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MULTIPLE POSITIVE SOLUTIONS FOR NONLINEAR SECOND ORDER M-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT: In this paper, the second order *m*-point boundary value problem

$$\begin{cases} u''(t) + q(t) f(t,u) = 0,\ 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0,\ u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = 0 \end{cases}$$

is studied, where $\alpha \geq 0$, $\beta \geq 0$, q is allowed to be singular at $t = 0$ and $t = 1$, f is allowed to change sign. By constructing available operator and using the Leggett-Williams fixed point theorem, the existence of at least three nontrivial positive solutions is established.

Keywords: *m*-point boundary value problem, singular, change of sign, three positive solutions.

AMS (MOS) Subject Classification: 34B10, 34B15.

1. INTRODUCTION

This paper deals with the following second order m-point boundary value problem

$$\begin{cases} u''(t) + q(t) f(t,u) = 0,\ 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0,\ u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = 0, \end{cases} \tag{1.1}$$

where $m \geq 3$, $\alpha \geq 0$, $\beta \geq 0$, $k_i > 0$ $(i = 1, 2, \ldots, m - 2)$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $q \in C((0, 1), [0,+\infty))$ is allowed to be singular at $t = 0$ and $t = 1$, $f \in C([0, 1] \times [0,+\infty), (-\infty, +\infty))$ is allowed to change sign.

The multi-point boundary value problems for ordinary differential equations arise in a variety of areas of applied mathematics and physics. The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev[8]. Since then, attention has been focused on the study of nonlinear multi-point boundary value problems as can be seen from for example, [1, 5, 6, 7, 10, 12, 13, 14] and their references. Recently, by using the Krasnosel'skii fixed point theorem, Ma[11] showed the existence of at least one positive solution for the three point boundary value problem

$$\begin{cases} u''(t) + a(t) f(u) = 0, 0 < t < 1, \\ u(0) = 0, u(1) = \alpha u(\eta), \end{cases}$$

where $a \in C([0, 1], [0, +\infty))$ and $f \in C([0, +\infty), [0, +\infty))$ is either superlinear or sublinear. In [15], the second order m-point boundary value problem

$$\begin{cases} \varphi''(t) + h(t) f(\varphi(t)) = 0, 0 < t < 1, \\ \varphi(0) = 0, \varphi(1) = \sum_{i=1}^{m-2} a_i \varphi(\xi_i), \end{cases}$$

was considered under some conditions concerning the first eigenvalue of the relevant linear operator, where $h \in C((0, 1), [0,+\infty))$ is allowed to be singular at $t = 0$, $t = 1$ and $f \in C([0,+\infty), [0,+\infty))$. By using the fixed point index theory , the existence of positive solutions was obtained. In [4], by constructing two cones, the author obtained two positive solutions for a three point boundary value problem with sign-changing nonlinearities

$$\begin{cases} x''(t) + f(t, x) = 0, 0 \le t \le 1, \\ x(0) - \beta x'(0) = 0, x(1) = \alpha x(\eta), \end{cases}$$

where $\beta > 0$ and $f \in C([0, 1] \times [0,+\infty), (-\infty,+\infty))$ dose not have any singularity.

In this paper, we study BVP (1.1) for the cases where $m \ge 3$, $\alpha \ge 0$, $\beta \ge 0$, but $\alpha + \beta > 0$, q is allowed to be singular at $t = 0$ and $t = 1$, and f is allowed to change sign. The existence of at least three nontrivial positive solutions is obtained by using the Leggett-Williams fixed point theorem.

2. SOME DEFINITIONS AND LEMMAS

Suppose P is a cone in a Banach space E. The map τ is a nonnegative continuous concave functional on P provided $\tau : P \to [0,+\infty)$ is continuous and $\tau(tu + (1 - t)v) \ge t\tau(u) + (1-t)\tau(v)$ for all $u, v \in P$ and $0 \le t \le 1$. Let constants a, b and $r > 0$ be given and let τ be a nonnegative continuous concave functional on P. Define P_r and $P_a(b)$ by

$$P_r = \{u \in P : \|u\| < r\}, P_a(b) = \{u \in P : a \le \tau(u), \|u\| \le b\}.$$

