Journal of Nonlinear Functional Analysis and Differential Equations
Vol. 3 No. 2 (July-December, 2018)

PRACTICAL STABILITY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAYS*

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ABSTRACT: This paper studies the practical stability problems for a class of impulsive functional differential equations with infinite delays. By using Liapunov functions and Razumikhin techniques or Liapunov functionals, some results on practical stability are obtained.

Keywords: Practical stability; Impulsive functional differential equation; Razumikhin technique.

1. INTRODUCTION

It is known that, the mathematical theory of impulsive differential equations has been developed very intensively, see [1-9] and references therein. One of the trends in the stability theory of differential equations is the so-called practical stability which is neither weaker nor stronger than Lyapunov stability (*cf.* [11]). Fundamental results in this direction were obtained in [10-11]. In recent years, the theory of practical stability of impulsive differential equations has been also intensively developed, see [12,13,14] and references therein. However, there are rare results of the practical stability for impulsive functional differential equations with infinite delays, compared with the results of impulsive ordinary differential equations or impulsive functional differential equations with finite delays (*cf.* [13,14]). As pointed out in [15,16,17], even though for functional differential equations without impulses, stability results established for equations with finite delays are not obviously true in general for infinite delays. The common and main difficulty is that the interval $(-\infty; t_0]$ is not compact, and the images of a solution map of closed and bounded sets in $C((-\infty, 0], R^n)$ space may not be compact. Same situation arises in $PC((-\infty, 0], R^n)$ space for impulsive functional differential equations with infinite delays (*cf.* [19,20]). Therefore, it is an interesting problem to extend the methods for investigating stability of functional differential equations with infinite delays to practical stability of impulsive functional differential equations with infinite delays.

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* This work is supported by the NNSF of China (10571050), Hunan Provincial Natural Science Foundation of (05JJ40013) and Scientific Research Fund of Hunan Provincial Education Department(05C413).

In this paper, we use Liapunov functions and Razumikin techniques or Liapunov functionals to study the practical stability of impulsive functional differential equations with infinite delays, some useful results are obtained.Two examples are given to illustrate our results.

Let $R = (-\infty, \infty)$, $R^+ = [0, \infty)$. For $x \in R^n$, $|\cdot|$ denotes the Euclidean norm of x, the ball $S(H)$ of R^n is denoted by $S(H) = \{x \in R^n : |x| < H \le \infty\}$. For $t \ge t^* > \alpha \ge -\infty$, $F(t, x(s); \alpha \le s \le t)$ or $F(t \le x(\cdot))$ is a Volterra-type functional, its values are in R^n and are determined by $t \ge t^*$ and the values of $x(s)$ for $[\alpha, t]$. In the case when $\alpha = -\infty$, the interval $[\alpha, t]$ is understood to be replaced by $(-\infty, t]$. We consider the impulsive Volterra-type functional differential equation with infinite delays

$$\begin{cases}(x'(t) = F(t, x(\cdot)), t \ne t_k, t > t^*, x \in R^n, \\ \Delta x(t_k) = x(tk) - x(t_k^-) = I_k\left(x(t_k^-)\right), k \in Z^+,\end{cases} \tag{1.1}$$

where Z^+ is the set of all positive integers, $x'(t)$ denotes the right-hand derivative of $x(t)$ at t, where $x\left(t_k^-\right) = \lim_{t \to t_k - 0} x\left(t\right)$. For $k \in Z^+$, $t^* < t_1 < t_2 < \ldots < t_k < t_{k+1}$ with $t_k \to \infty$ as $k \to \infty$, and $I_k : R^n \to R^n$ are some given functions.

Let $I \subset R$ be any interval. Define $PC(I, R^n) = \{x : I \to \infty\ R^n$, x is continuous everywhere except at the points $t = t_k \in I$ and $x\left(t_k^-\right), x\left(t_k^+\right) = \lim_{t \to t_k - 0} x\left(t\right)$ exist with $x\left(t_k^+\right) = x\left(t_k\right)\}$. For any $t \ge t^*$, $PC([\alpha, t], R^n)$ will be written as $PC(t)$. Define $PCB(t) = \{x \in PC(t) : x \text{ is bounded}\}$. For any $\phi \in PCB(t)$, the norm of ϕ is defined as

$$\|\phi\| = \|\phi\|^{[\alpha, t]} = \sup_{\alpha \le s \le t} \left|\phi\left(s\right)\right|.$$

For given $\sigma \ge t^*$ and $\phi \in PCB(t)$, the initial value problem of Eq. (1.1) is

$$\begin{cases}(x'(t) = F(t, x(\cdot)), t \ne t_k, t > t^*, x \in R^n, \\ \Delta x(t_k) = x(tk) - x(t_k^-) = I_k\left(x(t_k^-)\right), k \in Z^+, \\ x(t) = \phi(t), \alpha \le t \le \delta.\end{cases} \tag{1.2}$$

Definition 1.1: A function $x(t)$ is called a solution corresponding to σ of the initial value problem (1.2) if $x : [\alpha, \beta) \to R^n$ (for some $t^* < \beta \le \infty$) is continuous for $t \in [\alpha, \beta)\backslash\{t_k, k = 1, 2, \ldots\}$, $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^+) = x(t_k)$, and satisfies (1.2). We define by $x(t, \sigma, \phi)$ the solution of the initial value problem (1.2).

Under the following hypotheses (H_1)–(H_4), there is a unique solution of problems (1.2), we denote it by $x(t) = x(t, \delta, \phi)$ (see [9, 18]).

(H_1) F is continuous on $[t_{k-1}, t_k) - PC(t)$, $k = 1, 2, \ldots$, where $t_0 = t^*$. For all $\varphi \in PC(t)$ and $k \in Z^+$, the limit $\lim_{(t,\phi)\to(t_k^-,\varphi)} F(t,\phi) = F(t_k^-, \varphi)$ exists.

