ is measurable in $t \in T$ and only locally Lipschitz (hence in general nonsmooth) in the $x \in \mathbb{R}$ variable. Then $\partial j(t, x)$ denotes the generalized (Clarke) subdifferential of $j(t, \cdot)$. In the past the problem of existence of positive solutions for scalar boundary value problems was studied by many authors, primarily in the context of semilinear (i.e. $p = 2$) equations. We mention the works of the Erbe-Hu-Wang [5], Erbe-Wang [4], Liu-Li [7], Njoku-Zanolin [9], Wang [16], which deal with semilinear equations. In Erbe-Hu-Wang [5], Erbe-Wang [4] the authors deal with the the so-called sublinear and superlinear problem: i.e. if $f(x)$ represents the right hand side nonlinearity, they assume $\lim\limits_{x\to 0^+}\frac{f(x)}{x}=+\infty$, $\lim\limits_{x\to 0^+}\frac{f(x)}{x}=0$ (sublinear case) and $\lim\limits_{x\to 0^+}\frac{f(x)}{x}=0$, $\lim\limits_{x\to +\infty}\frac{f(x)}{x}=+\infty$ (superlinear case) and their approach uses degree theory and in particular the fixed point index and Krasnoselskii's theorem on fixed points for maps of the compression-expansion type. Similar is the work of Wang [16], who deals with the sublinear case for a semilinear elliptic equation on an annulus. Liu-Li and Njoku-Zanolin use weaker conditions on the asymptotic behavior at $+\infty$ and 0^+ of the ratio $\frac{f(x)}{x}$, however always avoiding any interaction with the spectrum of the negative scalar Laplacian (nonresonance). The method of Liu-Li is similar to that of Erbe-Hu-Wang and Erbe-Wang and it is based on degree theoretic arguments. The work of Njoku-Zanolin, combines degree theory with estimates for the time map. Recently the attention shifted to equations driven by the scalar p-Laplacian. In this direction we have the works of De Coster [3], Ben Naoum-De Coster [8] and Wang [17]. De Coster and Ben Naoum-De Coster use the method of upper-lower solutions in conjunction with the time map, while Wang extends the semilinear work of Erbe-Wang. None of the aforementioned works treats the resonant case. Resonant problems were first studied in the celebrated paper of Landesman- Lazer [6], who produced some sucfficient conditions for the existence of solutions to some smooth, semilinear Dirichlet problems. Since then these conditions are known as Landesman-Lazer conditions (LL-conditions for short). Resonant problems are important since they arise often in Mechanics.

In this paper we establish the existence of at least one nontrivial positive solution under complete resonance at $+\infty$. At 0^+ we allow only partial interaction with the spectrum of the negative scalar p-Laplacian with Dirichlet boundary conditions (nonuniform nonresonance).

2. MATHEMATICAL BACKGROUND

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X,X^*). A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ we can find a neighborhood of U of $x \in X$ and a constant $k_U > 0$ (depending on U) such that

$$|\varphi(y) - \varphi(z)| \leq k_U \| y - z\| \text{ for all } y, z \in U.$$

Recall that if $\psi : X \to \mathbb{R}$ is continuous convex, then it is locally Lipschitz. Similarly if $\psi \in C^1(X)$. If $\varphi : X \to \mathbb{R}$, is a locally Lipschitz function, the generalized directional derivative of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easily seen that $h \to \varphi^0(x;h)$ is continuous, sublinear and so it is the support function of a nonempty, w^*-compact and convex set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x; h) \text{ for all } h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is called the generalized subdifferential of φ. If φ is also convex, then the generalized subdifferential coincides with the subdifferential $\partial_c\varphi$ in the sense of convex analysis, which is defined by

$\partial_c\varphi(x) = \{x^* \in X^*: \langle x^*, h\rangle \leq \varphi(x + h) - \varphi(x) \text{ for all } h \in X\}$ for all $x \in X$.

If $\varphi \in C^1(X)$, then

$$\partial\varphi(x) = \{\varphi'(x)\} \text{ for all } x \in X.$$

If $\varphi,\psi : X \to \mathbb{R}$ are two locally Lipschitz functions, then

$$\partial(\varphi + \psi) \subseteq \partial\varphi + \partial\psi \text{ and } \partial(\lambda\varphi) = \lambda\partial\varphi \text{ for all } \lambda \in \mathbb{R}.$$

If $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function, then $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. In this case $c = \varphi(x)$ is a critical value of φ. If $x \in X$ is local extremum of φ(i.e. a local minimum or a local maximum), then $x \in X$ is a critical point of φ.

It is well-known that in the smooth critical point theory, crucial role plays a compactness type condition, known as the "Palais-Smale condition at the level $c \in \mathbb{R}$". Here we will use a generalized nonsmooth version of it. Let $\varphi : X \to \overline{R} = \mathbb{R}U\{+\infty\}$ be a functional, such that $\varphi = \Phi + \psi$,with $\Phi : X \to \mathbb{R}$ locally Lipschitz and ψ a proper, convex and lower semicontinuous functional. In the present nonsmooth constrained setting, this condition takes the following form:

> "A locally Lipschitz function $\varphi = \Phi + \psi$ satisfies the generalized nonsmooth Palais-Smale condition at the level $c \in \mathbb{R}$ (the generalized nonsmooth PS_c-condition for short), if any sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ and
>
> $$-\varepsilon_n\| y - x_n\|_X \leq \Phi^0(x_n; y - x_n) + \psi(y) - \psi(x_n) \ \forall y \in X,$$
>
> with $\varepsilon_n \downarrow 0$, has a strongly convergent subsequence."

The following is the generalized nonsmooth version of the well-known "Mountain Pass Theorem".

