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EXISTENCE OF SOLUTIONS FOR SOME DISCONTINUOUS PROBLEMS INVOLVING THE *P*-LAPLACIAN

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ABSTRACT: Some existence results for discontinuous problems involving the plaplacian operator are exposed. It is shown that in some particular case the solution is strictly positive.

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1. INTRODUCTION

In the last years the interest for the study of problems involving the p-laplacian operator have considerably increased. It is well known that $-\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, so for $p = 2$ we obtain the classical Laplace operator. The simpler form of a differential equation with the p-laplacian term is, for instance:

$$-\Delta_p u = f(x, u) \text{ in } \Omega,$$

where f is a C^1 function and is a bounded domain of IR^N. After this first statement, have been treated more general cases, for which is necessary the definition of weak solution of the equation. Anyway, these concepts are by now well known. When we deal with non-smooth nonlinear terms then the concept of weak solution fails and the integral equation related to it becomes a differential inclusion or, more generally, a hemi-variational inequality. To be more precise, two more general versions of the problem, which are that we will affront, are the following:

Find $u \in E$ satisfying

$$(P) \qquad \int_{\mathrm{IR}^N} (|\nabla u|^{p-2}\nabla u \cdot \nabla v + b|u|^{p-2}uv + J^0(x, u; v)) \geq 0 \text{ for all } v \in E,$$

where E is a Banach space, b and J are two non-smooth functions whose properties will be given in the sequel, and

Find $u \in W_0^{1,p}(\mathrm{IR}^N)$ satisfying

$$(P') \qquad \int_{\mathrm{IR}^N} (|\nabla u|^{p-2}\nabla u \cdot \nabla v + J^0(x, u; v)) \geq 0 \text{ for all } v \in W_0^{1,p}(\mathrm{IR}^N),$$

where J is non-smooth. Different techniques are available in treating problems with discontinuous nonlinearities; here we will use variational methods. In particular, we will apply a suitable version of the mountain pass theorem to the energy functional

I related to (P) and to (P'). Even in the classical case, namely when b and J are smooth, two difficulties arises:

(1) I doesn't satisfy a compactness condition of Palais-Smale type;

(2) for unbounded domains we lose the compactness of the embedding of $W_0^{1,p}(\Omega)$ in $Lt(\Omega)$.

To bypass the first problem we will use a version of the mountain pass theorem which yields only a Palais-Smale sequence of I, while for the second one we will exploit the properties of generalized gradient of non-smooth functions. When b and J are smooth functions, then (P) and (P') reduce to

$$\{-\Delta_p u + b(x)|u|^{p-2}u = J'(x, u) \text{ in } \mathrm{IR}^N \; u \in E,$$

and

$$\{-\Delta_p u = J'(x, u) \text{ in } \mathrm{IR}^N \; u \in W_0^{1,p}(\mathrm{IR}^N),$$

while for $p = 2$ we obtain an elliptic hemivariational inequality. Both these problems have been studied carefully, under various assumptions. One of the more interesting and general case occurs when is involved the critical Sobolev exponent $\left(\text{i.e. } \frac{N_p}{N-p}\right)$, that is when the growth of the nonlinear term needs not to be subcritical, but it is allowed to be critical too.

In this work, we seek nonnegative solutions of (P) and (P'). The starting point for this type of research stands both in [2, 13], where is proved the existence of positive solutions for (P) and for (P') respectively, in the smooth case, and in [3, 12, 20], where on examine elliptic problems in IR^N. As it is impossible to collect all the references for the results concerning these kind of problems, here we recall only the papers in which the assumptions are somewhat comparable with ours. The elliptic case of (P) is treated in [12], where b is assumed to be locally bounded, non negative and coercive, while the nonlinear term has the same kind of discontinuities that will be taken into account in Theorem 4.1. To the best of our knowledge, [20] is the first paper where the potential b may change sign; in treating (P) we will take this same assumption, together with another that weaken the classical one on the coercivity. Nevertheless, in [20] the growth of f is somewhat controlled by that of b, and the problem is classical. Between the papers dealing with the p-Laplacian, assumptions very similar to ours can be found in [13]: the nonlinear term is exactly $hu^q + u^{p^*-1}$, so the problem has a critical growth; in Theorem 3.2 we will take into account a more general situation of that cited above. If the result of previous paper can be totally comparable with our, moreover can be seen as one of the possible smooth versions of theorem 3.2, the same is not true for that exposed in [2], where the nonlinear term is $|u|^{p^*-2}u$, but the function on the left-hand side has an interval of growth of integrability, fact that exclude one of our assumptions on b.

Before introducing the preliminary results, that will be exposed in Section 2, we summarize the basic definitions that will be useful through all the paper.

Let $(E, \|\cdot\|)$ be a real Banach space. If $\rho > 0$, we define $B_\rho = \{x \in E : \|x\| < \rho\}$, $\overline{B}_\rho = \{x \in E : \|x\| \leq \rho\}$, and $\partial B_\rho = \{x \in E : \|x\| = \rho\}$.

We denote by E^* the dual space of E, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between E^* and E.

A function $g : E \to \mathbb{R}$ is called locally Lipschitz when to every $x \in E$ there correspond a neighbourhood U_x of x besides a constant $L_x \geq 0$ such that

$$|g(z) - g(w)| \leq L_x\|z - w\| \ \forall z, w \in U_x.$$

If $x, z \in E$, the symbol $g^0(x; z)$ indicates the generalized directional derivative of g at the point x along the direction z, namely

$$g^0(x; z) = \limsup_{w \to x,\, t \to 0^+} \frac{g(w + tz) - g(w)}{t}.$$

It is known (see [10], Proposition 2.1.1) that g^0 turns out upper semicontinuous on $E \times E$.

We denote by $\partial g(x)$ the generalized gradient of g at x, i.e.

$$\partial g(x) = \{x^* \in E^* : \langle x^*, z \rangle \leq g^0(x; Z) \ \forall z \in E\}.$$

Proposition 2.1.2 in [10] ensures that the set $\partial g(x)$ is nonempty, convex, and weak* compact. Hence, it makes sense to write

$$m_g(x) = \min\{\|x^*\|_{E^*} : x^* \in \partial g(x)\}.$$

We say that $\{x_n\} \subseteq E$ is a Palais-Smale sequence at a given level $d \in \mathbb{R}$ for g if

$$g(x_n) \to d \in \mathbb{R} \text{ and } m_g(x_n) \to 0,$$

while a critical point of g is any $x \in E$ satisfying $0 \in \partial g(x)$, which clearly means $g^0(x; z) \geq 0$ for all $z \in E$.

We denote by $|\cdot|$ and by $|\cdot|_t$ the usual norms on $\mathbb{R}^N$ and on $L^t(\mathbb{R}^N)$, for all $t \in [1, +\infty)$, while $W_0^{1,p}(\mathbb{R}^N)$ indicates the closure of $C_1^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{1,p} := \left(|\nabla u|_p^p + |u|_p^p\right)^{1/p},$$

or to the equivalent one

$$\|u\|_{1,p} = |\nabla u|_p.$$

From now on we take $p \in [2, N)$ and we put $p^* = \dfrac{N_p}{N-p}$. As $W_0^{1,p}(\mathrm{IR}^N)$ is embedded in $L^t(\mathrm{IR}^N)$ for any $t \in [p, p^*]$, for such t we denote by c_t the constant of the embedding:

$$|u|_t \le c_t \|u\|_{1,p} \ \forall u \in W_0^{1,p}(\mathrm{IR}^N).$$

Another embedding will play a basic role trough all the paper:

$$W_0^{1,p}(\mathrm{IR}^N) \hookrightarrow L^s(B_R) \text{ compactly, for any } s \in [p, p^*[,$$

so if $\{u_n\}$ is a sequence bounded in $W_0^{1,p}(\mathrm{IR}^N)$, then it converges strongly to some $u \in L^s(B_R)$; a diagonal argument shows that $u_n \to u$ a.e. in IR^N; the boundedness of $\{u_n\}$ in $L^t(\mathrm{IR}^N)$, together with its convergence a.e., allows to apply Remark 8 in [3], which guarantees that $u_n \to u$ weakly in $L^t(\mathrm{IR}^N)$ for any $t \in [p, p^*]$.