(H_2) F is locally Lipschitzian in ϕ in each compact set in $PCB(t)$. More precisely, for every $a \in [t^*, \beta)$ and every compact set $G \subset PCB(t)$ there exists a constant $L = L(a, G)$ such that

$$|F(t, \varphi(\cdot)) - F(t, \psi(\cdot))| \le L\|\varphi - \psi\|^{[\alpha, t]},$$

whenever $t \in [t^*, a]$ and $\varphi, \psi \in G$.

(H_3) For each $k \in Z^+$, $I_k(x) \in C(R^n, R^n)$, and there exists some $0 < H_1 \le H$ such that $x \in S(H_1)$ implies that $x + I_k(x) \in S(H)$ for all $k \in Z^+$.

(H_4) For $x \in PC([\alpha, \infty), R^n)$, the composite function $F(t, x(\cdot)) \in PC([t^*, \infty), R^n)$.

For any $t \ge t^*$ and $\rho > 0$, let

$$PCB_\rho(t) = \{\phi \in PCB(t) : \|\phi\| < \rho\}.$$

In [19, 20], uniformly asymptotical stability of (1.2) was studied, and some criteria were established. In this paper, we study practical stability of (1.2), the following definition will be used.

Definition 1.2: The impulsive functional differential equation (1.2) is said to be

(S1) practically stable if, given (λ, A) with $0 < \lambda < A$, we have $\phi \in PCB_\lambda(\delta)$ implies $|x(t, \delta, \phi)| < A$, $t \ge \delta$ for some $\delta \ge t^*$;

(S2) uniformly practically stable if (S1) holds for every $\delta \ge t^*$;

(S3) quasi-equi asymptotically stable in the large if for each $\varepsilon > 0$, $\alpha > 0$, $\delta \ge t^*$, there exists a positive number $T = T(\delta, \varepsilon, \alpha)$ such that $\phi \in PCB_\alpha(\delta)$ implies $|x(t, \delta, \phi)| < \varepsilon$ for $t \ge \delta + T$;

(S4) quasi-uniformly asymptotically stable in the large if the number T in (S3) is independent of δ;

(S5) practically asymptotically stable if (S1) and (S3) hold with $\alpha = \lambda$;

(S6) uniformly practically asymptotically stable if (S2) and (S4) hold at the same time with $\alpha = \lambda$;

(S7) practically unstable if (S1) does not hold.

We define the following Lyapunov like function and functional.

Definition 1.3: A function $V(t, x) : [t^*, \infty) \times S(H) \to R^+$ belongs to class ν_0 if

(1) V is continuous on each of the sets $[t_{k-1}, t_k) \times S(H)$ and for all $x \in S(H)$, $k \in Z^+$, the limit $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists.

(2) V is locally Lipschitzian in x and $V(t, 0) \equiv 0$.

Definition 1.4: A functional $V(t, \phi) : [t^*, \infty)\ PCB(t) \to R^+$ belongs to class $\nu0(\cdot)$ if

(1) V is continuous on each of the sets $[t_{k-1}, t_k) \times PCB(t)$ and for all $\varphi \in PCB(t)$, $k \in Z^+$, the limit $\lim_{(t,\phi)\to(t_k^-,\varphi)} V(t, \phi) = V(t_k^-, \varphi)$ exists.

(2) V is locally Lipschitzian in ϕ and $V(t, 0) \equiv 0$.

Definition 1.5: A functional $V(t, \phi)$ belongs to class $\nu_0^*(\cdot)$ if $V \in \nu_0(\cdot)$ and for any $x \in PC([\alpha, \infty), R^n)$, $V(t, x(\cdot))$ is continuous for $t \ge t^*$.

Remark: The class $v_0^*(\cdot)$ will play an important role in the application of the Lyapunov functional method to impulsive functional differential equations. For example, the functional denoted by

$$V(t, x(\cdot)) = \int_{-\infty}^{t} \int_{t}^{\infty} |c(u-s)|\, du\, |x(s)|^r ds,\ r \ge 1$$

belongs to class $v_0^*(\cdot)$ if $c(s)$ is piecewise right continuous and there exits a constant $K > 0$ such that for $t \ge t^*$,

$$\int_{-\infty}^{t} \int_{t}^{\infty} |c(u-s)|\, duds \le K.$$

Let $V \in \nu_0$, for any $(t, x) \in [t_{k-1}, t_k) \in S(H)$, the right hand derivative of V along the solution $x(t)$ of (1.2) is defined by

$$D^+V(t, x(t)) = \lim_{h\to 0^+} \sup \left\{V(t+h, x(t+h)) - V(t, x(t))\right\} / h.$$

Let $V \in \nu_0(\cdot)$, for any $(t, \phi) \in [t_{k-1}, t_k) \times PCB(t)$, the right hand derivative of V along the solution $x(t)$ of (1.2) is defined by

$$D^+V(t, x(\cdot)) = \lim_{h\to 0^+} \sup \left\{V(t+h, x(\cdot)) - V(t, x(\cdot))\right\} / h.$$

In order to prove our main results, we will also need the following Lemma.

Lemma 1.1 ([2]): Assume that $m \in PC(R^+, R)$ and $p \in C(R^+, R^+)$ such that

$$m(t) \le c + \int_{t_0}^{t} p(s)m(s)ds + \sum_{t_0 < t_k \le t} \beta_k m(t_k),\ t \le t_0.$$

where c, $\beta_k \ge 0$ are constants for $k = 1, 2, ...$, then

$$m(t) \le c \prod_{t_0 < t_k \le t} (1 + \beta_k) \exp\left(\int_{t_0}^{t} p(s)ds\right), t \ge t_0.$$

2. MAIN RESULTS

In this section, we develop Lyapunov-Razumikhin methods and derive some sufficient conditions for practical stability for Eq. (1.2).