Theorem 2.1. *If X is a reflexive Banach space,* $\varphi : \mathrm{X} \to \overline{\mathrm{R}} = \mathbb{R} \cup \{+\infty\}$ *is a functional, such that* $\varphi = \Phi + \psi$, *with* $\Phi : X \to \mathbb{R}$ *locally Lipschitz and* ψ *a proper, convex and lower semicontinuous functional, satisfies the generalized nonsmooth* PS_c*-condition and there exist* x_0, $x_1 \in X$ *and* $\rho > 0$ *such that*

(i) $\|x_1 - x_0\| > \rho$ and

(ii) $\max\{\varphi(x_0), \varphi(x_1)\} < c_0 = \inf\{\varphi(y) : \|y - x_0\| = \rho\}$

then φ *has a critical point* $x \in X$ *with critical value* $c = \varphi(x) \geq c_0$ *defined by the minimax relation*

$$c \overset{df}{=} \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)) < +\infty,$$

where

$$\Gamma \overset{df}{=} \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}.$$

Finally by λ_1 we denote the first (principal) eigenvalue of the negative p- Laplacian with Dirichlet boundary condition, i.e. of $(-\Delta_p, W_0^{1,p}(0,b))$. We know that $\lambda_1 > 0$ and it has the following variational characterization

$$\lambda_1 = \inf\left\{\frac{\| x' \|_p^p}{\| x \|_p^p} : x \in W_0^{1,p}(0,b),\ x \neq 0\right\}.$$

In fact we know the full spectrum of $(-\Delta_p, W_0^{1,p}(0,b))$ which is given by

$$\lambda_n = (\frac{n}{b})^p (p-1)\left[2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\right]^p$$

(see Gasinski-Papageorgiou [12], p.761). Also $u_1 \in C_0^1$ (T) is the principal eigen function.

3. EXISTENCE OF POSITIVE SOLUTIONS

Our hypotheses on the nonsmoth potential function are the following:

$H(j)$: $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(t, 0) = 0$ a.e. on T, $\partial j(t, 0) \subseteq \mathbb{R}_+$ a.e. on T and

(i) for all $x \in \mathbb{R}$, $t \to j(t, x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t, x)$ is a locally Lipschitz;

(iii) for every $M > 0$ there exists $\alpha_M \in L^\infty(T)_+$ such that for almost all $t \in T$, all $|x| \leq M$ and all $u \in \partial j(t, x)$, we have $|u| \leq \alpha_M(t)$;

(iv) $\lim_{x \to +\infty} \frac{u}{x^{p-1}} = \lambda_1$ uniformly for almost all $t \in T$ and all $u \in \partial j(t, x)$;

(v) Let $g_1(t, x) = \max[u : u \in \partial j(t, x)]$ and set

$$G_1(t,x) = \begin{cases} \frac{pj(t,x)}{x} - g_1(t,x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$ and suppose that there exists a function $G_+ \in L^1(T)$ such that

$$G_+(t) \leq \liminf_{x \to +\infty} G_1(t,x) \text{ uniformly for almost all } t \in T$$

and $\int_0^b G_+(t)u_1(t)dt > 0$;

(vi) $\limsup_{x \to 0^+} \frac{pj(t,x)}{x^p} \leq \eta(t)$ uniformly for almost all $t \in T$, where $\eta \in L^\infty(T)_+$ is such that $\eta(t) \leq \lambda_1$ a.e. on T and the inequality is strict on a set of positive measure.

Remark 3.1. *Hypothesis* $H(j)(iv)$, *says that we have complete resonance at* $+\infty$. *On the other hand hypothesis* $H(j)(vi)$ *implies that at* 0^+ *there is only partial interaction with the spectrum of the negative scalar p-Laplacian. Such a condition is often called* "*nonuniform nonresonance*". *If the potential j is time invariant, then this condition says that near* 0^+ *the* "*slope*" $\frac{pj(t,x)}{x^p}$ *stays strictly below* λ_1. *Finally hypothesis* $H(j)(v)$ *is a unilateral nonsmooth version of a generalized Landesman-Lazer type condition, first used in semilinear problems* (i.e. p = 2) *by Tang [14].*

By modifying $j(t, x)$ on a Lebesgue-null subset of T, if necessary, we may assume that $(t, x) \to j(t, x)$ is Borel measurable. Then by definition we have

$$j^0(t,x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0 \\ x', \lambda \in Q}} \frac{j(t,x'+\lambda h) - j(t,x')}{\lambda},$$

$$\Rightarrow (t, x) \to j^0(t, x; h) \text{ is Borel measurable.}$$

It follows that, if $\{h_m\}_{m\geq 1}$ is an enumeration of the rationals in $\mathbb{R}$, then

$$\mathrm{Gr}\partial j = \{(t, x, u) \in T \times \mathbb{R} \times \mathbb{R} : uh \leq j^0(t, x; h) \text{ for all } h \in \mathbb{R}\}$$

$$= \bigcap_{m\geq 1} \{t, x, u) \in T \times \mathbb{R} \times \mathbb{R} : uh_m \leq j^0(t, x; h_m)\}$$

(recall that $j^0(t, x, \cdot)$ is continuous)

$\in B(T) \times B(\mathbb{R}) \times B(\mathbb{R})$.

For every $\lambda \in \mathbb{R}$ we have that

$$\{(t, x) \in T \times \mathbb{R} : g_1(t, x) \geq \lambda\} = \mathrm{proj}_{T\times\mathbb{R}}\{(t, x, u) \in \mathrm{Gr}\partial j : u \geq \lambda\}.$$

Because $\partial j(t, x)$ is compact, from Hu-Papageorgiou [13], p.41 we have that

$\mathrm{proj}_{T\times\mathbb{R}}\{(t, x, u) \in \mathrm{Gr}j : u \geq \lambda\} \in B(T) \times B(\mathbb{R})$,

$\Rightarrow g_1$ is jointly measurable.