We first consider (P), that is the natural generalization of the classical problem involving the p-Laplacian:

$$-\Delta_p(u) + b(x)u^{p-1} = f(x, u) \text{ in } \mathrm{IR}^N,$$

in which the nonlinear term f is not necessarily continuous; after we will take into account the situation in which $b \equiv 0$. In all the two situations, we will impose two different growth conditions on f; we emphasize that for bounded domains the second one is more general than the first one, but the same is not true in IR^N. In Theorems 3.1, 3.3, 4.1 and 4.3 too, we take into account the case of a function $f: \mathrm{IR}^n \times \mathrm{IR} \to \mathrm{IR}$ which is measurable and satisfies the following growth condition with respect to the second variable, uniformly in the first one:
there exist $q \in]p-1, p^*-1[$ and $C > 0$ such that

$$|f(x, s)| \le C(|s|^{p-1} + |s|^q) \text{ a.e.} \tag{1}$$

Under the assumptions above the integral function

$$F(x, s) = \int_0^s f(x, t)\, dt$$

is measurable in the first variable and locally Lipschitz with respect to s. En effect, if we take any $s \in \mathrm{IR}$, and a $\delta > 0$, then

$$\left|F\left(x_1, s_1\right)-F\left(x, s_2\right)\right| \leq\left|\int_{s_1}^{s_2} f\left(x, s\right) ds\right| \leq C\left(\left(|s|+\delta\right)^{p-1}+\left(|s|+\delta\right)^q\right)\left|s_2-s_1\right|$$

$$= L_s|s_2 - s_1| \ \forall s_1,\ s_2 \in B(s, \delta).$$

In addiction to (1), we assume f satisfies:

$$\lim_{\varepsilon\to 0^+} ess \sup\left\{\left|\frac{f\left(x, s\right)}{s^{p-1}}\right| : \left(x, s\right)\in \mathrm{IR}^N \times\left(-\varepsilon, \varepsilon\right)\right\} = 0, \tag{2}$$

and

$$\exists\mu > p : 0 \leq \mu F\left(x, s\right) \leq s\underline{f}\left(x, s\right) \text{ a.e. in } \mathrm{IR}^N \times [0, +\infty], \tag{3}$$

where

$$\underline{f}\left(x, s\right) = \lim_{\varepsilon\to 0^+} ess \sin f\left\{f\left(x, t\right) : \left|t - s\right| < \varepsilon\right\},$$

$$\overline{f}\left(x, s\right) = \lim_{\varepsilon\to 0^+} ess \sup f\left\{f\left(x, t\right) : \left|t - s\right| < \varepsilon\right\}.$$

We denote by F^0 and ∂F the generalized directional derivative and the generalized gradient of F with respect to the second variable and we put

$$J(x, s) = -|s|^{p-1} - F(x, s) \text{ or } J(x, s) = -F(x, s),$$

accordingly to the situation in which we consider the critical or the subcritical case. It is well known (see, for instance [9], [15]) that

$$F^0\left(x, t; z\right) \leq \begin{cases} \overline{f}\left(x, t\right) z \text{ if } z > 0, \\ \underline{f}\left(x, t\right) z \text{ if } z < 0, \end{cases} \text{and } \partial F\left(x, t\right) \subseteq \left[\underline{f}\left(x, t\right), \overline{f}\left(x, t\right)\right].$$

After we will take into account the situation of a measurable function $f : \mathrm{IR}^N \times \mathrm{IR} \to \mathrm{IR}$ satisfying (3) in add to the following one:
there exist $q \in [0, p^* - 1[$, $r \in]p - 1, p^* - 1[$, $t \in [p, p^*]$, $t > q + 1$, $C > 0$ and a function $h \geq 0$, $h \in L\theta(\mathrm{IR}^N) \cap L^1(\mathrm{IR}^N)$, with $\theta\dfrac{t}{t-\left(q+1\right)}$, such that:

$$|f(x, s))| \leq h(x)|s|^q + C|s|^r \ \forall (x, s) \in \mathrm{IR}^N \times \mathrm{IR}. \tag{4}$$

As done above, we denote by F^0 and ∂F the generalized directional derivative and the generalized gradient of F with respect to the second variable; J too has the same structure. It is a simple matter to verify that F is locally Lipschitz, but not uniformly in x. In both cases the following Lemma hold:

Lemma 1.1: There exist $C_1, C_2 > 0$ such that $f(x, s) \geq C_1 s^{\mu-1} - C_2$ for a.e. $(x, s) \in \mathrm{IR}^N \times [0, +1]$.

Proof: The proof carries out in the same way in the two situations. For more details see [12], Lemma 5.

When we deal with the problem (P), we have to give some basic assumptions on b. Let $b : \mathrm{IR}^N \to \mathrm{IR}$ be a measurable function and B a nonnegative real number. We assume that

$$b(x) \geq -B \text{ a.e.} \tag{5}$$

In several papers on impose that $\lim_{x\to+\infty} b(x) = \infty$; here we weaken this assumption by taking condition (6) that we are going to give. If G is an open subset of IR^N and $s \in [p, p^*[$, we set

$$\mathcal{M}_s(G) = \left\{u \in W_0^{1,p}(G) : |u|_s = 1\right\}$$

and

$$\nu_s(G) = \inf_{u \in \mathcal{M}_s(G)} \int_G |\nabla u|^p + b|u|^p .$$

We require that for any $r > 0$ and any sequence $\{x_n\} \subseteq \mathrm{IR}^N$ which goes to infinity one has

$$\lim_{n\to+\infty} \nu_s(B_n) = +\infty, \tag{6}$$

where $B_n = B(x_n, r)$.

Let E be the space:

$$E = \left\{u \in W_0^{1,p}\left(\mathrm{IR}^N\right) : \int_{\mathrm{IR}^N} b(x)|u|^p < +\infty\right\}.$$

Our last assumption is the following:

$$\lambda_1 = \inf_{\substack{u\in E \\ \|u\|_p = 1}} \int |\nabla u|^p + b|u|^p > 0. \tag{7}$$

2. PRELIMINARIES

This section is devoted to the statement of all the auxiliary results that will be used in the proof of the main theorems. The following version of the mountain pass theorem for locally Lipschitz functions whose don't satisfy the classical Palais-Smale condition plays a basilar role in showing the existence of solution to (P) and to (P').

Theorem 2.1: Let E be a real reflexive Banach space and let $I : E \to \mathrm{IR}$ be a locally Lipschitz function, with $I(0) = 0$. Assume that there are $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ on ∂BP and that there is $e \in E$, with $|e| > \rho$, such that $I(e) \leq 0$. If we set

$$\Gamma := \{\gamma \in C^0([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}$$

and

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} f\left(\gamma\left(t\right)\right),$$

then there is a $(PS)_c$ sequence for I.

The following Lemmas and Propositions enable us to introduce the properties of E and yield a sufficient condition in order to (6) holds, in add to an equivalent one. In the sequel, we will omit the dependence from $x \in \mathrm{IR}^N$ of the functions involved.

Lemma 2.1: Assume (5) holds. If $u \in E$, then $|b|^{\frac{1}{p}} u \in L^p\left(\mathrm{IR}^N\right)$.