Lemma 2.1. (Leggett-Williams Fixed Point Theorem) *Let* $A : \overline{P_c} \to \overline{P_c}$ *be a completely continuous operator and* τ *be a nonnegative continuous concave functional* *on P such that* $\tau(u) \le \|u\|$ *for all* $u \in \overline{P_c}$. *If there exist real numbers* a, b *and* d *with* $0 < a < b < d \le c$ *such that*

(C_1) $\{u \in P_b(d) : \tau(u) > b\} \ne \phi$, *and* $\tau(Au) > b$ *for* $u \in P_b(d)$;

(C_2) $\|Au\| < a$ *for* $\|u\| \le a$;

(C_3) $\tau(Au) > b$ *for* $u \in P_b(c)$ *with* $\|Au\| > d$;

then A has at least three fixed point u_1, u_2 *and* u_3 *such that*

$$\|u_1\| < a,\ b < \tau(u_2), \text{ and } \|u_3\| > a \text{ with } \tau(u_3) < b.$$

The following conditions will be assumed throughout this paper:

(H_1) $f : [0, 1] \times [0,+\infty) \to (-\infty,+\infty)$ is continuous and $f(t, 0) \ge 0 (\not\equiv 0)$,

(H_2) $q : (0, 1) \to [0,+\infty)$, $q(t) \not\equiv 0$ on any subinterval of $(0, 1)$ and $\int_0^1 q(t)dt < +\infty$,

(H_3) $\alpha \ge 0$, $\beta \ge 0$, $\rho = \beta + \alpha > 0$ and $\Delta = \rho - \sum_{i=1}^{m-2} k_i(\beta + \alpha\xi_i) > 0$.

Lemma 2.2. *Suppose* (H_2) and (H_3) *hold. Then the problem*

$$\begin{cases} u''(t) + q(t) = 0, \ 0 < t < 1 \\ \alpha u(0) - \beta u'(0) = 0, \ u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = 0 \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)q(s)ds + B_1\psi(t),$$

where

$$\psi(t) = \beta + \alpha t, \varphi(t) = 1 - t,$$

$$G(t,s) = \begin{cases} \frac{1}{\rho}\varphi(t)\psi(s), & 0 \le s \le t \le 1, \\ \frac{1}{\rho}\varphi(s)\psi(t), & 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

$$B_1 = \frac{1}{\Delta}\sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i, s)q(s)ds. \tag{2.3}$$

Proof. It is easy to see that ψ and φ are two linearly independent solutions of the equation $u'' = 0$, so the solution of the equation $u''(t) + q(t) = 0$ can be expressed by

$$u(t) = \int_0^1 G(t,s)(s)ds + B_1\psi(t) + B_2\varphi(t), \tag{2.4}$$

where B_1 and B_2 are constants. The fact that when B_1 satisfies (2.3) and $B_2 = 0$, $u(t)$ defined by (2.4) is a solution of (2.1) is easy to check.

On the other hand, we will show that when $u(t)$ defined by (2.4) is a solution of (2.1), B_1 satisfies (2.3) and $B_2 = 0$. Suppose

$$u(t) = \int_0^1 G(t,s)(s)ds + B_1\psi(t) + B_2\varphi(t)$$

is a solution of (2.1), we have

$$u(t) = \int_0^1 \frac{1}{\rho}\varphi(t)\psi(s)q(s)ds + \int_t^1 \frac{1}{\rho}\varphi(s)\psi(t)q(s)ds + B_1\psi(t) + B_2\varphi(t),$$

$$u'(t) = \varphi'(t)\int_0^t \frac{1}{\rho}\psi(s)q(s)ds + \psi'(t)\int_t^1 \frac{1}{\rho}\varphi(s)q(s)ds + B_1\psi'(t) + B_2\varphi'(t),$$

$$u''(t) = \varphi''(t)\int_0^t \frac{1}{\rho}\psi(s)q(s)ds + \varphi'(t)\int_t^1 \frac{1}{\rho}\psi(t)q(t) + \psi''(t)\int_0^1 \frac{1}{\rho}\varphi(s)q(s)ds$$
$$-\psi'(t)\frac{1}{\rho}\varphi(t)q(t) + B_1\psi''(t) + B_2\varphi''(t).$$