We introduce the functional $\varphi_1 : W_0^{1,p}(0, b) \to \mathbb{R}$ defined by

$$\varphi_1(x) = \frac{1}{p}\| x' \|_p^p - \int_0^b j(t, x(t))dt.$$

We know that φ_1 is locally Lipschitz (see Clarke [2], p.85 or Denkowski-Papageorgiou [10], p.617). Also let $C = \{x \in W_0^{1,p}(0, b) : x(t) \geq 0 \text{ for all } t \in T\}$ (recall that $W_0^{1,p}(0, b) \subseteq C(T)$). Evidently C is a nonempty, closed and convex cone in the Sobolev space $W_0^{1,p}(0, b)$. Let $\varphi_2 \in \Gamma_0(W_0^{1,p}(0, b))$ be defined by

$$\varphi_2(x) = i_C(x) = \begin{cases} 0 & if \quad x \in C \\ +\infty & if \quad x \in C \end{cases}$$

(the indicator function of the set C).

Finally we set

$$\varphi = \varphi_1 + \varphi_2$$

Evidently $\varphi \in \Gamma_0(W_0^{1,p}(0, b))$. In the sequel $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 3.2. *If hypotheses H(j) hold, then* φ, *satisfies the generalized nonsmooth PS-condition.*

Proof. We consider a sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0,b)$ such that

$$|\varphi(x_n)| \leq M_1 \text{ for some } M_1 > 0 \text{ and all } n \geq 1$$

$$\text{and } \varphi_1^0(x_n; y-x_n) + \varphi_2(y) - \varphi_2(x_n) \geq -\varepsilon_n \| y - x_n \|$$

$$\text{for all } y \in W_0^{1,p}(0,b), \text{ with } \varepsilon_n \downarrow 0.$$

Evidently $\{x_n\}_{n\geq 1} \subseteq C$. Recalling that $\varphi_1^0(x_n;\cdot)$ is the support function of the weakly compact set $\partial\varphi_1(x_n)$, for every $y \in W_0^{1,p}(0,b)$, we can find $x_n^*, \in \partial\varphi_1(x_n)$ (depending in general on y), such that

$$\varphi_1^0(x_n; y-x_n) = \left\langle x_n^*, y - x_n \right\rangle.$$

Hereafter by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(0,b), W^{-1,q}(0,b) = W_0^{1,p}(0,b)^*)$. Consider the nonlinear operator $A : W_0^{1,p}(0,b) \to W^{-1,q}(0,b)$ defined by

$$\left\langle A(x), y \right\rangle = \int_0^b | x'(t) |^{p-2} x'(t) y'(t) dt \text{ for all } x, y \in W_0^{1,p}(0,b).$$

It is easy to see that A is monotone, demicontinuous, thus maximal monotone (see Denkowski-Papageorgiou [11], p.37). Moreover, we have

$$x_n^* = A(x_n) - u_n \text{ with } u_n \in S^q_{\partial j(.,x_n(.))}, n \geq 1$$

(see Clarke [2], p.83 or Denkowski-Papageorgiou [10], p.617).

We claim that the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0,b)$ is bounded. Suppose that the claim is not true. Then by passing to a subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Set $v_n = \dfrac{x_n}{\| x_n \|}, n \geq 1$. Evidently $\|v_n\| = 1$, $n \geq 1$ and so by passing to a further subsequence if necessary, we may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,p}(0,b) \text{ and } v_n \to v \text{ in } C_0(T)$$

(recall that $W_0^{1,p}(0,b)$ is embedded compactly in $C_0(T)$).

By virtue of hypotheses $H(j)(iii)$ and (iv), *given* $\varepsilon > 0$, we can find $\alpha_\varepsilon \in L^\infty(T)_+$, such that for almost all $t \in T$, all $x \geq 0$ and all $u \in \partial j(t, x)$, we have

$$u \leq (\lambda_1 + \varepsilon)x^{p-1} + \alpha_\varepsilon(t). \tag{3.1}$$

Let $|\cdot|_1$ denote the Lebesgue measure on $\mathbb{R}$. By hypothesis $H(j)(ii)$ for all $t \,\varepsilon\, T\backslash D$, $|D|_1 = 0$ the function $j(t, \cdot)$ is locally Lipschitz. So we can find $E(t) \subseteq \mathbb{R}$ with $|E(t)|_1 = 0$ such that $j(t, \cdot)$ is differentiable at every $x \in \mathbb{R}\backslash E(t)$. Then for all $t \,\varepsilon\, T\backslash D$ and all $x \geq 0$, we have

$$j(t,x) - j(t,0) = j(t,x) = \int_0^x j_r'(t,r)dr.$$

Recall that $j_r'(t,r) \in \partial j(t,r)$ for all $t \in T\backslash D$ and all $r \in \mathbb{R}\backslash E(t)$. So using (3.1), we have

$$j(t,x) \leq \int_0^x \alpha_\varepsilon(t)dr + \int_0^x (\lambda_1 + \varepsilon)r^{p-1}dr$$

$$= \alpha_\varepsilon(t)x + \frac{1}{p}(\lambda_1 + \varepsilon)x^p$$

for almost all $t \in T$ and all $x \geq 0$. Because $\{x_n\}_{n\geq 1} \subseteq C$, we can write that for almost all $t \,\varepsilon\, T$ and all $n \geq 1$, we have

$$\frac{j(t,x_n(t))}{\| x_n \|^p} \leq \frac{\alpha_\varepsilon(t)}{\| x_n \|^{p-1}} v_n(t) + \frac{\lambda_1 + \varepsilon}{p} v_n(t)^p,$$

$$\Rightarrow \int_0^b \frac{j(t,x_n(t))}{\| x_n \|^p} dt \leq \int_0^b \frac{\alpha_\varepsilon(t)}{\| x_n \|^{p-1}} v_n(t) + \frac{\lambda_1 + \varepsilon}{p} \| v_n \|_p^p,$$

$$\Rightarrow \limsup_{n\to\infty} \int_0^b \frac{j(t,x_n(t))}{\| x_n \|^p} dt \leq \frac{\lambda_1 + \varepsilon}{p} \| v \|_p^p$$