Proof: Let be $u \in E$. By using (5) we obtain

$$\begin{aligned}\int_{\mathrm{IR}^N} \left|b^{\frac{1}{p}} u\right| &= \int_{b\leq 0} -b|u|^p + \int_{b\geq 0} b|u|^p \\ &= -\int_{b\leq 0} b|u|^p - \int_{b\leq 0} b|u|^p + \int_{\mathrm{IR}^N} b|u|^p \\ &= -2\int_{b\leq 0} b|u|^p + \int_{\mathrm{IR}^N} b|u|^p \leq 2B\int_{b\leq 0} |u|^p + \int_{\mathrm{IR}^N} b|u|^p \\ &\leq 2B\|u\|_p^p + \int_{\mathrm{IR}^N} b|u|^p < +\infty.\end{aligned}$$

Proposition 2.1: Assume (5) and (7) hold. Then

$$\|u\| = \left(\int_{IR^N} |\nabla u|^p + b|u|^p\right)^{\frac{1}{p}}$$

is a norm on E. Furthermore there is $\tilde{c} > 0$ such that $|u|_{1,p} \leq \tilde{c}||u||$ for any $u \in E$ and $||\cdot||_{1,p}$ and $||\cdot||$ are equivalent.

Proof: The first statement is obvious, so we begin by proving the inequality $||u||_{1,p} \leq \tilde{c}||u||$ for any $u \in E$. Assume, on the contrary, that there exists a sequence $\{u_n\} \subseteq E$ such that

$$||u_n||_{1,p} = 1 \text{ and } \int_{IR^N} |\nabla u_n|^p + b|u_n|^p \leq \frac{1}{n} \text{ for any } n \in N.$$

Put $v_n = \dfrac{u_n}{|u_n|^p}$. Then $||v_n||_p = 1$. As $\int_{\mathrm{IR}^N} |\nabla v_n|^p + b|v_n|^p \geq \lambda_1$, from this we deduce that

$\int_{IR^N} |\nabla u_n|^p + b|u_n|^p \geq \lambda_1 |u_n|_p^p$, so $|u_n|_p^p \leq \dfrac{1}{n\lambda_1}$ and $u_n \to 0$ in $L^p(\mathrm{IR}^N)$. Due to (5), we get

$$-B|u_n|_p^p \leq \int_{IR^N} b|u_n|^p \leq \frac{1}{n} - \int_{IR^N} |\nabla u_n|^p = \frac{1}{n} - 1$$

and passing to the limit for $n \to +\infty$ we obtain the contradiction $0 \leq -1$. We pass now to verify that E is a Banach space with respect to $||\cdot||$. If $\{u_n\}$ is a Cauchy sequence with respect to $||\cdot||$, then there is $u \in W_0^{1,p}\left(\mathrm{IR}^N\right)$ such that $u_n \to u$ in $W_0^{1,p}\left(\mathrm{IR}^N\right)$. Due to Lemma 2.1 $|b|^{\frac{1}{p}} u_n \in L^p\left(\mathrm{IR}^N\right)$ for any $n \in N$ and the same arguments of the above Lemma show that $\left\{|b|^{\frac{1}{p}} u_n\right\}$ is a Cauchy sequence in $L^p(\mathrm{IR}^N)$, so $|b|^{\frac{1}{p}} u_n \to v$ in $L^p(\mathrm{IR}^N)$ and a.e., up to a subsequence. As $u_n \to u$ a.e. we deduce that $v = |b|^{\frac{1}{p}}$, a, e; so $|b|^{\frac{1}{p}} u_n \to |b|^{\frac{1}{p}} u$ in $L^p(\mathrm{IR}^N)$; bearing in mind that $\nabla u_n \to \nabla u$ in $L^p(\mathrm{IR}^N)$, we conclude that $u_n \to u$ in E.

Lemma 2.2: Let $\Omega \subseteq \mathrm{IR}^N$ be an open set and let $p \leq s \leq t < \dfrac{N_p}{N-p}$. Then there is $\alpha_1 \in [0, 1]$ such that

$$\nu_t(\Omega) \geq C(s, t, N, \lambda)(\nu_s(\Omega))^{\alpha_1}.$$

If $p < t < s < \frac{N_p}{N-p}$ then there is $\alpha_2 > 0$ such that

$$\nu_t(\Omega) \geq C(s, t, N, \lambda)(\nu_s(\Omega))^{\alpha_2}.$$

In particular $\nu_p \Rightarrow \nu_1$ for any $t \in [p, p^*[$.

Proof: Due to previous Lemma and Gagliardo-Niremberg inequality we can write

$$\nu_t(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{|u|_t^p} \geq \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{C|u|_s^{\alpha p}\, |u|^{p(1-\alpha)}} \geq \frac{1}{C} \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \left(\frac{\|u\|^p}{|u|_s^p} \right)^{\alpha}$$

$$= \frac{1}{C}\left(\nu_s\left(\Omega\right)\right)^{\alpha}.$$

The proof of the second inequality carries out in the same way.

Lemma 2.3: Condition (6) holds if and only if

$$\lim_{R \to +\infty} \nu_s\left(IR^N \setminus \overline{B}_R\right) = +\infty. \tag{8}$$

Proof: We observe that $\nu_s(\Omega_2) \leq \nu_s(\Omega_1)$ for all $\Omega_1 \subseteq \Omega_2$. If (8) holds then for any $M > 0$ we can find $\overline{R} > 0$ such that for any $R > \overline{R}$ one has $\nu_s\left(\mathrm{IR}^N \setminus \overline{B}_R\right) > M$. If we fix $r > 0$ and a sequence $\{x_n\} \subseteq \mathrm{IR}^N$ which goes to infinity, than $B_n \subseteq \mathrm{IR}^N \setminus \overline{B}_R$ for n sufficiently large, so, for such n, $\nu_s(B_n) > M$ and the conclusion follows immediately. For the second part of the proof we suppose by contradiction that we can find two sequences $\{R_n\} \subseteq \mathrm{IR}$ and $\{u_n\} \subseteq E$ such that $R_n \to +\infty$,

$$\operatorname{supp} u_n \subseteq \mathrm{IR}^N \setminus \overline{B}_{R_n},\ \|u_n\|^p \leq C \text{ and } |u_n|_s = 1 \text{ for any } n \in N.$$

As $\{u_n\}$ doesn't go to 0 in $L^s(\mathrm{IR}^N)$, due to a classical compactness result (see [14]) we can find a sequence $\{x_n\} \subseteq \mathrm{IR}^N$ and a number $r_0 > 0$ such that $\int_{B(x_n, r_0)} |u_n|^s \geq c_0 > 0$. Being $\operatorname{supp} u_n \subseteq \mathrm{IR}^N \setminus \overline{B}_{R_n}$, for any $n \in N$ there is a point $y_n \in B(x_n, r_0)$ such that $|y_n| \geq R_n$ so $R_n \leq |y_n| \leq r_0 + |x_n|$ and $|x_n| \to +\infty$. Now, for any $n \in N$ we take a function $\varphi_n \in C_c^\infty(\mathrm{IR}^N)$ with the following characteristics:

$$0 \leq \varphi_n \leq 1,\ \varphi_n \equiv 0 \text{ in } \mathrm{IR}^N \setminus B(x_n, 2\, r_0),\ \varphi_n \equiv 1 \text{ in } B(x_n, r_0).$$

The function $v_n = \varphi_n u_n$ is bounded on E, supp $v_n \subseteq B(x_n, 2r_0)$ and $\int_{B(x_n, 2r_0)} |v_n|^s \geq \int_{B(x_n, r_0)} |v_n|^s = \int_{B(x_n, r_0)} |u_n|^s c_0 > 0$. From these inequalities we infer that $\nu_s(B_n) \leq L$ for any $n \in N$ and this contradicts ours assumptions.