Let the sets K, $K_i (i = 1, 2, 3, 4)$ be defined by $K = \{\alpha \in C(R^+, R^+)$: strictly increasing and $\alpha(0) = 0\}$, $K_1 = \{\alpha \in C(R^+, R^+) : \alpha(0) = 0, \alpha(s) > 0$ for $s > 0\}$, $K_2 = \{\alpha \in K_1, \alpha(s)$ is non-decreasing in $s\}$, $K_3 = \{\alpha \in K, \alpha(s) > s$ for $s > 0\}$, $K_4 = \{\alpha \in K_1, \alpha(s)$ is non-increasing in $s\}$.

Theorem 2.1: Assume that

(i) $0 < \lambda < A$ are given;

(ii) there exist functions $V \in v_0$, $a, b \in K$, $c \in K_2$, $p \in K_3$ and $q \in K_4$ such that

$$a(|x|) \le V(t, x) \le b(|x|), \text{ for all } (t, x) \le [\alpha, \infty) \times S(H);$$

(iii) for any solution $x(t)$ of (1.2), $V(s, x(s)) \le p(V(t, x(t))$, for $\max\{\alpha, t - q(V(t, x(t)))\} \le s \le t$, implies that

$$D^+V(t, x(t)) \le g(t)c(V(t, x(t))), t \ne t_k,$$

where $g : [t^*, \infty) \to R^+$, locally integrable;

(iv) for all $k \in Z^+$ and $x \in S(H_1)$; $V(t_k, x + I_k(x)) \le h_k(V(t_k^-, x))$, where $h_k \in C(R^+, R^+)$ with $h_k(s) \cdot p^{-1}(s)$ for $s \ge 0$ and $k \in Z^+$, where p^{-1} is the inverse of the function p;

(v) $\mu = \sup_{k \in Z^+}\{t_k - t_{k-1}\} < \infty$, and there exists a constant $\overline{A} > 0$ such that for all $k \in Z^+$, $0 < u \le b(A)$,

$$\int_{p^{-1}(u)}^{u} \frac{ds}{c(s)} - \int_{t_{k-1}}^{t_k} g(s)ds > \overline{A};$$

(vi) $p(b(\lambda)) < a(A)$, $A \le H_1$.

Then the Eq. (1.2) is uniformly practically asymptotically stable with respect to (λ, A).

Proof: We first show the uniformly practical stability. Let $\delta \geq t^*$, $\phi \in PCB_\lambda(\delta)$ and $x(t) = x(t, \delta, \phi)$ be the solution of (1.2), $V(t) = V(t, x(t))$. Suppose $\delta \in [t_{m-1}, t_m)$ for some $m \in Z^+$, $t_0 = t^*$. Then we have, for $\in \leq t \leq \delta$,

$$a(|x(t)|) \leq V(t) \leq b(|x(t)|) \leq b(\lambda) \leq p(b(\lambda)) < \alpha(A). \tag{2.1}$$

We will prove that

$$|x(t)| < A \text{ for } t \geq \delta. \tag{2.2}$$

If (2.2) does not hold, then there exist some $t \in [\delta, \infty)$ such that $|x(t)| \geq A$. Let $\hat{t} = \inf\left\{t \geq \delta \middle| |x(t)| \geq A\right\}$. Since $|x(\delta)| < A$, then $\hat{t} > \delta, |x(t)| < A \leq H_1$, for $t \in [\delta, \hat{t})$, and either $|x(\hat{t})| = A$, or $|x(\hat{t})| > A$ and $\hat{t} = t_k$ for some k. In the latter case, $|x(\hat{t}^-)|$ $\leq A \leq H_1$, by assumption (H_3), we have $|x(\hat{t})| = |x(t_k)| = |x(t_{\bar{k}}^-) + I_k(x(t_k^-))| \leq H$. Thus, in either case, $V(t)$ is defined for $t \in [\alpha, \hat{t}]$, and for $t \in [\alpha, \hat{t}]$, we have

$$a(|x(t)|) \leq V(t) \leq b(|x(t)|). \tag{2.3}$$

Let $\tilde{t} = \inf\left\{t \in [\delta, \hat{t}] \middle| V(t) \geq a(A)\right\}$. Since $V(\delta) < a(A)$ and $V(\hat{t}) \geq a(A)$, then $\tilde{t} \in (\delta, \hat{t}]$, and $V(t) < a(A)$ for $t \in (\delta, \tilde{t})$. We claim that $V(\tilde{t}) = a(A)$ and $\tilde{t} \neq t_k$ for any k. If $\tilde{t} = t_k$ for some k, from assumption (iv), we have

$$0 < a(A) \leq V(\tilde{t}) = V(t_k) \leq h_k(V(t_k^-)) \leq p^{-1}(V(t_k^-)) < V(t_k^-) = V(\tilde{t}^-) < a(A),$$

a contradiction, thus $\tilde{t} \neq t_k$ for any k, and $V(\tilde{t}) = a(A)$ since $V(t)$ is continuous on $\tilde{t}$.