Because $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ to obtain

$$\limsup_{n\to\infty}\int_0^b \frac{j(t,x_n(t))}{\|x_n\|^p}dt \le \frac{\lambda_1}{p}\|v\|_p^p \tag{3.2}$$

From the choice of the sequence $\{x_n\}_{n\le 1} \subseteq C$, we have

$$\frac{|\varphi(x_n)|}{\|x_n\|^p} \le \frac{M_1}{\|x_n\|^p},$$

$$\Rightarrow \frac{1}{p}\|v_n'\|_p^p \le \frac{M_1}{\|x_n\|^p} + \int_0^b \frac{j(t,x_n(t))}{\|x_n\|^p}dt. \tag{3.3}$$

Passing to the limit as $n \to \infty$, exploiting the weak lower semicontinuity of the norm functional in a Banach space and using (3.2), we obtain

$$\|v'\|_p^p \le \lambda_1 \|v\|_p^p,$$

$$\Rightarrow v = u_1 \text{ or } v = 0 \text{ (recall that } v \,\varepsilon\, C).$$

If $v = 0$, then we have that $v_n' \to 0$ in $L^p(T)$ (see (3.3)) and so $v_n \to 0$ in $W_0^{1,p}(0,b)$ a contradiction to the fact that $\|v_n\| = 1$ for all $n \ge 1$. So $v = u_1$. Recall that $u_1(t) > 0$ for all $t \in$ (0, b). From the choice of $\{x_n\}_{n\ge 1} \subseteq C$ and using as a test function $y = 0 \in C$, we have

$$\langle A(x_n), x_n\rangle - \int_0^b u_n x_n dt \le \varepsilon_n \|x_n\|,$$

$$\Rightarrow \|x_n'\|_p^p - \int_0^b u_n x_n dt \le \varepsilon_n \|x_n\|. \tag{3.4}$$

Also from the inequality $|\varphi(x_n)| \le M_1$ for all $n \ge 1$, we have

$$-\|x_n'\|_p^p + \int_0^b pj(t,x_n(t))dt \le pM_1. \tag{3.5}$$

Adding (3.4) and (3.5), we obtain

$$\int_0^b \left(pj(,x_n(t)) - u_n x_n\right)dt \le pM_1 + \varepsilon_n \|x_n\|,\ n \ge 1,$$

$$\Rightarrow \int_0^b \left(\frac{pj(t,x_n(t))}{\|x_n\|} - u_n v_n\right)dt \le \frac{pM_1}{\|x_n\|} + \varepsilon_n,\ n \ge 1. \tag{3.6}$$

We introduce the functions

$$h_n(t) = \begin{cases} \dfrac{j(t, x_n(t))}{x_n(t)} & if \quad x_n(t) > 0 \\ 0 & if \quad x_n(t) = 0 \end{cases}, \ n \geq 1 \ \ (recall\ x_n \in C).$$

Then we have

$$\varepsilon_n + \frac{pM_1}{\| x_n \|} \geq \int_0^b \left(\frac{pj(t, x_n(t))}{\| x_n \|} - u_n v_n \right) dt$$

$$= \int_{\{x_n > 0\}} \left(\frac{pj(t, x_n(t))}{\| x_n \|} - u_n x_n \right) dt \ \ \text{(since } j(t, 0) = 0 \text{ a.e. on } T\text{)}$$

$$\geq \int_0^b p h_n v_n dt - \int_0^b g_1(t, x_n(t)) v_n(t) dt$$

$$= \int_0^b G_1(t, x_n(t)) v_n(t) dt. \tag{3.7}$$

By virtue of hypothesis $H(j)(v)$, given $\varepsilon > 0$, we can find $M_2 = M_2(\varepsilon) > 0$ such that for almost all $t \in T$ and all $x \geq M_2$, we have

$$G_1(t, x) \geq G_+(t) - \varepsilon. \tag{3.8}$$

On the other hand from the mean value theorem for locally Lipschitz functions (see Clarke [2], p.41 or Denkowski-Papageorgiou [10], p.607), for almost all $t \in T$ and all $x \in (0, M_2)$, we have

$$j(t, x) = \hat{u}x \text{ with } \hat{u} \in \partial j(t, \lambda(t)x),\ \lambda(t) \in (0, 1),$$

$$\Rightarrow \frac{pj(t, x)}{x} - g_1(t, x) = \hat{u}x - g_1(t, x)$$

$$\geq -M_{2\alpha M_2}(t) - \alpha_{M_2}(t)$$

$$\Rightarrow G_1(t, x) \geq (-M_2 - 1)\alpha_{M_2}(t). \tag{3.9}$$

From (3.8) and (3.9), we see that for almost all $t \in T$ and all $x \geq 0$, we have

$$G_1(t, x) \geq \beta(t) \text{ with } \ \beta \in L^1(T)_+. \tag{3.10}$$

Note that $u_1(t) > 0$ for all $t \in (0, b)$, implies that $x_n(t) \to +\infty$ for all $t \in (0, b)$. So using Fatou's lemma ((3.10) permits its use), we obtain

$$\liminf_{n\to\infty} \int_0^b G_1(t, x_n(t))v_n(t)dt$$

$$\geq \int_0^b \liminf_{n\to\infty} G_1(t, x_n(t))v_n(t)dt$$

$$\geq \int_0^b G_+(t)v(t)dt = \int_0^b G_+(t)u_1(t)dt. \tag{3.11}$$

So if we pass to the limit as $n \to \infty$ in (3.7) and use (3.11), we have

$$\int_0^b G_+(t)u_1(t)dt \leq 0,$$

a contradiction to hypothesis $H(j)(v)$.