Thus

$$u''(t) = \frac{1}{\rho}[\psi(t)\varphi'(t) - \varphi(t)\psi'(t)]q(t) = -q(t).$$

From

$$u(0) = \beta\int_0^1 \frac{1}{\rho}\varphi(s)q(s)ds + B_1\beta + B_2,$$

$$u'(0) = \alpha\int_0^1 \frac{1}{\rho}\varphi(s)q(s)ds + B_1\alpha + B_2,$$

we have $B_2\rho = 0$, thus $B_2 = 0$. Since $u(1) = B_1\rho$, we have

$$B_1\rho = \sum_{i=1}^{m-2} k_i u(\xi_i) = \sum_{i=1}^{m-2} k_i \left[\int_0^1 G(\xi_i, s)q(s)ds + B_1\psi(\xi_i)\right].$$

Therefore,

$$B_1 = \frac{1}{\Delta}\sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i, s)q(s)ds.$$

The proof of Lemma 2.2 is completed.

Lemma 2.3. *Suppose* (H_2) and (H_3) *hold. Then the unique solution* $u(t)$ *of* (2.1) *satisfies*

$$u(t) \geq 0, 0 \leq t \leq 1 \ \ \textit{and} \ \ \min_{\sigma\leq t\leq 1-\sigma} u(t) \geq \gamma \| u \|,$$

where

$$\sigma \in \left(0, \frac{1}{2}\right), \gamma = \min\left\{\sigma, \frac{\beta+\alpha\sigma}{\beta+\alpha}\right\}.$$

Proof. From (H_2), (H_3), we obtain $0 \leq G(t, s) \leq G(s, s)$ for $t \in [0, 1]$ and $B_1 \geq 0$. So by Lemma 2.2, we know $u(t) \geq 0$, for $t \in [0, 1]$, and

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)q(s)ds + B_1\psi(t) \\ &\leq \int_0^1 G(s,s)q(s)ds + B_1\psi(t) \\ &\leq \int_0^1 G(s,s)q(s)ds + (\beta+\alpha)B_1, t \in [0,1], \end{aligned}$$

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \begin{cases} \dfrac{\varphi(t)}{\varphi(s)}, & 0 \leq s \leq t \leq 1 \\[2ex] \dfrac{\psi(t)}{\psi(s)}, & 0 \leq s \leq t \leq 1 \end{cases} \\[2ex] &\geq \begin{cases} \sigma, & 0 \leq s \leq t \leq 1-\sigma \\[1ex] \dfrac{\beta+\alpha\sigma}{\beta+\alpha}, & \sigma \leq t \leq s \leq 1 \end{cases} \\[2ex] &\geq \gamma. \end{aligned}$$

Therefore, for all $t \in [\sigma, 1-\sigma]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)q(s)ds + B_1\psi(t) \\ &= \int_0^1 \frac{G(t,s)}{G(s,s)} G(s,s)q(s)ds + B_1\psi(t) \\ &\geq \gamma \int_0^1 G(s,s)q(s)ds + B_1\psi(t) \\ &\geq \gamma \int_0^1 G(t,s)q(s)ds + \frac{\beta+\alpha\sigma}{\beta+\alpha}(\beta+\alpha)B_1 \\ &\geq \gamma \left[\int_0^1 G(t,s)q(s)ds + (\beta+\alpha)B_1 \right] \\ &\geq \gamma \| u \|. \end{aligned}$$

The proof of Lemma 2.3 is completed.

Let $\| u \| = \max_{0 \le t \le 1} | u(t) |$ for all $u \in C[0, 1]$, $P = \{u \in C[0, 1] : u(t) \geq 0, t \in [0, 1]\}$.

Then P is a cone on $C[0, 1]$. For $u \in P$, we define $\tau(u) = \max_{\sigma \le t \le 1-\sigma} u(t)$. Then τ is a nonnegative continuous concave functional on P and $\tau(u) \leq \|u\|$. Define

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t,s)q(s)f(s,u(s))ds \\ &\quad + \frac{1}{\Delta} \sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i, s)q(s)f(s,u(s))ds\psi(t), \ \ u \in P, t \in [0,1], \end{aligned}$$

$(Tu)(t) = \max\{(Au)(t), 0\}$, $u \in P$, $t \in [0, 1]$.