We next consider two possible cases:

Case (1): $t_{m-1} \leq \delta < \tilde{t} < t_m$. Let $t^* = \sup\{t \in [\delta, \tilde{t}] \,|V(t) \leq p^{-1}(a(A))\}$. From assumption (vi), we know that $b(\lambda) < p^{-1}(a(A))$, and so, $V(\delta) \leq b(\lambda) < p^{-1}(a(A))$, $V(\tilde{t}) = a(A) > p^{-1}(a(A))$, hence, $t^* \in [\delta, \tilde{t})$. Moreover, $V(t) \geq p^{-1}(a(A))$ for $t \in [t^*, \tilde{t}]$, $V(t^*) = p^{-1}(a(A))$ since $V(t)$ is continuous on $[\delta, \tilde{t}]$. Therefore, for $t \in [t^*, \tilde{t}]$ and $\max\{\alpha, t - q(V(t))\} \leq s \leq t$, we have $p^{-1}(V(s)) \leq p^{-1}(a(A)) \leq V(t)$, i.e. $V(s) \leq p(V(t))$. From assumption (iii), we have

$$\int_{V(t^*)}^{V(\tilde{t})} \frac{ds}{c(s)} \le \int_{t^*}^{\tilde{t}} g(s)\,ds \le \int_{t_{m-1}}^{t_m} g(s)\,ds.$$

On the other hand, from assumption (v), we have

$$\int_{V(t^*)}^{V(\tilde{t})} \frac{ds}{c(s)} = \int_{p-1a(A)}^{a(A)} \frac{ds}{c(s)} > \int_{t_{m-1}}^{t_m} g(s)\,ds + \overline{A} > \int_{t_{m-1}}^{t_m} g(s)\,ds \ge \int_{V(t^*)}^{V(\tilde{t})} \frac{ds}{c(s)},$$

a contradiction.

Case (2): $t_k < \tilde{t} < t_{k+1}$ for some $k \ge m$. By assumption (iv) we have $V(t_k) \le h_k V(t_k^-)) \le p^{-1}(V(t_k^-)) \le p^{-1}(a(A))$. Let $t^{**} = \sup\{t \in [t_k, \tilde{t}\,] \,|V(t) \le p^{-1}(a(A))\}$. Similar to case (1), we have $t^{**} \in [t_k, \tilde{t}\,)$, $V(t) \ge p^{-1}(a(A))$ for $t \in [t^{**}, \tilde{t}\,]$, $V(t^{**}) = p^{-1}(a(A))$. Hence, for $t \in [t^{**}, \tilde{t}\,]$ and $\max\{\alpha, t - q(V(t))\} \le s \le t$, we have $p^{-1}(V(s)) \le p^{-1}(a(A)) \le V(t)$, i.e. $V(s) \le p(V(t))$. By assumption (iii) we get

$$\int_{V(t^{**})}^{V(\tilde{t})} \frac{ds}{c(s)} \le \int_{t^{**}}^{\tilde{t}} g(s)\,ds \le \int_{t_k}^{t_{k+1}} g(s)\,ds,$$

on the other hand, from assumption (v), we have

$$\int_{V(t^{**})}^{V(\tilde{t})} \frac{ds}{c(s)} = \int_{p-1(a(A))}^{a(A)} \frac{ds}{c(s)} > \int_{t_k}^{t_{k+1}} g(s)\,ds + \overline{A} > \int_{t_k}^{t_{k+1})} g(s)\,ds \ge \int_{V(t^{**})}^{V(\tilde{t})} \frac{ds}{c(s)},$$

this is also a contradiction. Therefore, (2.2) holds, which completes the proof for the uniformly practical stability.

Next, we shall prove that Eq. (1.2) is quasi-uniformly asymptotically stable in the large. By the preceding argument, for given $0 < \lambda < A$ with $p(b(\lambda)) < a(A)$ and $A < H_1$, $\phi \in PCB_\lambda(\delta)$ implies $|x(t)| < A$ and $V(t) \le b(A)$ for $t \ge \delta$. Now, let $\varepsilon > 0$ be given, we suppose ε so small that $a(\varepsilon) < b(A)$. We then will prove that there exists a positive number $T(\varepsilon, \lambda)$ such that $\phi \in PCB_\lambda(\delta)$ implies $|x(t)| \le \varepsilon$ for $t \ge \delta + T$. To this end, set $M = M(\varepsilon, \lambda) = \sup_{p^{-1}(a(\varepsilon)) \le s \le b(A)}\{c^{-1}(s)\}$, choose $d = d(\varepsilon, \lambda)$ such that $0 < d < \overline{A}/M$. We first claim that

$$p^{-1}(u) < u - d, \text{ for } a(\varepsilon) \le u \le b(A). \tag{2.4}$$

Since $a(\varepsilon) \le u \le b(A)$, we have $p^{-1}(a(\varepsilon)) \le p^{-1}(u) < u \le b(A)$. By assumption (v), one can get

$$\overline{A} < \int_{p^{-1}(u)}^{u} \frac{ds}{c(s)} \leq M\left[u - p^{-1}(u)\right],$$

and so $p^{-1}(u) < u - \overline{A}/M < u - d$, thus (2.4) holds.

Let N be the smallest positive integer such that $b(A) \leq a(\varepsilon) + Nd$. Set $T = T(\varepsilon, \lambda) = \mu + (\tau + \mu)(N - 1)$, where $\tau = q(p^{-1}(a(\varepsilon)))$. Let m_i $(i = 1, 2, ..., N)$ be Defined by $m_1 = m$, m_i satisfies: $t_{m_{i-1}} < t_{m_{i-1}} + \tau \leq t_{m_i}$, for $i = 2, 3, ..., N$. Then $t_{m_1} = t_m \leq \delta + \mu$, $t_{m_i} \leq t_{m_{i-1}} + \mu \leq t_{m_{i-1}} + \tau + \mu$, $(i = 2, 3, ..., N)$. Therefore $t_{m_N} \leq \delta + \mu + (\tau + \mu)(N - 1) = \delta + T$. We will use mathematical induction to prove that

$$V(t) \leq b(A) - id,\ t \geq t_{m_i},\ i = 1, 2, ..., N. \tag{2.5$_i$}$$

We first prove that

$$V(t) \leq b(A) - d,\ t \geq t_{m_1}. \tag{2.5$_1$}$$

If $(2.5)_1$ does not hold, then there exists some $t > t_{m_1} = t_m$ such that $V(t) > b(A) - d$. Set $\overline{t} = \inf\left\{t \geq t_m \middle| V(t) > b(A) - d\right\}$. Thus $\overline{t} \in \left[t_k, t_{k+1}\right)$ for some $k \geq m$. Since $a(\varepsilon) \leq b(A)$, by Eq. (2.4), we have $p^{-1}(b(A)) < b(A) - d$. From assumption (iv), we get

$$V(t_k) \leq h_k(V(t_k^-)) \leq p^{-1}(V(t_k^-)) \leq p^{-1}(b(A)) < b(A) - d.$$