This proves that the sequence $\{x_n\}_{n\geq 1} \subseteq C$ is bounded. Hence by passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,p}(0,b) \text{ and } x_n \to x \text{ in } C_0(T).$$

Recall that from the sequence $\{x_n\}_{n\geq 1} \subseteq C$, we have

$$\langle A(x_n), x_n - x\rangle - \int_0^b u_n(x_n - x)dt \leq \varepsilon_n \| x_n - x \|.$$

Remark that $\int_0^b u_n(x_n - x) \to 0$ as $n \to \infty$ (see hypothesis $H(j)(iii)$). So we have

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leq 0.$$

But A being maximal monotone, it is generalized pseudomonotone and so we have

$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle,$

$\Rightarrow \|x'_n\|_p \to \|x'\|_p.$

Since $x'_n \xrightarrow{w} x'$ in $L^p(T)$ and the latter is uniformly convex, from the Kadec-

Klee property we have $x'_n \to x'$ in $L^p(T)$. This means that $x_n \to x$ in $W_0^{1,p}(0,b)$ and so we conclude that φ satisfies the generalized nonsmooth PS-condition.

Proposition 3.3. *If hypotheses $H(j)$ hold, then* $\varphi\,(tu_1) \to -\infty$ as $t \to +\infty$.

Proof. Because of hypothesis $H(j)(v)$, given $\varepsilon > 0$, we can find $M_2 = M_2(\varepsilon) > 0$ such that for all $t \in T \setminus N$, with $|N|_1 = 0$ and all $x \geq M_1$, we have

$$K_\varepsilon^+(t) = G_+(t) - \varepsilon \leq G_1(t,x),$$

$$\Rightarrow \frac{G_1(t,x)}{x^p} \geq \frac{K_\varepsilon^+(t)}{x^p} = \frac{d}{dx}\left(-\frac{1}{p-1}\frac{K_\varepsilon^+(t)}{x^{p-1}}\right). \tag{3.12}$$

Recalling the definition of $G_1(t, x)$ (see hypothesis $H(j)(v)$), we see that for all $t \in T\backslash N$, $|N|_1 = 0$, all $x \geq M_1$ and all $u \in \partial j(t, x)$, we have

$$\begin{aligned}\frac{G_1(t,x)}{x^p} &= \frac{pj(t,x)}{x^{p+1}} - \frac{g_1(t,x)}{x^p} \\ &\leq \frac{pj(t,x)}{x^{p+1}} - \frac{u}{x^p}.\end{aligned} \tag{3.13}$$

Note that for all $t \in T\backslash N$, the function $x \to \dfrac{pj(t,x)}{x^p}$ is locally Lipschitz on $[M_2, +\infty)$. So we have (see Clarke [2], p.48 or Denkowski-Migorski-Papageorgiou [10], p.612)

$$\begin{aligned}\partial\left(\frac{j(t,x)}{x^p}\right) &\subseteq \frac{\partial j(t,x)x^p - pj(t,x)x^{p-1}}{x^{2p}} \\ &= \frac{\partial j(t,x)}{x^p} - \frac{pj(t,x)}{x^{p+1}},\end{aligned}$$

$$\Rightarrow \max\left[\hat{u} : \hat{u} \in \partial\left(\frac{j(t,x)}{x^p}\right)\right] \leq \frac{g_1(t,x)}{x^p} - \frac{pj(t,x)}{x^{p+1}} \leq -\frac{G_1(t,x)}{x^p} \tag{3.14}$$

(see (3.13)).

Fix $t \in T\backslash N$, $|N|_1 = 0$. The function $x \to \dfrac{j(t, x)}{x^p}$ is differentiable at all $x \in [M_2, +\infty)\backslash E(t)$, $|E(t)|_1 = 0$. We set

$$\widehat{u}_0(t, x) = \begin{cases} \dfrac{d}{dx}\left(\dfrac{j(t, x)}{x^p}\right) & if \quad x \in [M_2, +\infty) \setminus E(t) \\ 0 & if \quad x \in E(t) \end{cases}.$$

We know that $\widehat{u}(t, x) \in \partial\left(\dfrac{j(t, x)}{x^p}\right)$ for all $x \in [M_2, +\infty)\backslash E(t)$. Therefore

$$\widehat{u}(t, x) \leq \frac{d}{dx}\left(\frac{1}{p-1}\frac{K_\varepsilon^+(t)}{x^{p-1}}\right) \text{(see (3.12) and (3.14)).} \tag{3.15}$$

Let $y, z \in [M_2, +\infty)$ with $y < z$. Integrating (3.15) over $[y, z]$, we obtain

$$\frac{j(t, z)}{z^p} - \frac{j(t, y)}{y^p} \leq \frac{K_\varepsilon^+(t)}{p-1}\left(\frac{1}{z^{p-1}} - \frac{1}{y^{p-1}}\right) \text{ for all } t \in T\backslash N,\ |N|_1 = 0. \tag{3.16}$$

By virtue of hypotheses $H(j)(iii)$, (iv), given $\varepsilon > 0$, we can find α_ε ε $L^\infty(T)_+$ such that for almost all t ε T, all $z \geq 0$ and all $u \in \partial j(t, x)$, we have

$$(\lambda_1 - \varepsilon)z^{p-1} - \alpha_\varepsilon(t) \leq u \leq (\lambda_1 + \varepsilon)z^{p-1} + \alpha_\varepsilon(t).$$

Since $j'_z(t, z)$ exists for all $z \in \mathbb{R}\backslash C(t)$, $|C(t)|_1 = 0$ and $j'_z(t, z) \in \partial j(t, z)$, then for almost all $t \in T$ and all $z \in \mathbb{R}_+ \backslash C(t)$, we have

$$(\lambda_1 - \varepsilon)z^{p-1} - \alpha_\varepsilon(t) \leq j'_z(t, z) \leq (\lambda_1 + \varepsilon)z^{p-1} + \alpha_\varepsilon(t).$$

Integrating this inequality over $[0, x]$, $x > 0$, we obtain

$$\frac{1}{p}(\lambda_1 - \varepsilon)x^p - \alpha_\varepsilon(t)x \leq j(t, x) \leq (\lambda_1 + \varepsilon)x^p + \alpha_\varepsilon(t)x$$

(recall that $j(t, 0) = 0$ a.e. on T).