For $u \in C[0, 1]$, define $\mu : C[0, 1] \to P$ by $(\mu u)(t) = \max \{u(t), 0\}$. From the definition, we have $T = \mu \circ A$.

Lemma 2.4. *If* $A : P \to C[0, 1]$ *is a completely continuous operator, then* $T : P \to P$ *is a* completely continuous operator.

Proof. The complete continuity of A implies that A is continuous and maps each bounded subset in P to a relatively compact set. Denote μy by $\overline{y}$, $y \in C[0,1]$. Given a function $h \in P$, for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|Ah - Ag\| < \varepsilon, \text{ for } g \in P, \|g - h\| < \delta.$$

Since

$$|(\mu Ah)(t) - (\mu Ag)(t)| = |\max\{(Ah)(t), 0\} - \max\{(Ag)(t), 0\}|$$
$$\leq |(Ah)(t) - (Ag)(t)| < \varepsilon,$$

we have

$$\|(\mu A)(h) - (\mu A)(g)\| < \varepsilon, \textit{for } g \in P, \|g - h\| < \delta,$$

and then $\mu \circ A$ is continuous.

For an arbitrarily given bounded set $D \subset P$ and $\varepsilon > 0$, there are y_i, $i = 1, \ldots, m$ such that

$$AD \subset \bigcup_{i=1}^{m} B(y_i, \varepsilon),$$

where $B(y_i, \varepsilon) = \{u \in P : \|u - y_i\| < \varepsilon\}$. Then, for each $\overline{y}(t) \in (\mu \circ A)D$, there is $y \in AD$ such that $\overline{y}(t) = \max\{y(t), 0\}$. We choose one $y_i \in \{y_1, \ldots, y_m\}$ such that $\|y - y_i\| < \varepsilon$. The fact

$$\max_{0 \leq t \leq 1} |\overline{y}(t) - \overline{y}_i(t)| \leq \max_{0 \leq t \leq 1} |y(t) - y_i(t)|$$

implies $\overline{y} \in B(\overline{y}_i, \varepsilon)$. Then $(\mu \circ A)\, D$ has a finite ε–net, and therefor, $(\mu \circ A)\, D$ is relatively compact. So T is a completely continuous operator.

Lemma 2.5. *Suppose* (H_1), (H_2) *and* (H_3) *hold. Then* $T: P \to P$ *is a completely continuous operator*.

Proof. First of all, we show that $A : P \to C[0, 1]$ is a completely continuous operator. Let $D \subset P$ denote a bounded set. Then there exists $M_1 > 0$ such that $\|u\| \leq M_1$ for all $u \in D$. Since f is continuous, there exists $M_2 > 0$ such that $|f(t, u)| \leq M_2$ for $(t, u) \in [0, 1] \times [0, M_1]$. *By* (H_2), for $u \in D$, we have

$$|(Au)(t) \leq \int_0^1 G(t,s)q(s)\,|f(s,u(s))|\,ds + \frac{1}{\Delta}\sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i, s)q(s)\,|f(s,u(s))|\,ds$$

$$\leq M_2 \int_0^1 G(t,s)q(s)ds + \frac{1}{\Delta}\sum_{i=1}^{m-2} k_i M_2 \int_0^1 G(\xi_i, s)q(s)ds < +\infty.$$

Thus $AD = \{Au : u \in D\} \subset C[0, 1]$ is a bounded set. For $u \in D$,

$$|(Au)'(t)| = \left| -\frac{1}{\rho}\int_0^t (\beta+\alpha s)q(s)f(s,u(s))ds + \frac{1}{\rho}\int_t^1 \alpha(1-s)q(s)f(s,u(s))ds \right.$$

$$\left. + \frac{\alpha}{\Delta}\sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i,s)q(s)f(s,u(s))ds \right|$$

$$\leq \frac{M_2}{\rho}\int_0^t (\beta+\alpha s)q(s)ds + \frac{M_2}{\rho}\int_t^1 \alpha(1-s)q(s)ds$$

$$+ \frac{\alpha M_2}{\Delta}\sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i,s)q(s)ds$$

$$= g_1(t) + g_2(t) + g_3,$$

where

$$g_1(t) = \frac{M_2}{\rho}\int_0^t (\beta+\alpha s)q(s)ds,$$

$$g_2(t) = \frac{M_2}{\rho}\int_t^1 \alpha(1-s)q(s)ds,$$

$$g_3 = \frac{\alpha M_2}{\Delta}\sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i,s)q(s)ds.$$