Therefore $\overline{t} \in \left[t_k, t_{k+1}\right)$. By the continuity of $V(t)$ on $\overline{t}$, we get

$$V(\overline{t}) = b(A) - d,\ V(t) \leq b(A) - d < b(A), \text{ for } t \in [t_k, \overline{t}].$$

Let $\overline{t} = \sup\{t \in [t_k, \overline{t}]\} V(t) \leq p^{-1}(b(A))\}$. Since $V(\overline{t}) = b(A) - d > p^{-1}(b(A)) \geq V(t_k)$, then $\hat{t} \in \left[t_k, \overline{t}\right)$. Moreover, $V(\hat{t}) = P^{-1}(b(A))$, and $V(t) \geq p^{-1}(b(A))$ for $t \in \left[\hat{t}, \overline{t}\right]$. Thus for $t \in \left[\hat{t}, \overline{t}\right]$ and $\max\{\alpha, t - q(V(t))\} \leq s \leq t$, we have

$$p^{-1}(V(s)) \leq p^{-1}(b(A)) \leq V(t),$$

which implies $V(s) \leq p(V(t))$ for $t \in \left[\hat{t}, \overline{t}\right]$ and $\max\{\alpha, t - q(V(t))\} \leq s \leq t$. By assumption (iii), we get $D^+V(t) \leq g(t)c(V(t))$, $t \in \left[\hat{t}, \overline{t}\right]$. And so

$$\int_{V(\hat{t})}^{V(\overline{t})} \frac{ds}{c(s)} \leq \int_{\hat{t}}^{\overline{t}} g(s)\,ds \leq \int_{t_k}^{t_{k+1}} g(s)\,ds. \tag{2.6}$$

On the other hand,

$$\int_{V(\hat{t})}^{V(\bar{t})} \frac{ds}{c(s)} = \int_{p^{-1}(b(A))}^{b(A)-d} \frac{ds}{c(s)} = \int_{p^{-1}(b(A))}^{b(A)} \frac{ds}{c(s)} - \int_{b(A)-d}^{b(A)} \frac{ds}{c(s)}. \tag{2.7}$$

Since $p^{-1}(a(\varepsilon)) \le p^{-1}(b(A)) \le b(A) - d \le b(A)$, then $\int_{b(A)-d}^{b(A)} \frac{ds}{c(s)} \le Md$. From (2.6), (2.7) and assumption (v), we get

$$\int_{V(\hat{t})}^{V(\bar{t})} \frac{ds}{c(s)} > \bar{A} + \int_{t_k}^{t_{k+1}} g(s)\,ds - Md > \int_{t_k}^{t_{k+1}} g(s)\,ds \ge \int_{V(\hat{t})}^{V(\bar{t})} \frac{ds}{c(s)}$$

This is a contradiction, so $(2.5)_1$ holds.

Now, suppose that $(2.5)_i$ holds for some $1 \le i < N$, we then prove that

$$V(t) \le b(A) - (i+1)d,\ t \ge t_{m_{i+1}}. \tag*{$(2.5)_{i+1}$}$$

If $(2.5)_{i+1}$ does not hold, then there exist some $t \ge t_{m_{i+1}}$ such that

$$V(t) > b(A) - (i+1)d.$$

Set $\bar{t} = \inf\{t \ge t_{m_{i+1}} | V(t) > b(A) - (i+1)d\}$, then $\bar{t} \in [t_k, t_{k+1})$ for some $k \ge m_{i+1}$. Since $a(\varepsilon) \le b(A) - id < b(A)$, from (2.4), we have $p^{-1}(b(A) - id) < b(A) - (i+1)d$. By assumption (iv), we get

$$V(t_k) \le h_k(V(t_k^-)) \le p^{-1}(V(t_k^-)) \le p^{-1}(b(A) - id) < b(A) - (i+1)d.$$

Thus $\bar{t} \in (t_k, t_{k+1})$. By the continuity of $V(t)$ on $\bar{t}$, we have

$$V(\bar{t}) = b(A) - (i+1)d,\ V(t) \le b(A) - (i+1)d \text{ for } t \in [t_k, \bar{t}].$$

Let $\hat{t} = \sup\{t \in [t_k, \bar{t}] | V(t) \le p^{-1}(b(A) - id)\}$. Since $V(\bar{t}) = b(A) - (i+1)d > p^{-1}(b(A) - id) \ge V(t_k)$, then $\hat{t} \in [t_k, t)$, by the continuity of $V(t)$ on $\hat{t}$, we know

$$V(\hat{t}) = p^{-1}(b(A) - id),\ V(t) \ge p^{-1}(b(A) - id), \text{ for } t \in [\hat{t}, \bar{t}].$$