Therefore we have

$$\lim_{x\to\infty} \frac{j(t, x)}{x^p} = \frac{1}{p}\lambda_1 \text{ for almost all } t \in T.$$

Set $j_1(t, x) = j(t, x) - \frac{\lambda_1}{p} | x |^p$. Then we have

$$\lim_{x\to\infty} \frac{j_1(t, x)}{x^p} = 0 \text{ uniformly for almost all } t \in T. \tag{3.17}$$

Returning to (3.16), we see that we can rewrite it as

$$\frac{j_1(t,z)}{z^p} - \frac{j_1(t,y)}{y^p} \le \frac{K_\varepsilon^+(t)}{p-1}\left(\frac{1}{z^{p-1}} - \frac{1}{y^{p-1}}\right). \tag{3.18}$$

Passing to the limit as $z \to +\infty$ in (3.18) and using (3.17), we obtain

$$\frac{j_1(t,y)}{y^p} \ge \frac{K_\varepsilon^+(t)}{p-1}\frac{1}{y^{p-1}},$$

$$\Rightarrow \frac{j_1(t,y)}{y^p} \ge \frac{K_\varepsilon^+(t)}{p-1},$$

$$\Rightarrow \liminf_{y\to+\infty} \frac{j_1(t, y)}{y} \ge \frac{K_\varepsilon^+(t)}{p-1} \text{ for almost all } t \in T. \tag{3.19}$$

Now suppose that the claim of the Proposition was not true. Then we can find a sequence $\lambda_n \to \infty$ and $M_3 > 0$ such that

$$\varphi(\lambda_n u_1) \ge -M_3 \text{ for all } n \ge 1$$

$$\Rightarrow \frac{\lambda_n^p}{p} \| u'_1 \|_p^p - \frac{\lambda_n^p}{p} \| u_1 \|_p^p - \int_0^b j_1(t, \lambda_n u_1(t))dt \ge -M_2$$

$$\Rightarrow -\int_0^b \frac{j_1(t,\lambda_n u_1(t))}{\lambda_n(t)u_1(t)} u_1(t)\, dt \ge -\frac{M_2}{\lambda_n}$$

(recall $u_1(t) > 0$ for all $t \in (0, b)$).

Passing to the limit as $n \to \infty$, using (3.13) and recalling that $\varepsilon > 0$ was arbitrary, we infer that

$$\int_0^b G_+(t)u_1(t)\, dt \le 0$$

a contradiction to the hypothesis $H(j)(v)$. This proves that $\varphi|_{\mathbb{R}u_1}$ is anticoercive.

Our goal is to show that the Mountain Pass geometry is satisfied. To do this we need the following Lemma

Lemma 3.4. *There exists* $\xi_0 > 0$ *such that for all* $x \in W_0^{1,p}(0,b)$*, we have*

$$\| x' \|_p^p - \int_0^b \eta(t) \,|\, x(t) \,|^p \; dt \geq \xi_0 \,\| x' \|_p^p .$$

Proof. Let $\psi :\; W_0^{1,p}(0, b)\mathbb{R}$ be defined

$$\psi(x) = \| x' \|_p^p - \int_0^b \eta(t) \,|\, x(t) \,|^p \; dt,\; x \in W_0^{1,p}(0, b).$$

By virtue of the variational characterization of $\lambda_1 > 0$ and hypothesis $H(j)(vi)$, we have that $\psi \geq 0$. Suppose that the claim of the Lemma is not true. Then exploiting the p-homogeneity of ψ we can find $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0, b)$ with $\| x'_n \|_p = 1$ such that ψ $x_n \downarrow 0$ as $n \to \infty$. by virtue of the Poincaré inequality and by passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \;\text{ in }\; W_0^{1,p}(0, b), \;\text{ and }\; x_n \to x \;\text{ in }\; C_0(T).$$

Because of the weak lower semicontinuity of the norm functional in a Banach space, we have

$$\| x'_n \|_p^p \leq \int_0^b \eta(t) x(t) \,|^p \; dt \leq \lambda_1 \,\| x \|_p^p,$$

$$\Rightarrow \| x' \|_p^p = \lambda_1 \,\| x \|_p^p$$

$$\Rightarrow x = \pm u_1 \text{ or } x = 0. \tag{3.20}$$

We can not have $x = 0$, since then $\| x'_n \|_p \to 0$ as $n \to \infty$, a contradiction to the fact that $\| x'_n \|_p = 1$ for all $n \geq 1$. Therefore $x = \pm u_1$. Because $u_1(t) > 0$ for all $t \in (0, b)$, from (3.20) it follows that

$$\| x' \|_p^p < \lambda_1 \,\| x \|_p^p$$

a contradiction to the variational characterization of $\lambda_1 > 0$.

Using this lemma, we can prove the next Proposition, which assures us that the Mountain Pass geometry is in place.

Proposition 3.5. *If hypotheses H(j) hold and $r > p$, then we can find* $\xi_1, \xi_2 > 0$ *such that* $\varphi(x) \geq \xi_2 \| x \|^p - \xi_1 \| x \|^r$ *for all* $x \in W_0^{1,p}(0, b)$.