From

$$\int_0^1 |g_1(t)|\,dt = \int_0^1 \left(\frac{M_2}{\rho}\int_0^t (\beta+\alpha s)q(s)ds \right) dt$$

$$= \frac{M_2}{\rho}\int_0^1 (1-s)(\beta+\alpha s)q(s)ds$$

$$< +\infty,$$

$$\int_0^1 |g_2(t)|\,dt = \int_0^1 \left(\frac{M_2}{\rho}\int_t^1 \alpha(1-s)q(s)ds \right) dt$$

$$=\frac{M_2}{\rho}\int_0^1 \alpha s(1-s)q(s)ds <+\infty,$$

$$\int_0^1 |g_3|\,dt=\int_0^1\left(\frac{\alpha M_2}{\Delta}\sum_{i=1}^{m-2}k_i\int_0^1 G(\xi_i,s)q(s)ds\right)dt$$

$$=\frac{M_2}{\Delta}\sum_{i=1}^{m-2}k_i\int_0^1 G(\xi_i,s)q(s)ds <+\infty,$$

we obtain $0<\int_0^1 |(Au)'(t)|\,dt<+\infty.$ It is easy to show that AD is equicontinuous, that is, A is compact. Let u_n, $u\in P$ and $u_n\to u(n\to+\infty)$. Then

$$|(Au_n)(t)-(Au(t)|\le\int_0^1 G(t,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds$$
$$+\frac{1}{\Delta}\sum_{i=1}^{m-2}k_i\int_0^1 G(\xi_i,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds$$
$$\le\int_0^1 G(s,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds$$
$$+\frac{1}{\Delta}\sum_{i=1}^{m-2}k_i\int_0^1 G(\xi_i,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds$$
$$=\int_0^{1/n} G(s,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds$$
$$+\int_{1/n}^{1-1/n} G(s,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds$$
$$+\int_{1-1/n}^{1} G(s,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds$$
$$+\frac{1}{\Delta}\sum_{i=1}^{m-2}k_i\int_0^{1/n} G(\xi_i,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds$$
$$+\frac{1}{\Delta}\sum_{i=1}^{m-2}k_i\int_{1/n}^{1-1/n} G(\xi_i,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds$$
$$+\frac{1}{\Delta}\sum_{i=1}^{m-2}k_i\int_{1-1/n}^{1} G(\xi_i,s)q(s)\,|f(s,u_n(s))-f(s,u(s))|\,ds,\ n>2.$$

From (H_1), (H_2) (H_3), it is easy to get

$$\| Au_n - Au \| = \max_{0 \le t \le 1} |(Au_n)(t) - (Au)(t)| \to 0,\ n \to \infty,$$

therefore, A is continuous. By using the Ascoli-Arzela Theorem, we obtain $A : P \to C[0, 1]$ is a completely continuous operator. Finally, from Lemma 2.4, we have $T = \mu \circ A : P \to P$ is a completely continuous operator.

3. MAIN RESULT

Let

$$\Lambda = \max_{0 \le t \le 1} \int_0^1 G(t,s)q(s)ds + \frac{\rho}{\Delta} \sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i, s)q(s)ds$$

$$\lambda = \max_{0 \le t \le 1-\sigma} \int_\sigma^{1-\sigma} G(t,s)q(s)ds + \frac{\psi(\sigma)}{\Delta} \sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i, s)q(s)ds.$$

Then $0 < \lambda < \Lambda$.