Thus for $t \in [\hat{t}, \bar{t}]$. and $t_{m_i} \le s \le t$, we have

$$P^{-1}(V(s)) \le p^{-1}(b(A) - id) \le V(t), \text{ i.e. } V(s) \le p(V(t)).$$

From inequality $t_{m_i} \le t_{m_{i+1}} - \tau$ and the definition of τ, we know that $\max\{\alpha, t - q(V(t))\} \le s \le t$. By assumption (iii), we can get

$$D^{+}V(t) \le g(t)c(V(t)),\ t \in \left[\hat{t}, \overline{t}\right].$$

And so

$$\int_{V(\hat{t})}^{V(\overline{t})} \frac{ds}{c(s)} \le \int_{\hat{t}}^{\overline{t}} g(s)\,ds \le \int_{t_k}^{t_{k+1}} g(s)\,ds. \tag{2.8}$$

On the other hand,

$$\int_{V(\hat{t})}^{V(\overline{t})} \frac{ds}{c(s)} = \int_{p^{-1}(b(A)-id)}^{b(A)-(i+1)d} \frac{ds}{c(s)} = \int_{p^{-1}(b(A)-id)}^{b(A)-id} \frac{ds}{c(s)} - \int_{b(A)-(i+1)d}^{b(A)-id} \frac{ds}{c(s)}. \tag{2.9}$$

Since

$$p^{-1}(a(\varepsilon)) \le p^{-1}(b(A) - id) < b(A) - (i+1)d \le b(A) - id < b(A),$$

it follows that $\int_{b(A)-(i+1)d}^{b(A)-id} \frac{ds}{c(s)} \le Md.$

From (2.8), (2.9) and assumption (v), we get

$$\int_{V(\hat{t})}^{V(\overline{t})} \frac{ds}{c(s)} > \overline{A} - Md + \int_{t_k}^{t_{k+1}} g(s)\,ds > \int_{t_k}^{t_{k+1}} g(s)\,ds \ge \int_{V(t)}^{V(t)} \frac{ds}{c(s)}.$$

This is a contradiction, and so $(2.5)_{i+1}$ holds. By the induction, we know that $(2.5)_i$ holds for all $i = 1, 2, \ldots, N$. Thus, when $i = N$, we obtain

$$a(|x(t)|) \le V(t) \le b(A) - Nd \le a(\varepsilon),\ t \ge t_{m_N},$$

By the definition of t_{m_N}, we have $t_{m_N} \le + T$, therefore

$$a(|x(t)|) \le V(t) \le b(A) - Nd \le a(\varepsilon),\ t \ge \delta + T,$$

which implies $|x(t)| \le \varepsilon$ for $t \ge \delta + T$. The proof is complete.

It is easily seen that with minor modification in the proof for the uniformly practical stability of Theorem 2.1, we can obtain the following corollary.

Corollary 2.1: Assume that

(i) $0 < \lambda < A$ are given;

(ii) there exist functions $V \in v_0$, $a, b \in K$, $c \in K_2$, and $p \in K_3$ such that

$$a(|x|) \le V(t, x) \le b(|x|), \text{ for all } (t, x) \in [\alpha, \infty) \times S(H);$$

(iii) for any solution $x(t)$ of (1.2), $V(s, x(s)) \le p(V(t, x(t))$, for $\alpha \le s \le t$, implies that

$$D^+V(t, x(t)) \le g(t)c(V(t, x(t))),\ t \ne t_k,$$

where $g : [t^*, \infty) \to R^+$, locally integrable;

(v) for all $k \in Z^+$, $0 < u \le b(A)$,

$$\int_{p-1(u)}^{u} \frac{ds}{c(s)} - \int_{t_{k-1}}^{t_k} g(s)\,ds > 0;$$

(vi) $p(b(\lambda)) < a(A)$, $A \le H_1$.

Then the Eq. (1.2) is uniformly practically stable with respect to (λ, A).

Theorem 2.2: Assume that

(i) $0 < \lambda < A$ be given;

(ii) there exist $V_1(t, x) \in v_0$, $V_2(t, \phi) \in v_0^*(\cdot)$, $W_i(i = 1, 2 ..., 5) \in K$, $\Phi : [0, \infty) \to [0, \infty)$ which is $L_1[0, \infty)$ and bounded, such that

$$W_1(|\phi(t)|) \le V(t, \phi(\cdot)) \le W_2\left(|\phi(t)| + W_3\left(\int_\alpha^t \Phi(t-s) W_4(|\phi(s)|)\,ds\right)\right),$$

where $V(t, \phi(\cdot)) = V_1(t, \phi(t)) + V_2(t, \phi(\cdot)) \in v_0(\cdot)$;

(iii) for any $x \in R^n$, and each $k \in Z^+$,

$$V_1(t_k, x + I_k(x)) \le (1 + b_k)V_1(t_k^-, x), \text{ where } b_k \ge 0 \text{ with } \sum_{k=1}^{\infty} b_k < \infty;$$

(iv) for any solution $x(t)$ of (1.2), the right hand derivative of $V(t, x(\cdot))$ along the solution satisfies $D^+V(t, x(\cdot)) \le -W_5(|x|)$;

(v) $M[W_2(\lambda) + W_3(LW_4(\lambda))] < W1(A)$; where $M = \prod_{k=1}^{\infty}(1 + b_k)$, $L = \int_0^{\infty} \Phi(u)\,du.$

Then Eq. (1.2) is uniformly practically stable with respect to (λ, A).