Proof. By virtue of hypotheses $H(j)(iii)$ and (vi), given $\varepsilon > 0$, we can find $\widehat{\alpha}_\varepsilon \in L^\infty(T)_+$ such that for almost all $t \in T$ and all $x \geq 0$, we have

$$j(t, x) \leq \frac{1}{p}(\eta(t) + \varepsilon)x^p + \widehat{\alpha}_\varepsilon(t)x^r. \tag{3.21}$$

So for all $x \in W_0^{1,p}(0, b)$ we have

$$\varphi = \frac{1}{p} \| x' \|_p^p - \int_0^b j(t, x(t))dt + \varphi_2(x)$$

$$\geq \frac{1}{p} \| x' \|_p^p - \frac{1}{p} \int_0^b \eta(t) \,| x(t) |^p \, dt - \frac{\varepsilon}{p} \| x \|_p^p - \xi_1 \| x \|^r$$

(see (3.21) and recall that $\varphi_2 = i_C$)

$$\geq \xi_0 \| x' \|_p^p - \frac{\varepsilon}{\lambda_1 p} \| x' \|_p^p - \xi_1 \| x \|^r \text{ (see Lemma 3.4).}$$

Choose $\varepsilon > 0$ such that $\varepsilon < \xi_0 \lambda_1 p$. Then by virtue of the Poincar´e inequality, we have

$$\varphi(x) \geq \xi_2 \| x \|^p - \xi_1 \| x \|^\eta$$

for some $\xi_2 > 0$ and all $x \in W_0^{1,p}(0, b)$.

Now we are ready to prove an existence theorem for positive solutions of problem (1.1).

Theorem 3.6. *If hypotheses H(j) hold, then problem (1.1) has a nontrivial solution* $x \in C_0^1(T)$ *such that* $x(t) \geq 0$ *for all* $t \in T$.

Proof. Because of Proposition 3.5 we see that $\rho > 0$ small, we have

$$\inf [\varphi(x) : \| x \| = \rho] = \xi_3 > 0.$$

On the other hand Proposition 3.3 implies that we can find $\lambda > 0$ large enough such that $\varphi(\lambda u_1) \leq \varphi(0) = 0$. These facts together with Proposition 3.2, permit the use of the generalized nonsmooth Mountain Pass Theorem, which gives us an $x \in C$ such that

$$\varphi(x) \geq \xi_3 > 0 = \varphi(0) \text{ (hence } x \neq 0)$$

and $\varphi_1^0(x; h) + \varphi_2(x+h) - \varphi_2(x) \geq 0$ for all $h \in W_0^{1,p}(0, b)$.

Set $\psi_1(h) = \varphi_1^0(x; h)$ and $\psi_2(h) = \varphi_2(x+h) - \varphi_2(h)$. Note that $\psi_1 : W_0^{1,p}(0,b) \to \mathbb{R}$ is continuous, sublinear, while $\psi_2 \in \Gamma_0(W_0^{1,p}(0, b))$. Moreover, remark that

$$\partial_c \psi_1(0) = \partial \varphi_1(x) \text{ and } \partial_c \psi_2(0) = \partial_c \varphi_2(x)$$

Then we have

$$0 \leq \psi_1(h) + \psi_2(h) \text{ for all } h \in W_0^{1,p}(0, b),$$

$$\Rightarrow 0 \in \partial_c(\psi_1 + \psi_2)(0) \text{ (recall that } \psi_1(0) = \psi_2(0) = 0).$$

But from Convex Analysis (see Denkowski-Migorski-Papageorgiou [10], p.549), we have

$$\partial_c(\psi_1 + \psi_2)(0) = \partial_c \psi_1(0) + \partial_c \psi_2(0) = \partial\varphi_1(x) + \partial\varphi_2(x),$$

$$\Rightarrow 0 \in \partial\varphi_1(x) + N_C(x) \text{ (since } \partial_c\varphi_2(x) = N_C(x)),$$

$$\Rightarrow x^* + v^* = 0 \text{ for some } x^* \in \partial\varphi(x),\ v^* \partial N_C(x).$$

We know that $x^* = A(x) - u$, with $u \in L^q(T)$, $u(t) \in \partial j(t, x(t))$ a.e. on T and $\langle v^*, y - x\rangle \leq 0$ for all $y \in C$. So we have

$$\langle x^*, y - x\rangle \geq 0 \text{ for all } y \in C. \tag{3.22}$$

Let $\theta \in W_0^{1,p}(0, b)$ and $\varepsilon > 0$. Use as a test function

$$y = (x + \varepsilon\theta)^+ = (x + \varepsilon\theta) + (\mathrm{x} + \varepsilon\theta)^- \in C.$$

Using this in (3.22), we obtain

$$\langle x^*, \varepsilon\theta\rangle \geq -\langle x^*, (x + \varepsilon\theta)^-\rangle$$

$$= -\left\langle A(x), (x+\varepsilon\theta)^- \right\rangle + \int_0^b u(x+\varepsilon\theta)^- dt. \tag{3.23}$$

Set $T_-^\varepsilon = \{t \in T : (x+\varepsilon\theta)(t) < 0\}$. Then we have

$$-\left\langle A(x), (x+\varepsilon\theta)^- \right\rangle = -\int_0^b |x'|^{p-2} x'[(x+\varepsilon\theta)^-]' dt$$

$$= \int_{T_-^\varepsilon} |x'|^{p-2} x'(x+\varepsilon\theta)' dt$$

(since $[(x+\varepsilon\theta)^-]'(t) = \begin{cases} -(x+\varepsilon\theta)'(t) & \text{if } t \in T_-^\varepsilon \\ 0 & \text{if } t \notin T_-^\varepsilon \end{cases}$,

$$\geq \varepsilon \int_{T_-^\varepsilon} |x'|^{p-1} x'\theta' dt. \tag{3.24}$$

Because of hypothesis $H(j)(iv)$, we can find $M_4 > 0$ such that for almost all $t \in T$ and all $x \geq M_4$, we have $\partial j(t, x) \subseteq \mathbb{R}_+$. Then

$$\int_0^b u(x+\varepsilon\theta)^- dt = -\int_{T_-^\varepsilon} u(x+\varepsilon\theta) dt$$

$$= -\int_{T_-^\varepsilon \cap \{x < M_4\}} u(x+\varepsilon\theta) dt - \int_{T_-^\varepsilon \cap \{x \geq M_4\}} u(x+\varepsilon\theta) dt \tag{3.25}$$