Theorem 3.1. *Suppose* (H_1), (H_2) and (H_3) *hold. In addition, there exist real numbers a, b, c such that* $0 < a < b \le \min\{\gamma, \lambda / \Lambda\}c$ *and f satisfies the following conditions:*

(H_4) $f(t, u) \le c / \Lambda$ *for all* $(t, u) \in [0, 1] \times [0, c]$,

(H_5) $f(t, u) < a / \Lambda$ *for all* $(t, u) \in [0, 1] \times [0, a]$,

(H_6) $f(t, u) > b / \lambda$ *for all* $(t, u) \in [\sigma, 1 - \sigma] \times [b, b / \gamma]$,

(H_7) $f(t, u) \ge 0$ *for all* $(t,u) \in [0, 1] \times [b, c]$.

Then the m-point boundary value problem (1.1) *has at least three nontrivial positive solutions* u_1, u_2, u_3, *such that*

$$0 < \| u_1 \| < a,\ b < \min_{\sigma \le t \le 1-\sigma} u_2 \ \ and \ \ \| u_3 \| > a \ \ with \ \ \min_{\sigma \le t \le 1-\sigma} u_3 < b.$$

Proof. At first, we show that $T : \overline{P_c} \to \overline{P_c}$ is a completely continuous operator. If $u \in \overline{P_c}$, then $\|u\| \le c$. From (H_4), we obtain

$$\| Tu \| = \max_{0 \le t \le 1} |\max\{(Au)(t),\ 0\}|$$

$$= \max_{0\le t\le 1}\left|\max\left\{\int_0^1 G(t,s)q(s)f(s,u(s))ds + \frac{1}{\Delta}\sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i,s)q(s)f(s,u(s))ds\psi(t), 0\right\}\right|$$

$$\le \max_{0\le t\le 1}\frac{c}{\Lambda}\left[\int_0^1 G(t,s)q(s)ds + B_1\rho\right] = c.$$

Thus $T:\left(\overline{P_c}\right)\subset \overline{P_c}$. From Lemma 2.5, we have $T:\overline{P_c}\to\overline{P_c}$ is a completely continuous operator.

Next, we show that T has a fixed point u_1, and u_1 is a solution of (1.1). For $||u|| \le a$, from (H_5), we obtain

$$\| Tu \| = \max_{0\le t\le 1} |\max\{(Au)(t), 0\}|$$

$$= \max_{0\le t\le 1}\left|\max\left\{\int_0^1 G(t,s)q(s)f(s,u(s))ds + \frac{1}{\Delta}\sum_{i=1}^{m-2} k_i \int_0^1 G(\xi_i,s)q(s)f(s,u(s))ds\psi(t), 0\right\}\right|$$

$$\le \max_{0\le t\le 1}\frac{a}{\Lambda}\left[\int_0^1 G(t,s)q(s)ds + B_1\rho\right] = a.$$

Combining with (C_2) in Lemma 2.1 and (H_1), (H_2), we conclude that T has a fixed point u_1, $0 < ||u_1|| < a$. Next, we show that u_1 is a solution of (1.1), that is, u_1 is a fixed point of A. Suppose this is not true, then there is $t_0 \in (0, 1)$ such that $u_1(t_0) \ne (Au_1)(t_0)$. It must be $(Au_1)(t_0) < 0 = u_1(t_0)$. Let (t_1, t_2) be the maximal interval such that $t_0 \in (t_1, t_2)$, $(Au_1)(t) < 0$ for all $t \in (t_1, t_2)$. It is easy to see that $[t_1, t_2] \ne [0, 1]$ by (H_1). If $t_2 < 1$, then $u_1(t) \equiv 0$ for all $t \in [t_1, t_2]$, and $(Au_1)(t) < 0$, for all $t \in (t_1, t_2)$, and $(Au_1)(t_2) \ge 0$. Thus $(Au_1)'(t_2) \ge 0$. (H_1) and (H_2) imply $(Au_1)''(t) = -q(t)f(t, 0) \le 0$ for $t \in (t_1, t_2)$. So, $(Au_1)'(t) \ge 0$, for $t \in [t_1, t_2]$. We obtain $t_1 = 0$. On the other hand, $\alpha(Au_1)(0) - \beta(Au_1)'(0) = 0$. If $\alpha = 0$, then