We omit the proof of the next Lemma, because it is the same of that given in [20], Lemma 2.3.

Lemma 2.4: Let $\{\omega_n\} \subseteq \mathrm{IR}^N$ be a sequence of open sets, with $\lim_{n\to+\infty} |w_n| = 0$. Then, for any $C > 0$ and for any $s \in [p, p^*[$ one has

$$\lim_{n\to+\infty}\left(\sup_{\|u\|_{1,p}\leq C}\int_{\omega_n}|u|^s\right) = 0.$$

Lemma 2.5: Assume that for any M, $r > 0$, $\{x_n\} \subseteq \mathrm{IR}^N$, with $|x_n| \to +\infty$, one has $\lim_{n\to+\infty}|\Omega_M \cap B_n| = 0$, where $\Omega_M = b^{-1}([-\infty, M])$. Then (6) holds for any $s \in [p, p[$.

Proof: Thanks to Lemma 2.3 it suffices to prove (6) only for $s = p$. Suppose by contradiction that (6) is not satisfied for $s = p$. We can find a number $r > 0$ and two sequences $\{x_n\} \subseteq \mathrm{IR}^N$ and $\{u_n\} \subseteq W_0^{1,p}(B_n)$ such that

$$|x_n| \to +\infty,\ \int_{B_n} |\nabla u_n|^p + b|u_n|^p \leq C, \text{ and } \int_{B_n}|u_n|^p = 1.$$

Now, we fix $M > 0$ and we put $\Omega_M, n = \Omega_M \cap B_n$; bearing in mind (5), we get

$$C \geq \|u_n\|^p \geq \int_{B_n} b|u_n|^p = \int_{\Omega M,n} b|u_n|^p + \int_{B_n\setminus\Omega_{M,n}} b|u_n|^p \geq \int_{\Omega M,n} b|u_n|^p + M\int_{B_n\setminus\Omega_{M,n}}|u_n|^p$$

$$\geq -B\int_{\Omega M,n}|u_n|^p + M\int_{B_n\setminus\Omega_{M,n}}|u_n|^p$$

$$= -(B+M)\int_{\Omega M,n}|u_n|^p + M\int_{B_n}|u_n|^p$$

$$= -(B+M)\int_{\Omega_{M,n}}|u_n|^p + M. \tag{9}$$

Recalling that $\lim_{n\to+\infty} |\Omega_{M,n}| = 0$, for n sufficiently large we can write

$$\int_{\Omega M,n}|u_n|^p < \frac{M}{2(B+M)},$$

so we infer $C \geq \dfrac{M}{2}$ which is an absurd, as M is arbitrary.

Lemma 2.6: If (5), (6) and (7) hold, then E is compactly embedded in $L^t(\mathrm{IR}^N)$ for any $t \in [p, p^*]$.

Proof: Let be $u_n \to 0$ in E; then $\|u_n\| \leq K$, $u_n \to 0$ in $L^t(B_R)$ for any $R > 0$ and a.e. If we choose $\varphi \in C^\infty(\mathrm{IR}^N)$, $\varphi \equiv 0$ on B_R, $\varphi \equiv 1$ on $\mathrm{IR}^N \setminus B_{R+1}$, $0 \leq \varphi \leq 1$, then we obtain

$$|u_n|_t \leq |(1-\varphi)u_n|_t + |\varphi u_n|_t$$

$$= \left(\int_{B_{R+1}} |u_n|^t\right)^{\frac{1}{t}} + \left(\int_{\mathrm{IR}^N \setminus B_R} \varphi^t |u_n|^t\right)^{\frac{1}{t}} \leq \left(\int_{B_{R+1}} |u_n|^t\right)^{\frac{1}{t}} + \left(\int_{\mathrm{IR}^N \setminus B_R} |u_n|^t\right)^{\frac{1}{t}}.$$

Due to (6), fixed any $M > 0$ the following inequality holds for a suitable $R > 0$:

$$\nu_t(\mathrm{IR}^N \setminus B_R) > M;$$

corresponding to such R and to $\varepsilon > 0$, $\varepsilon < \min\left\{\frac{1}{M}, \frac{K}{B}\right\}$, we can find $\bar{n} \in N$ such that $\int_{B_{R+1}} |u_n| t < \varepsilon$ for any $n > \bar{n}$. Bearing in mind the definition of ν_t, (5), and the fact that $\int_\Omega |u_n|^t < \varepsilon$ for any $\Omega \subseteq B_{R+1}$, we can write

$$\begin{aligned}
\int_{\mathrm{IR}^N \setminus B_R} |u_n|^t &\leq \frac{1}{\nu_t\left(\mathrm{IR}^N \setminus B_R\right)} \int_{\mathrm{IR}^N \setminus B_R} |\nabla u_n|^t + b|u_n|^t \\
&\leq \frac{1}{\nu_t\left(\mathrm{IR}^N \setminus B_R\right)} \left[\int_{\mathrm{IR}^N} |\nabla u_n|^t + b|u_n|^t - \int_{B_R} b|u_n|^t\right] \\
&\leq \frac{\|u_n\|^t + B\int_{B_R} b|u_n|^t}{\nu_t\left(\mathrm{IR}^N \setminus B_R\right)} \leq \frac{K + B\varepsilon}{M} < \frac{2K}{M},
\end{aligned} \tag{10}$$

so

$$|u_n|^t \leq \left(\frac{1}{M}\right)^{\frac{1}{t}} + \left(\frac{2K}{M}\right)^{\frac{1}{t}} \; \forall n > \bar{n} \tag{11}$$

and $u_n \to 0$ in $L^t(\mathrm{IR}^N)$.

After stating those preliminary results, we pass now to the construction of the energy functionals: we treat in parallel all the situations that may occur because their behavior is similar. For our convenience we assume $f(x, s) \equiv 0$ for any $s < 0$. In the first case, namely when (5)–(7) and (1)–(3) hold, the functional

$$\Phi(u) = \int_{\mathrm{IR}^N} \left(\int_0^{u(x)} f(x,t)\,dt \right) = \int_{\mathrm{IR}^N} F(x, u(x))\,dx$$

is well defined and locally Lipschitz on E. In fact, if we take $u_0 \in E$, > 0, and $u, v \in B(u_0, \delta)$, then the embedding of E in $L^t(\mathrm{IR}^N)$ and Hölder's inequality ensure that we can find $L_{u_0} > 0$ such that

$$|\Phi(u) - \Phi(v)| \le L_{u_0} \|u - v\|.$$

As our goal is to treat the case when $b \equiv 0$ too, we explicitly observe that, independently from conditions (5)–(7), can be defined on the whole space $W_0^{1,p}(\mathrm{IR}^N)$ and, beside to be locally Lipschitz in this space too, the following well known properties hold:

- $\Phi_E^0(u; v) \le \Phi^0_{W_0^{1,p}}(u; v) \, \forall u, v \in E;$
- $\Phi^0_{W_0^{1,p}}(u; v) \le \int_{\mathrm{IR}^N} F^0(x, u(x); v(x))\,dx \le \int_{v<0} \underline{f}(x, u(x))\, v(x)\,dx$

  $+\int_{v>0} \overline{f}(x, u(x))\, v(x)\,dx \qquad \forall u, v \in W_0^{1,p}(\mathrm{IR}^N);$
- $\partial\Phi_E(u) \subseteq \partial\Phi_{W_0^{1,p}}(u).$