Proof: Let $\delta \ge t^*$, $\phi \in PCB_\lambda(\delta)$ and $x(t) = x(t, \delta, \phi)$ be the solution of (1.2), $V_1(t) = V_1(t, x(t))$, $V_2(t) = V(t, x(\cdot))$. Then

$$V(\delta) \le W_2(\lambda) + W_3\left(W_4(\lambda)\int_\alpha^t \Phi(t-s)\,ds\right)$$

$$= W_2(\lambda) + W_3\left(W_4(\lambda)\int_0^{t-\alpha}\Phi(u)\,du\right)$$

$$\le W_2(\lambda) + W_3\left(W_4(\lambda)\int_0^{\infty}\Phi(u)\,du\right)$$

$$= W2(\lambda) + W_3(LW_4(\lambda)).$$

From assumption (iv), we have

$$D^+V(t) \le -W_5(|x(t)|). \tag{2.10}$$

By integrating the both sides of (2.10) form δ to $t > \delta$, we get

$$V(t) \le V(\delta) - \int_\delta^t W_5(|x(t)|)\,ds + \sum_{\delta<t_k\le t}\left[V(t_k) - V(t_k^-)\right].$$

Since $V_2(t)$ is continuous, it follows that

$$V(t_k) - V(t_k^-) = V_1(t_k) - V_1(t_k^-) \le b_kV_1(t_k^-) \le b_kV_1(t_k^-) + b_kV_2(t_k^-) = b_kV(t_k^-),$$

and so

$$V(t) \le V(\delta) + \sum_{\delta<t_k\le t} b_kV(t_k^-).$$

By Lemma 1.1, we see that

$$V(t) \le V(\delta)\prod_{\delta<t_k\le t}(1+b_k) \le MV(\delta),\ t \ge \delta.$$

Form assumption (ii) and (v), we obtain

$$W_1(|x(t)|) \le V(t) \le M[W_2(\lambda) + W_3(L\tau W_4(\lambda))] < W_1(A),\ t \ge \delta,$$

which implies $|x(t)| < A$, $t \ge \delta$, and so Eq. (1.2) is uniformly practically stable. The proof is complete.

3. EXAMPLES

As the applications of our main results, in this section, we consider the following examples.

Example 3.1: Consider the following impulsive functional differential equation

$$\begin{cases} x'(t) = a(t)x(t) + b(t)x(t-\tau) + \int_{-\infty}^{t} k(u-t)x(u)du, t \geq 0,\ t \neq t_k, \\ x(t_k) - x\left(t_k^-\right) = I_k\left(x\left(t_k^-\right)\right), k \in Z^+. \end{cases} \tag{3.1}$$

Where $\tau > 0$, a, b and k are continuous functions, $a(t) \leq a$, $b(t) \leq b$, $I_k(x) \in C(R, R)$ and $|x + I_k(x)| \leq \beta|x|$ for all $k \in Z^+$, a, b, β are some constants, $\int_{-\infty}^{0} |k(u)| du < \infty$. For given (λ, A) with $0 < \lambda < A < H_1$, let the following conditions hold.

(A_1) $0 < \beta < 1$ and $a + b\beta^{-1} + \beta^{-1}\int_{-\infty}^{0} |k(u)| du < 0$;

(A_2) there exists $\gamma > 0$ such that

$$t_{k+1} - t_k < \gamma < -\frac{\ln\beta}{a + b\beta^{-1} + \beta^{-1}\int_{-\infty}^{0} |k(u)| du + A}, k \in Z^+;$$

(A_3) $\lambda/\beta < A$.

Then Eq. (3.1) is uniformly practically asymptotically stable with respect to (λ, A).

Indeed, let $\delta \geq t^*$, $\phi \in PCB_\lambda(\delta)$, $x(t) = x(t, \delta, \phi)$ be the solution of (3.1), we choose the functions in Theorem 2.1 as following:

$$V(t, x(t)) = \frac{1}{2}x^2, a(s) = b(s) = \frac{1}{2}s^2, g(s) = 2\left(a + \beta^{-1+\beta^{-1}}\int_{-\infty}^{0} |k(u)| du + A\right), c(s) = s,$$

$h_k(s) = h(s) = \beta^2 s$, $p(s) = \frac{1}{\beta^2}s$, then $p(s) > s$ for $s > 0$, and $h_k(s) \leq p^{-1}(s)$ for $s \geq 0$. By Corollary 2.1, one can easily see that Eq. (3.1) is uniformly practically stable. Therefore, $|x(t)| \leq \lambda < A$ for $-\infty \leq t < \delta$, and $|x(t)| \leq A$ for $t \geq \delta$, thus, $\|x(t)\|^{(-\infty, t]} \leq A$.

Since $\int_{-\infty}^{0} |k(u)| du < \infty$, we know that there exists a continuous function $q : (0, \infty) \to (0, \infty)$, $q(s) > \tau$ for $s > 0$, $q(s)$ is non-increasing, such that $\int_{-\infty}^{-q(s)} |k(u)| du \leq \sqrt{2s}$.

(1) Clearly, $a(|x|) \leq V(t, x(t)) \leq b(|x|)$ holds;

(2) for the solution $x(t)$ of (3.1), if $V(s, x(s)) \leq p(V(t, x(t)))$, for $\max\{-\infty, q(V(t, x(t)))\} \leq s \leq t$, then we have $|x(s)| \leq \beta^{-1}|x(t)|$, and so

$$
\begin{aligned}
D^+V(t, x(t)) &= a(t)x^2(t) + b(t)x(t)x(t-\tau) + x(t)\int_{-\infty}^{t} k(u-t)x(u)du \\
&\le ax^2(t) + b|x(t)||x(t-\tau)| + |x(t)| \int_{-\infty}^{t} |k(u-t)|\,|x(u)|du \\
&\le ax^2(t) + b\beta^{-1}x^2(t) + |x(t)| \int_{-\infty}^{t-q(V(t,x(t)))} |k(u-t)||x(u)|du \\
&\quad + |x(t)|\int_{t-q(V(t,x(t)))}^{t} |k(u-t)||x(u)|du \\
&\le (a + b\beta^{-1})x^2(t) + A|x(t)|\int_{-\infty}^{t-q(V(t,x(t)))} |k(u)|du \\
&\quad + \left(\beta^{-1}\int_{-\infty}^{0}|k(u)|du\right)x^2(t) \\
&\le \left(a + b\beta^{-1} + \beta^{-1}\int_{-\infty}^{0}|k(u)|du + A\right)x^2(t) \\
&= g(t)c(V(t, x(t))).
\end{aligned}
$$