$$(Au_1)'(0) \ge (Au_1)'(t_0) > 0 = (Au_1)'(0),$$

which is a contradiction. If $\beta = 0$, then $(Au_1)(0) \le (Au_1)(t_0) < 0 = (Au_1)(0)$, which is a contradiction. If $\alpha\,\beta \ne 0$, then

$$(Au_1)'(0) = \frac{\alpha}{\beta}(Au_1)(0) < 0,$$

which is a contradiction. If $t_1 > 0$, then $u_1(t) \equiv 0$ for all $t \in [t_1, t_2]$, and $(Au_1)(t) < 0$, for all $t \in (t_1, t_2)$ and $(Au_1)(t_1) = 0$. Thus $(Au_1)'(t_1) \le 0$. (H_1) and (H_2) imply $(Au_1)''(t)$

$= -q(t)\ f\ (t,\ 0) \le 0$ for $t \in (t_1,\ t_2)$. We obtain $t_2 = 1$. On the other hand, $(Au)(1) - \sum_{i=1}^{m-2} k_i (Au)(\xi_i) = 0,$ so there is i, $1 \le i \le m - 2$, such that $(Au_1)(\xi_i) < 0$. Let $j = \min\ \{i : (Au_1)(\xi_i) < 0,\ 1 \le i \le m - 2\}$, then $(Au_1)(\xi_i) < 0$ for $j \le i \le m - 2$. So $\xi_i \in (t_1,\ 1)$ for $j \le i \le m - 2$. As $(Au_1)(t)$ is concave on $[t_1,\ 1]$, we obtain

$$\frac{(Au_1)(\xi_i)}{\xi_i - t_1} \ge \frac{(Au_1)(1)}{1 - t_1},\ j \le i \le m - 2,$$

thus,

$$(Au_1)(\xi_i) \ge \frac{\xi_i - t_1}{1 - t_1}(Au_1)(1) \ge (\xi_i)(Au_1)(1),\ j \le i \le m - 2.$$

If $j = 1$, then

$$(Au_1)(1) = \sum_{i=1}^{m-2} k_i (Au_1)(\xi_i) \ge (Au_1)(1) \sum_{i=1}^{m-2} k_i \xi_i.$$

So $\sum_{i=1}^{m-2} k_i \xi_i \ge 1,$ furthermore,

$$\rho = \beta + \alpha \le \sum_{i=1}^{m-2} k_i (\beta + \alpha) \xi_i \le \sum_{i=1}^{m-2} k_i \psi(\xi_i),$$

which is a contradiction with (H_3). If $2 \le j \le m - 2$, then

$$\begin{aligned}(Au_1)(1) &= \sum_{i=1}^{m-2} k_i (Au_1)(\xi_i) = \sum_{i=1}^{j-1} k_i (Au_1)(\xi_i) + \sum_{i=j}^{m-2} k_i (Au_1)(\xi_i) \\ &\ge \sum_{i=1}^{j-1} k_i (Au_1)(\xi_i) + (Au_1)(1) \sum_{i=j}^{m-2} k_i \xi_i \\ &> (Au_1)(1) \sum_{i=1}^{j-1} k_i \xi_i + (Au_1)(1) \sum_{i=j}^{m-2} k_i \xi_i \\ &= (Au_1)(1) \sum_{i=1}^{m-2} k_i \xi_i.\end{aligned}$$

So $\sum_{i=1}^{m-2} k_i\xi_i > 1,$ furthermore,

$$\rho = \beta + \alpha < \sum_{i=1}^{m-2} k_i(\beta+\alpha)\xi_i \le \sum_{i=1}^{m-2} k_i\psi(\xi_i),$$

which is a contradiction with (H_3). Therefor, u_1 is a solution of (1.1).