The first inequality blows from the definition of generalized directional derivative, the proof of the second one is substantially the same of Lemma 2.6 in [15], while the third one can be found in [9]. It is worthwhile to punctuating that for any $w \in \partial\Phi(u)$ and any $v \in W_0^{1,p}(\mathrm{IR}^N)$ one has

$$\int_{v<0} \overline{f}v \le \int_{v<0} \overline{f}v + \int_{v>0} \underline{f}v \le \langle w, v \rangle \le \int_{v>0} \overline{f}v + \int_{v<0} \underline{f}v \le \int_{v>0} \overline{f}v, \tag{12}$$

$$|\langle w, v \rangle| \le C\left(\int_{\mathrm{IR}^N} |u|^{p-1} |v| + |u|^q |v| \right) \le C\left(|u|_p^{p-1} |v|_p + |u|_{q+1}^q |v|_q \right) \tag{13}$$

and

$$\int_{u>0} \underline{f}u \le \langle w, u \rangle \le \int_{u>0} \overline{f}u.$$

If we assume (4) and (3) in place of (1)–(3) then

$$\Psi(u) = \int_{\mathrm{IR}^N} F(x, u(x))\,dx$$

is well defined and locally Lipschitz on E. In fact, if we take $u_0 \in E$, $\delta > 0$, and $u, v \in B(u_0, \delta)$ then

$$|\Psi(u) - \Psi(v)| \leq (|h|_\theta\, a_1 + a_2)\, \|u - v\|,$$

where a_1, a_2 depend from δ, t, q, r and c. Now, as above, Ψ can be defined on $W_0^{1,p}(\mathrm{IR}^N)$ and, beside to be locally Lipschitz in this space too, arguing in a standard way and owing to the Fatou's Lemma we obtain the same inequalities written above for Ψ, in add to (12), while (13) takes now the form:

$$\left|\langle w, v\rangle\right| \leq \left(\int_{\mathrm{IR}^N} h(x)|u|^q |v| + C|u|^r |v|\right) \leq |h|_\theta |u|_t^q |v|_t + C|u|_{r+1}^r |v|_{r+1} \tag{14}$$

3. THE CRITICAL CASE

We begin by taking into account the case in which the nonlinearity can be critical, but it is necessary to point out that the term with critical growth is classical, so (P) and (P') can be written as:

Find $u \in E$ satisfying

(P) $\quad \int_{\mathrm{IR}^N} (|\nabla u|^{p-2}\nabla u \cdot \nabla v + b|u|^{p-2}uv - |u|^{p^*-2}uv + (-F)^0(x, u; v) \geq 0 \;\forall v \in E,$

and

Find $u \in W_0^{1,p}(\mathrm{IR}^N)$ satisfying

(P') $\quad \int_{\mathrm{IR}^N} (|\nabla u|^{p-2}\nabla u \cdot \nabla v - |u|^{p^*-2}uv + (-F)^0(x, u; v) \geq 0 \;\forall v \in W_0^{1,p}(\mathrm{IR}^N).$

When we deal with (P) the energy functional is either

$$I(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*}|u|_{p^*}^{p^*} - \Phi(u) \qquad \forall u \in E,$$

or

$$I(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*}|u|_p^{p^*} - \Psi(u) \quad \forall u \in E.$$

Classical properties of generalized directional derivatives, together with all the others cited in the previous section, allow us to majoring $I^0(u; v)$ in the following way:

$$I^0(u;v) = \int_{\mathrm{IR}_N}(|\nabla u|^{p-2}\nabla u\cdot\nabla v + b|u|^{p-2}uv - |u|^{p^*-2}uv) + (-\Phi)^0(u;v)$$
$$\geq \int_{\mathrm{IR}_N}(|\nabla u|^{p-2}\nabla u\cdot\nabla v + b|u|^{p-2}uv - |u|^{p^*-2}uv) - \langle w, v\rangle,$$

where $w \in \partial\phi(u)$, while for $I_1^0(u; v)$ we actually have

$$I_1^0(u;v) = \int_{\mathrm{IR}_N}(|\nabla u|^{p-2}\nabla u\cdot\nabla v + b|u|^{p-2}uv - |u|^{p-2}uv) + (-\Psi)^0(u;v)$$
$$\geq \int_{\mathrm{IR}_N}(|\nabla u|^{p-2}\nabla u\cdot\nabla v + b|u|^{p-2}uv - |u|^{p^*-2}uv) - \langle w, v\rangle,$$

where $w \in \partial\Phi(u)$.

When $b \equiv 0$, namely when the problem becomes (P'), both I and I_1 are defined on $W_0^{1,p}\left(\mathrm{IR}^N\right)$, but their structure is the same, that is the norm of u to the p-power minus a function. We begin the section with two lemmas regarding the Palais-Smale sequences of I and of I_1. It is superfluous to stress that we will give the proofs for the first case only, because when $b \equiv 0$ they carry out in the same way.

Lemma 3.1: Any $(PS)_c$ sequence of I and of I_1 is bounded.

Proof: We begin by examining the behavior of the $(PS)_c$ sequences of I. If $\{u_n\}$ is a $(PS)_c$ sequence for I, then $m_I(u_n) \to 0$, so for any $n \in N$ there exist $\omega_n \in \partial I(u_n)$, as well as $\omega_n \in \partial(u_n)$ such that

- $\langle\omega_n, v\rangle = \int_{\mathrm{IR}_N}(|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla v + b|u_n|^{p-2}u_n v - |u_n|^{p^*-2}u_n v) - \langle w_n, v\rangle \ \forall v \in E$;
- $||\omega_n||_{E^*} \to 0$;

Thanks to (3), we can write

$$\Phi(u_n) \leq \frac{1}{\mu}\int_{\mathrm{IR}^N} u_n(x)\underline{f}(x, u_n(x))\,dx \leq \frac{1}{\mu}\langle w_n, u_n\rangle \ \forall n \in N;$$

if we choose $\nu = \min\{\mu, p^*\}$ then we obtain the inequalities below, from which the boundedness of $\{u_n\}$ follows immediately:

$$I(u_n) = I(u_n) - \frac{1}{\nu}\langle\omega_n, u_n\rangle + \frac{1}{\nu}\langle\omega_n, u_n\rangle \geq \frac{\nu-p}{\nu p}\|u_n\|^p - \Phi(u_n) + \frac{1}{\nu}\langle w_n, u_n\rangle$$

$$+\frac{p^*-\nu}{p^*\nu}|u_n|_{p^*}^{p^*} - \frac{1}{\nu}\|\omega_n\|_{E^*}\|u_n\| \geq \frac{\nu-p}{\nu p}\|u_n\|^p - \frac{1}{\nu}\|\omega_n\|_{E^*}\|u_n\|.$$

Arguing in the same way, we obtain

$$I_1(u_n) \geq \frac{\nu - p}{\nu p}\|u_n\|^p - \frac{1}{\nu}\|\omega_n\|_{E^*}\|u_n\|,$$

and the Lemma is totally proved.

Lemma 3.2: Any $(PS)_c$ sequence, nonnegative a.e., satisfies $\nabla u_n \to \nabla u$ a.e. in IR^N.