Note that $$= -\int_{T_-^\varepsilon \cap \{x \geq M_4\}} u(x+\varepsilon\theta) dt \geq 0. \tag{3.26}$$

Also since by hypothesis we have $\partial j(t, 0) \subseteq \mathbb{R}_+$ a.e. on T, we see that $u(t) \geq 0$ a.e. on $T_-^\varepsilon \cap \{x = 0\}$. Moreover, since by hypothesis we have $x(t) \geq 0$ for all $t \in T$ (recall that $x \in C$), we have that $\theta(t) < 0$ for all $t \in T_-^\varepsilon$. Hence

$$-\int_{T_-^\varepsilon \cap \{x=0\}} u(x+\varepsilon\theta) dt = -\varepsilon \int_{T_-^\varepsilon \cap \{x=0\}} u\theta dt \geq 0. \tag{3.27}$$

Then we can write that

$$-\int_{T_-^\varepsilon \cap \{x < M_4\}} u(x+\varepsilon\theta) dt =$$

$$-\int_{T_-^\varepsilon \cap \{x=0\}} u(x+\varepsilon\theta) dt - -\int_{T_-^\varepsilon \cap \{0 < x < M_4\}} u(x+\varepsilon\theta) dt$$

$$\geq -\int_{T_-^\varepsilon \cap \{0 < x < M_4\}} u(x+\varepsilon\theta) dt \quad \text{(see (3.27))}$$

$$\geq \xi_4 \int_{T_-^\varepsilon \cap \{0<x<M_4\}} (x+\varepsilon\theta)dt \text{ (for some } \xi_4 > 0$$

(see hypothesis $H(j)(iii)$),

$$\geq \xi_4 \int_{T_-^\varepsilon \cap \{0<x<M_4\}} \theta dt \text{ (because } x(t) \geq 0 \text{ for all } t \in T). \tag{3.28}$$

Using (3.26) and (3.28) in (3.25), we obtain

$$\int_0^b u(x+\varepsilon\theta)^- dt \geq \varepsilon\xi_4 \int_{T_-^\varepsilon \cap \{0<x<M_4\}} \theta dt. \tag{3.29}$$

Returning to (3.23) and using (3.24) and (3.29), we obtain

$$\langle x^*, \theta\rangle \geq \int_{T_-^\varepsilon} \cdots |x'|^{p-2} x'\theta' dt + \xi_4 \int_{T_-^\varepsilon \cap \{0<x<M_4\}} \theta dt.$$

Remark that $|T_-^\varepsilon \cap \{0<x\}|_1 \to 0$ as $\varepsilon \downarrow 0$. So in the limit as $\varepsilon \downarrow 0$, we obtain

$$\langle x^*, \theta\rangle \geq 0 \text{ for all } \theta \in W_0^{1,p}(0,b),$$
$$\Rightarrow x^* = 0,$$
$$\Rightarrow A(x) = u.$$

From this equality it follows, via Green's identity, that

$$-(|x'(t)|^{p-2}x'(t))' = u(t) \text{ a.e. on } T,\ x(0) = x(b) = 0,$$
$$\Rightarrow |x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,q}(0,b),$$
$$\Rightarrow |x'(\cdot)|^{p-2}x'(\cdot) \in C(T),$$
$$\Rightarrow x' \in C(T).$$

Therefore $x \in C^1(T)$, $x \neq 0$, $x(t) \geq 0$ for all $t \in T$ and is a solution of problem (1.1).

If we strengthen our hypotheses,we can produce a strictly positive solution of (1.1).

Namely our hypotheses on $j(t, x)$ are the following:

<u>$H(j)'$</u>: $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(t, 0) = 0$ a.e. on T, $\partial j(t, 0) \subseteq \mathbb{R}_+$ a.e. hypotheses $H(j)(i) \to (vi)$ hold and

(vii) for almost all $t \in T$, all $x \geq 0$ and all $u \in \partial j(t, x)$, we have $u \geq -\xi_5 x^{p-1}$ with $\xi_5 > 0$.

Theorem 3.7. *If hypotheses H(j) hold, then problem (1.1) has a solution* $x \in C_0^1(T)$ *such that* $x(t) > 0$ for all $t \in (0, b)$.

Proof. Let $x \in C_0^1(T)$ be the nontrivial positive solution obtained in Theorem 3.6. We have

$-(|x'(t)|^{p-2}x'(t))' = u(t) \geq -\xi_5 x(t)^{p-1}$ a.e. on T (see hypothesis $H(j)'(vii)$) $\Rightarrow (|x'(t)|^{p-2}x'(t))' \leq \xi_5 x(t)^{p-1}$ a.e. on T.

Invoking Theorem 5 of Vazquez [15], we obtain $x(t) > 0$ for all $t \in (0, b)$.

Remark 3.8. *A nonsmooth locally Lipschitz function satisfying hypotheses H(j) is the following (for simplicity we drop the t-dependence)*

$$j_1(x) = \begin{cases} e^{-x} - 1 & \text{if } \ x < 0 \\ \lambda_1 x^r & \text{if } \ x \in [0,1] \quad \text{with } \ r > p. \\ \lambda_1 x^p + x \ln x & \text{if } \ x > 1 \end{cases}$$

The function $j_2(x)$ that follows satisfies hypotheses $H(j)'$.

$$j_2(x) = \begin{cases} x e^x & \text{if } \ x < 0 \\ c \sin x^p & \text{if } \ x \in [0,1] \ . \\ \lambda_1 x^p + x^r - c \sin 1 & \text{if } \ x > 0 \end{cases}$$

with $r < p$, $cp < \lambda_1$.

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