We now verify that (C_1) of Lemma 2.1 is satisfied. It is easy to see $\{u \in P_b(b/\gamma)$: $\tau(u) > b\} \ne \phi$. If $u \in P_b(b/\gamma)$, then $b \le u(s) \le b/\gamma$ for $s \in [\sigma, 1-\sigma]$. From (H_6), we obtain

$$\begin{aligned}\tau(Au) &= \min_{\sigma\le t\le 1-\sigma} \max\{(Au)(t), 0\}\\ &= \min_{\sigma\le t\le 1-\sigma}\left[\int_0^1 G(t,s)q(s)f(s,u(s))ds + \frac{1}{\Delta}\sum_{i=1}^{m-2} ki\int_0^1 G(\xi_i,s)q(s)f(s,u(s))ds\psi(t)\right]\\ &\ge \min_{\sigma\le t\le 1-\sigma}\int_\sigma^{1-\sigma} G(t,s)q(s)f(s,u(s))ds + \frac{1}{\Delta}\sum_{i=1}^{m-2} ki\int_0^1 G(\xi_i,s)q(s)f(s,u(s))ds\psi(\sigma)\\ &> \frac{b}{\lambda}\left[\min_{\sigma\le t\le 1-\sigma}\int_\sigma^{1-\sigma} G(t,s)q(s)ds + B_1\psi(\sigma)\right] = b.\end{aligned}$$

Finally, we verify that (C_3) of Lemma 2.1 is satisfied. Suppose $u \in P_b(c)$ with $\|Tu\| > b/\gamma$. From (H_7) and Lemma 2.3, we obtain

$$\tau(Au) = \min_{\sigma\le t\le 1-\sigma}(Tu)(t) \ge \gamma\|Tu\| > b.$$

By Lemma 2.1, T has at least three fixed point. So, the m-point boundary value problem (1.1) has at least three positive solutions u_1, u_2 and u_3 such that

$$0 < \|u_1\| < a,\ b < \min_{\sigma\le t\le 1-\sigma} u_2 \text{ and } \|u_3\| > a \text{ with } \min_{\sigma\le t\le 1-\sigma} u_3 < b.$$

The proof of theorem 3.1 is completed.

4. AN EXAMPLE

Example 4.1. Consider the boundary value problem

$$\begin{cases} u'' + q(t)f(t,u) = 0, & 0 < t < 1,\\ u(0) - u'(0) = 0, & u(1) - u\left(\frac{1}{4}\right) = 0, \end{cases} \tag{4.1}$$

where $q(t)=\frac{1}{\sqrt{t}}$,

$$f(t,u)=\begin{cases}-1.3u+0.3+0.1t & 0\le t\le 1, 0\le u<1,\\ 3u-4+0.1t, & 0\le t\le 1, 1\le u<2,\\ 2+0.1t, & 0\le t\le 1, 2\le u<6,\\ -u+0.1t+8, & 0\le t\le 1, u\ge 6.\end{cases} \tag{4.2}$$

Conclusion. Problem (4.1) has at least three nontrivial positive solutions.

Proof. Let $\sigma=\frac{9}{25}, a=1, b=2, c=6.$ From (4.1), we know $m=3$, $\alpha=1$, $\beta=1$, $k_1=1$, $\xi_1=\frac{1}{4}$. Clearly, (H_1), (H_2) and (H_3) hold. After some simple calculation, we have $\rho=2, \Delta=\frac{3}{4}, B_1=\frac{8}{9}, \gamma=\frac{9}{25}, \Lambda=\frac{200}{81}, \lambda=\frac{37024}{28125}, \frac{c}{\Lambda}=\frac{243}{100}, \frac{a}{\Lambda}=\frac{81}{200}, \frac{b}{\lambda}=\frac{28125}{18512}$.
Combining with (4.2), we obtain

$$f(t,u)\le\frac{243}{100}\quad for\quad (t,u)\in[0,1]\times[0,6],$$

$$f(t,u)<\frac{81}{200}\quad for\quad (t,u)\in[0,1]\times[0,1],$$

$$f(t,u)>\frac{28125}{18512}\quad for\quad (t,u)\in\left[\frac{9}{25},\frac{16}{25}\right]\times\left[2,\frac{50}{9}\right],$$

$$f(t,u)\ge 0\ for\ (t,u)\in[0,1]\times[2,6].$$

Thus, by an application of Theorem 3.1, we get that problem (4.1) has at least three nontrivial positive solutions u_1, u_2 and u_3 such that

$$0<\|u_1\|<1, 2<\min_{t\in[9/25,16/25]}u_2, \|u_3\|>1 \text{ with } \min_{t\in[9/25,16/25]}u_3<2.$$

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