Proof: We give only a sketch of the proof, which follows essentially the lines of that of Lemma 7 in [13], so we refer to that paper for more details. Let $\{u_n\}$ be a $(PS)_c$ sequence for I, non negative a.e. Then we can suppose that $u_n \rightharpoonup u$ in $L^t(\mathrm{IR}^N)$ for any $t \in [p, p^*]$ and a.e., as well as $u \geq$ a.e.; furthermore

$$u_n^{p*} \rightharpoonup u^{p*} + \sum_{i=1}^{j} \nu_i \delta_{x_i} \equiv \nu,$$

while for $\{\nabla u_n\}$ we may assume that

$$|\nabla u_n|^p \rightharpoonup \mu$$

for some measure μ. If there is i such that $\nu_i > 0$ then we choose an $\varepsilon_0 > 0$ such that

$$B_{\varepsilon_0}(x_i) \cap B_{\varepsilon_0}(x_k) = \emptyset \;\forall_i \neq k,\; |x_k| < \frac{1}{2\varepsilon_0} \quad \forall k = 1, ..., j$$

and we put

$$A_\varepsilon := B_{\frac{1}{2\varepsilon}} \setminus \bigcup_{k=1}^{j} B_\varepsilon(x_k).$$

Let ρ and ε be two real numbers satisfying $0 < \varepsilon < \rho$ and ϕ a function belonging to $C_0^\infty(\mathrm{IR}^N)$ such that $\phi \equiv 1$ on $B_{\frac{1}{2}}, \phi \equiv 0$ on B_1^c. Starting from we construct a new function: $\psi_\varepsilon(x) = \phi(\varepsilon x) - \sum_{k=1}^{j} \phi\left(\frac{x - x_k}{\varepsilon}\right)$. We can write

$$\begin{aligned}
0 &\leq \int_{A_\rho} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot (\nabla u_n - \nabla u) \\
&\leq \int_{\mathrm{IR}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot (\nabla u_n - \nabla u)\psi_\varepsilon \\
&= \int_{\mathrm{IR}^N} (|\nabla u_n|^p \psi_\varepsilon - |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u \psi_\varepsilon - |\nabla u|^{p-2}\nabla u_n \cdot \nabla u \psi_\varepsilon + |\nabla u|^p \psi_\varepsilon)
\end{aligned}$$

$$= \int_{\mathbb{R}^N}\left(|\nabla u_n|^p\,\psi_\varepsilon + bu_n^p\psi_\varepsilon - u_n^{p^*}\psi_\varepsilon\right) - \langle w_n, u\psi_\varepsilon\rangle$$

$$-\left[\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\,\nabla u_n\cdot\nabla u\psi_\varepsilon + bu_n^{p-1}u\psi_\varepsilon - u_n^{p^*-1}u\psi_\varepsilon\right) - \langle w_n, u\psi_\varepsilon\rangle\right]$$

$$-\int_{\mathbb{R}^N}\left(|\nabla u|^{p-2}\,\nabla u_n\cdot\nabla u\psi_\varepsilon - |\nabla u|^p\,\psi_\varepsilon\right)$$

$$+\int_{\mathbb{R}^N}\left(-bu_n^p\psi_\varepsilon + bu_n^{p-1}u\psi_\varepsilon + u_n^{p^*}\psi_\varepsilon - u_n^{p^*-1}u\psi_\varepsilon\right) + \langle w_n, (u_n - u)\,\psi_\varepsilon\rangle$$

$$= (\langle \omega_n, u_n\psi_\varepsilon\rangle - \int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\psi_\varepsilon u_n) - (\langle\omega_n, u\psi_\varepsilon\rangle$$
$$- \int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\psi_\varepsilon u) - I_{3,n,\varepsilon} + I_{4,n,\varepsilon}$$
$$= I_{1,n,\varepsilon} - I_{2,n,\varepsilon} - I_{3,n,\varepsilon} + I_{4,n,\varepsilon}.$$

It is obvious that $\lim_{n\to+\infty}\langle\omega_n, u_n\psi_\varepsilon\rangle = 0$, as well as $\lim_{n\to+\infty}\langle\omega_n, u_n\psi_\varepsilon\rangle = 0$, so, owing to Claim 3 at p. 65 in [13], we can affirm that

$$\lim_{n\to+\infty} I_{1,n,\varepsilon} = \lim_{n\to+\infty} I_{2,n,\varepsilon} = 0.$$

It is a simple matter to see that $\lim_{n\to+\infty} I_{3,\ n,\ \varepsilon} = 0$, so we have only to examine $I_{4,n,\varepsilon}$. Now, (13) yields

$$\left|\langle w_n, (u_n - u)\,\psi_\varepsilon\rangle\right| \le C\left(\int_{\mathbb{R}^N} u_n^{p-1}\,|u_n - u|\,\psi_\varepsilon + u_n^q\,|u_n - u|\,\psi_\varepsilon\right),$$

so, bearing in mind the properties of $\{u_n^{p-1}\,|u_n - u|\}$, $\{u_n^q\,|u_n - u|\}$ and of ψ_ε we conclude that

$$\lim_{n\to+\infty}\langle w_n, (u_n - u)\,\psi_\varepsilon\rangle = 0$$

and, finally

$$\lim_{n\to+\infty} I_{4,n,\varepsilon} = 0.$$

When we deal with I_1, we proceed in the same way. The only difference stands in $(I_1)_{4,n,\varepsilon}$, for which we have

$$(I_1)_{4,n,\varepsilon} = \int_{\mathbb{R}^N}\left(-bu_n^p\psi_\varepsilon + bu_n^{p-1}u\psi_\varepsilon - u_n^{p^*}\psi_\varepsilon - u_n^{p^*-1}u\psi_\varepsilon\right) + \langle w_n, (u_n - u)\,\psi_\varepsilon\rangle,$$

with $w_n \in \partial(u_n)$ and

$$\left|\left\langle w_n, (u_n - u)\psi_\varepsilon\right\rangle\right| \leq \left(\int_{\mathrm{IR}^N} h(x)u_n^q |u_n - u|\psi_\varepsilon + Cu_n^r |u_n - u|\psi_\varepsilon\right). \qquad (16)$$

Now, arguing as above, we conclude that

$$\lim_{n\to+\infty} (I_1)_{4,n,\varepsilon} = 0.$$

We have so proved that

$$\lim_{n\to+\infty} \int_{A_\rho} \left(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u\right)\cdot\left(\nabla u_n - \nabla u\right) = 0.$$

The same result holds also when $\nu_i \equiv 0$ for any $i = 1, ..., j$. Finally, the same reasoning made at p. 66 in [13] leads to the conclusion. It is although obvious that when $b \equiv 0$ we proceed exactly in the same manner.

Theorem 3.1: Assume (1)–(3) and (5)–(7) hold. Then (P) has a solution which is non negative a.e.

Proof: The proof is quite standard and we split it into two parts. In the first of them we show that we can apply Theorem 2.1 to the energy functional related to (P), and in the second one we prove that the critical point obtained turns out a non negative solution to (P). For the sake of completeness we remember that

$$I : E \to \mathrm{IR} \quad I(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*}|u|_{p^*}^{p^*} - \Phi(u)$$

is locally Lipschitz on E; we have to verify the mountain pass geometry. Obviously $I(0) = 0$. Let be $v \in C_c^\infty(\mathrm{IR}^N)\setminus\{0\}$, $v \geq 0$ a.e. Then

$$I(tv) = \frac{t^p}{p}\|v\|^p - \frac{t^{p^*}}{p^*}|v|_{p^*}^{p^*} - \Phi(tv) \text{ for any } t \geq 0$$

and Lemma 1.1 allows us to write

$$I(tv) \leq \frac{t^p}{p}\|v\|^p - \int_{\mathrm{IR}^N}\left(\int_0^{tv(x)}\left(C_1 s^{\mu-1} - C_2\right)ds\right) = \frac{t^p}{p}\|v\|^p$$

$$-\frac{C_1 t^\mu}{\mu}\int_{\mathrm{IR}^N} v^\mu + C_2 t\int_{\mathrm{IR}^N} v = \frac{t^p}{p}\|v\|^p - \frac{C_1 t^\mu}{\mu}|v|_\mu^\mu + C_2 t|v|_1. \qquad (17)$$