Thus, condition (iii) of Theorem 2.1 is satisfied;

(3) for all $k \in Z^+$ and $x \in S(H_1)$, we have

$$
V\left(t_k, x + I_k(x)\right) = \frac{1}{2}\left(x + I_k(x)\right)^2 \le \frac{1}{2}\beta^2 x^2 = h\left(V\left(t_k^-, x\right)\right).
$$

The condition (iv) of Theorem 2.1 is satisfied;

(4) Let $\overline{A} = -2\ln\beta - 2\left(a + b\beta^{-1} + \beta^{-1}\int_{-\infty}^{0}|k(u)|du + A\right)\gamma$, from condition (A_2), we know $\overline{A} > 0$ and for any $k \in Z^+, 0 < u \le \frac{1}{2}A^2$,

$$
\begin{aligned}
\int_{p^{-1}(u)}^{u}\frac{ds}{c(s)} - \int_{t_k}^{t_{k+1}} g(s)ds &= \int_{\beta^2 u}^{u}\frac{ds}{s} - 2\int_{t_k}^{t_{k+1}}\left(a + b\beta^{-1} + \beta^{-1}\int_{-\infty}^{0}|k(u)|du + A\right)ds \\
&= -2\ln\beta - 2\left(a + b\beta^{-1} + \beta^{-1}\int_{-\infty}^{0}|k(u)|du + A\right)(t_{k+1} - t_k)
\end{aligned}
$$

$$> -2\ln\beta - 2\left(a + b\beta^{-1} + \beta^{-1}\int_{-\infty}^{0}\left|k(u)\right|du + A\right)\gamma$$

$$= \bar{A}.$$

Thus, condition (v) of Theorem 2.1 is also satisfied;

(5) Since $\frac{\lambda}{\beta} < A$, then $p(b(\lambda)) < a(A)$. The condition (vi) of the Theorem 2.1 is satisfied.

From (1)–(5), we know that all conditions of Theorem 2.1 are satisfied. By Theorem 2.1, Eq. (3.1) is uniformly practically asymptotically stable with respect to (λ, A).

Example 3.2: Consider the scalar impulsive functional differential equation

$$\begin{cases} x'(t) = -x^3(t) + \int_{-\infty}^{t} C(t-s)v\left(x(s)\right)ds,\ t \geq 0,\ t \neq t_k, \\ x(t_k) - x\left(t_k^-\right) = I_k\left(x\left(t_k^-\right)\right),\ k \in Z^+, \end{cases} \tag{3.2}$$

with $\int_{t}^{\infty}\left|C(u)\right|du \in L^1[0,\infty), \left|v(x)\right| \leq \beta\left|x\right|^3$ for $0 \leq \beta \leq 1$. Suppose that $|x + I_k(x)| \leq (1 + b_k)|x|$ with $b_k \geq 0$, $\sum_{k=1}^{\infty} b_k < \infty$. Denote $M = \prod_{k=1}^{\infty}\left(1 - b_k\right)$. For given (λ, A) with $0 < \lambda < A$. Let the following conditions hold:

(i) $\int_{0}^{\infty}\left|C(u)\right|du < 1.$

(ii) $M(\lambda + L\lambda^3) < A$, where $L = \int_{0}^{\infty}\int_{u}^{0}\left|C(s)\right|dsdu.$

Then Eq. (3.2) is uniformly practically stable with respect to (λ, A).

In fact, we let $V_1(t, \phi(t)) = |\phi(t)| \in v_0$, $V_2(t, \phi(\cdot)) = \int_{-\infty}^{t}\int_{t-s}^{\infty}\left|C(u)\right|du\left|\phi(s)\right|^3 ds \in v_0^*(\cdot)$. $W_1(s) = W_2(s) = W_3(s) = s$, $W_4(s) = s^3$, $W_5(s) = \left(1 - \int_{0}^{\infty}\left|C(u)\right|du\right)s^3$, $\Phi(t-s) = \int_{t-s}^{\infty}\left|C(u)\right|du$, $\delta \geq 0$, $\varphi \in PCB_\lambda(\delta)$, $x(t) = x(t, \delta, \varphi)$ is the solution of (3.1). Then

(1) $W_1(|x(t)|) \le V(t, x(\cdot)) \le W_2(|x(t)| + W_3\left(\int_{-\infty}^{t} \Phi(t-s) W_4(|x(s)|)\, ds\right).$

(2) $V_1(t_k, x + I_k(x)) = |x + I_k(x)| \le (1 + b_k)|x| = (1 + b_k)V_1(t_k^-, x)$, for $k \in Z^+$.

(3) For any solution $x(t)$ of (3.2)

$$
\begin{aligned}
D^+V(t, x(\cdot)) &\le -|x(t)|^3 + \int_{-\infty}^{t} |C(t-s)||v(x(s))|\, ds + \int_0^{\infty} |C(u)||du||x|^3 \\
&\quad - \int_{-\infty}^{t} |C(t-s)||x(s)|^3\, ds \\
&\le -\left(1 - \int_0^{\infty} |C(u)|\, du\right)|x(t)|^3 \\
&\le -W_5(|x(t)|.
\end{aligned}
$$

(4) By condition (ii), we obtain $M(W_2(\lambda) + W_3(LW_4(\lambda))) < W_1(A)$.

From (1)–(4), we know that all conditions of Theorem 2.2 are satisfied. By Theorem 2.2, Eq. (3.2) is uniformly practically stable with respect to (λ, A).

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