Being $\mu > p$ we can conclude that there is $\bar{t} = 0$ such that

$$I(tv) \le 0 \qquad \forall t \ge \bar{t}. \tag{18}$$

In a standard way we obtain that for any $\varepsilon > 0$ there is a number $A_\varepsilon > 0$ such that

$$f(x, s) < \varepsilon s^{p-1} + A_\varepsilon s^q \ \forall s > 0$$

and from this last inequality we infer

$$\Phi(u) \le \frac{\varepsilon}{p}|u|_p^p + \frac{A_\varepsilon}{q+1}|u|_{q+1}^{q+1} \le \frac{\varepsilon(\tilde{c}c_p)^p}{p}\|u\|^p + \frac{A_\varepsilon(\tilde{c}c_{q+1})^{q+1}}{q+1}\|u\|^{q+1} \ \forall u \in E,$$

where c_p and c_q are the constants of the embeddings of $W_0^{1,p}(\mathrm{IR}^N)$ in $L^p(\mathrm{IR}^N)$ and in $L^q(\mathrm{IR}^N)$, while $\tilde{c}$ is that obtained in Proposition 2.1. If $\|u\| = \rho$ then

$$I(u) \ge \frac{\rho^p}{p}(1 - \varepsilon K_1) - \frac{\rho^{p^*}}{p^*}c_{p^*}^{p^*} - \frac{\rho^{q+1}}{q+1}A_\varepsilon K_2$$

and for ε and ρ sufficiently small one has

$$I(u) \ge \beta > 0. \tag{19}$$

So, we can apply Theorem 2.1 to I, which guarantees the existence of a $(PS)_c$ sequence for I. Now, we point out two facts:

- if $\gamma \in \Gamma$ then $|\gamma| \in \Gamma$ too, because $e = tv \ge 0$ a.e.;
- $I(|u|) \le I(u)$.

By virtue of the remarks above we can assume that the $(PS)_c$ sequence obtained through the mountain pass theorem is non negative a.e. Let $\{u_n\}$ be such a sequence. Due to Lemma 3.1 $\{u_n\}$ is bounded, so we can extract a subsequence which converges weakly in E and in $W_0^{1,p}(\mathrm{IR}^N)$, and a. e. in IR^N to a function $u \ge 0$ a.e. Lemma 3.2 guarantees that $\nabla u_n \to \nabla u$ a.e.; being $\{\nabla u_n\}$ bounded in $L^p(\mathrm{IR}^N)$ we can apply Remark 8 of [13] and we obtain

$$\lim_{n\to+\infty} \int_{\mathrm{IR}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi = \int_{\mathrm{IR}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \quad \forall\varphi \in W_0^{1,p}(\mathrm{IR}^N). \tag{20}$$

An analogous result holds for $b|u_n|^{p-2}u_n$ and for $|u_n|^{p-2}u_n$:

$$\lim_{n\to+\infty} \int_{\mathrm{IR}^N} b|u_n|^{p-2}u_n\varphi = \int_{\mathrm{IR}^N} b|u|^{p-2}u\varphi \quad \forall\varphi \in E, \tag{21}$$

$$\lim_{n\to+\infty} \int_{\mathrm{IR}^N} |u_n|^{p^*-2}u_n\varphi = \int_{\mathrm{IR}^N} |u|^{p^*-2}u\varphi \quad \forall\varphi \in W_0^{1,p}\left(\mathrm{IR}^N\right), \tag{22}$$

while for $\{w_n\}$ we actually have

$$\lim_{n\to+\infty} \langle -w_n, v\rangle = \langle -w, v\rangle \ \forall v \in W_0^{1,p}\left(\mathrm{IR}^N\right)$$

and

$$\begin{aligned}\langle -w_n, v\rangle &= \langle w_n, -v\rangle \le \int_{\mathrm{IR}^N} F^0(x, u_n; -v)\\ &= \int_{\mathrm{IR}^N}(-F)^0(x, u_n; v)\ \forall n \in N, \forall v \in W_0^{1,p}\left(\mathrm{IR}^N\right).\end{aligned}$$

Due to the properties of f and of $\{u_n\}$, after passing to the limit we can write

$$\langle -w, v\rangle \le \int_{\mathrm{IR}^N}(-F)^0(x, u(x); v(x))dx\ \forall v \in W_0^{1,p}\left(\mathrm{IR}^N\right). \tag{23}$$

Finally, we remember that $\|\omega_n\|_{E^*} \to 0$, and this, together with (20), (21), (22) and (23) yields

$$\int_{\mathrm{IR}^N}(|\nabla u|^{p-2}\nabla u\nabla v + b|u|^{p-2}uv - |u|^{p-2}uv + (-F)^0(x, u; v) \ge 0\ \forall v \in E$$

and u turns out a non negative solution to (P).

Theorem 3.2: Assume (3), (4) and (5)–(7) hold. Suppose furthermore that

$$|h|_\theta \le \frac{q+1}{4p\left(\tilde{c}c_t\right)^{q+1}} s^{p-q-1}$$

for some $s \in \mathrm{IR}$ satisfying

$$\frac{\left(\tilde{c}c_{p^*}\right)p^*}{p^*} s^{p^*-p} + \frac{C\left(\tilde{c}c_{r+1}\right)^{r+1}}{r+1} s^{r+1-p} < \frac{1}{2p},$$

whenever $q + 1 - p \le 0$. Then (P) has a solution which is non negative a.e.

Proof: We begin by proving that I_1 has a mountain pass geometry. This part of the proof is very similar to that of Lemma 3 in [13] so we omit some detail. Firstly we observe that:

$$\left|F\left(x, s\right)\right| \le \frac{h\left(x\right)|s|^{q+1}}{q+1} + \frac{C|s|^{r+1}}{r+1} \text{ for any } s \in \mathrm{IR},$$

so

$$I(u) \geq \frac{1}{p}\|u\|^p - \frac{(\tilde{c}c_{p^*})^{p^*}}{p^*}\|u\|^{p^*} - \frac{|h|_\theta(\tilde{c}c_t)^{q+1}}{q+1}\|u\|^{q+1} - \frac{C(\tilde{c}c_{r+1})^{r+1}}{r+1}\|u\|^{r+1}$$

$$= \|u\|^p\left(\frac{1}{p} - \frac{(\tilde{c}c_{p^*})^{p^*}}{p^*}\|u\|^{p^*-p} - \frac{|h|_\theta(\tilde{c}c_t)^{q+1}}{q+1}\|u\|^{q+1-p} - \frac{C(\tilde{c}c_{r+1})^{r+1}}{r+1}\|u\|^{r+1-p}\right)$$

$$= \|u\|^p\left(\frac{1}{p} - \alpha\|u\|^{p^*-p} - \beta\|u\|^{q+1-p} - \gamma\|u\|^{r+1-p}\right).$$

Now, bearing in mind the condition imposed on $|h|$ Ø when $q + 1 - p \leq 0$, surely we can find a $\rho > 0$ such that $I(u) > 0$ whenever $\|u\| = \rho$. The proof of

$$I(e) \leq 0 \tag{24}$$

is the same of that of previous theorem. Now, Theorem 2.1 yields a $(PS)_c$ sequence of I_1. Arguing exactly as above we obtain that the sequence found converges to a point u which is a non negative solution to (P).

Theorem 3.3: Assume (1)–(3) hold. Then (P') has a solution which is non negative a.e.

Theorem 3.4: Assume (3), (4) hold, in add to

$$|h|_\theta \leq \frac{q+1}{4pc_t^{q+1}} s^{p-q-1}$$

for some $s \in$ IR satisfying

$$\frac{c_{p^*}^{p^*}}{p^*}s^{p^*-p} + \frac{Cc_{r+1}^{r+1}}{r+1}s^{r+1-p} < \frac{1}{2p},$$

whenever $q + 1 - p \leq 0$. Then (P') has a solution which is non negative a.e.

Remark 3.1: All previous results don't guarantee that u is a nontrivial solution, but under some additional assumption on f we obtain a nontrivial, positive solution for (P').

Corollary 3.1: Assume (3), (4) hold, with $C = 0$, in add to

$$|h|_\theta \leq \frac{q+1}{4pc_t^{q+1}} s^{p-q-1}$$

for some $s \in$ IR satisfying

$$\frac{c_{p^*}^{p^*}}{p^*} s^{p^*-p} < \frac{1}{2p},$$

whenever $q + 1 - p \leq 0$. Suppose furthermore that $f(x, s) > 0$ whenever $s > 0$. Then (P') has a positive solution.

Proof: The only difference with respect to previous results stands in the choice of the function e. In fact in this case we take

$$\omega_\varepsilon(x) = \frac{\left[N\varepsilon \frac{N-p}{p-1}\right]^{\frac{N}{p^2}}}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}.$$

It is well known that $\|\omega_\varepsilon\|^p = |\omega_\varepsilon|_{p^*}^{p^*} = c_{p^*}^{-N}$. As

$$I(t\omega_\varepsilon) = \left(\frac{t^p}{p} - \frac{t^{p^*}}{p^*}\right) c_{p^*}^{-N} - \Psi(t\omega_\varepsilon) < \left(\frac{t^p}{p} - \frac{t^{p^*}}{p^*}\right) c_{p^*}^{-N} < \frac{c_{p^*}^{-N}}{N} \tag{25}$$

it is obvious that we can take $t > 0$ such that $I(t\omega_\varepsilon) < 0$, as well as $c < \frac{c_{p^*}^{-N}}{N}$. Now, assume by contradiction that the nonnegative $(PS)_c$ sequence obtained trough the mountain pass theorem converges weakly to $u \equiv 0$. Then the following equalities hold true:

$$\lim_{n\to+\infty} \langle w_n, u_n \rangle = \lim_{n\to+\infty} \Psi(u_n) = 0;$$

from these we deduce that

$$\lim_{n\to+\infty} \|u_n\|^p = \lim_{n\to+\infty} |u_n|_{p^*}^{p^*} = l,$$

so

As $I(u_n) \to c$ we obtain

$$\frac{1}{N} = c < \frac{c_{p^*}^{-N}}{N},$$

which contradicts previous inequality.

Remark 3.2: The result above is the nonsmooth version of Theorem in [13], where the function f is exactly $h(x)s^q$.

4. THE SUBCRITICAL CASE

When we deal with a problem with subcritical growth we can give a more precise result; in fact in this case we can guarantee that the solution is nontrivial. It is obvious that from now on we take:

$$I(u) = \frac{1}{p}\|u\|^p - \Phi(u),$$

and

$$I_1(u) = \frac{1}{p}\|u\|^p - \Psi(u).$$

Lemma 4.1: Any $(PS)_c$ sequence of I and of I_1 is bounded.

Proof: Arguing exactly as in Lemma 3.1 we obtain the inequalities below for any $(PS)_c$ sequence of I and of I_1:

$$\begin{aligned} I(u_n) &= \frac{1}{p}\|u_n\|^p - \Phi(u_n) \\ &= \frac{\mu - p}{\mu p}\|u_n\|^p + \frac{1}{\mu}\|u_n\|^p - \frac{1}{\mu}\langle w_n, u_n\rangle + \frac{1}{\mu}\langle w_n, u_n\rangle - \Phi(u_n) \\ &= \frac{\mu - p}{\mu p}\|u_n\|^p + \frac{1}{\mu}\langle \omega_n, u_n\rangle + \frac{1}{\mu}\langle w_n, u_n\rangle - \Phi(u_n) \\ &\geq \frac{\mu - p}{\mu p}\|u_n\|^p - \frac{1}{\mu}\|\omega_n\|_{E^*}\|u_n\|, \end{aligned}$$

$$I_1(u_n) \geq \frac{\mu - p}{\mu p}\|u_n\|^p - \frac{1}{\mu}\|\omega_n\|_{E^*}\|u_n\|,$$

Lemma 4.2: Any $(PS)_c$ sequence, nonnegative a.e., satisfies $\nabla u_n \to \nabla u$ a.e. in IR^N.

Theorem 4.1: Assume (1)–(3) and (5)–(7) hold. Then (P) has a solution which is positive a.e.

Proof: We consider

$$I : E \to \mathrm{IR} \quad I(u) = \frac{1}{p}\|u_n\|^p - \Phi(u).$$

Arguing exactly as in Theorem 3.1 we find u, a nonnegative solution to (P). Our goal is to show that $u \neq 0$. Being $\{u_n\}$ a $(PS)_c$ sequence, for n sufficiently large we are able to write

$$0 < \frac{c}{2} \leq I(u_n) - \frac{1}{p}\langle \omega_n, u_n \rangle = 0 - \Phi(u_n) + \frac{1}{p}\langle w_n, u_n \rangle \leq \frac{1}{p}\langle w_n, u_n \rangle$$

$$\leq \frac{1}{p}\int_{\mathrm{IR}^N} u_n(x)\overline{f}(x, u_n(x))\,dx \leq \frac{C}{p}\left(|u_n|_p^p + |u_n|_{q+1}^{q+1}\right) \quad (26)$$

so we infer that $\{u_n\}$ doesn't go to 0 in $L^p(\mathrm{IR}^N)$ or in $L^{q+1}(\mathrm{IR}^N)$. Due to Theorem 2.6 $u_n \to u$ both in $L^p(\mathrm{IR}^N)$ and in $L^{q+1}(\mathrm{IR}^N)$, so it is immediately seen that $u > 0$ a.e.

Remark 4.1: The solution u found throw previous theorem satisfies the following differential inclusion:

$$-\Delta_p(u) + b|u|^{p-2}u \in \left[\underline{f}(x, u(x)), f(x, u(x))\right] \text{ a.e.}$$

and hence the result can be seen as a generalization of Theorem 2 in [12].

Theorem 4.2: Assume (3), (4) and (5)–(7) hold. Suppose furthermore that

$$|h|_\theta \leq \frac{q+1}{4p(\tilde{c}c_t)^{q+1}}\left[\frac{r+1}{2pC(\tilde{c}c_{r+1})^{r+1}}\right]^{\frac{p-q-1}{r+1-p}}$$

whenever $q + 1 - p \leq 0$. Then (P) has a solution which is positive a.e.

Theorem 4.3: Assume (1)–(3) hold. Then (P') has a solution which is non negative a.e.

Theorem 4.4: Assume (3), (4) hold, in add to

$$|h|_\theta \leq \frac{q+1}{4pc_t^{q+1}}\left[\frac{r+1}{2pCc_{r+1}^{r+1}}\right]^{\frac{p-q-1}{r+1-p}}$$

whenever $q + 1 - p \leq 0$. Then (P') has a solution which is non negative a.e.

Corollary 4.1: Assume (3), (4) hold, with $C = 0$ in add to

$$|h|_\theta \leq \frac{q+1}{2pc_t^{q+1}} s^{p-q-1},$$

for some $s > 0$, whenever $q + 1 - p \leq 0$. Then (P') has a solution which is non negative a.e.

Proof: Let $u \geq 0$ be the nonnegative solution to (P'). Then

$$\begin{aligned} 0 < \frac{c}{2} &\leq I_1(u_n) - \frac{1}{p}\langle \omega_n, u_n \rangle \\ &= 0 - \Psi(u_n) + \frac{1}{p}\langle w_n, u_n \rangle \\ &\leq \frac{1}{p}\langle w_n, u_n \rangle \leq \frac{1}{p}\int_{\mathrm{IR}^N} h(x) u_n^{q+1} dx \end{aligned} \tag{27}$$

so passing to the limit we infer that

$$0 < \frac{c}{2} \leq \int_{\mathrm{IR}^N} h(x) u^{q+1} dx \leq \frac{|h|_\theta}{p} |u|_t^{q+1},$$

and hence $u \neq 0$.

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