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A COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS ON NORMED VECTOR SPACES AND ITS APPLICATIONS

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ABSTRACT: In this paper, a common fixed point theorem for two pairs of weakly compatible mappings on a normed vector space is proved. We have relaxed the condition of compatibility in main theorem of Pathak and Fisher [10], by weak compatibility and removed the requirement of gauge function (ϕ). Two applications are given. First is related to dynamic programming and another is related to product space. By an example we have shown that weak compatibility of a pair of mappings opens the possibility of having fixed point(s), which other types of compatibility (except pointwise R-weakly commuting [6]) do not have.

Keywords and Phrases: Compatible mappings, compatible mappings of type (A), (B) and (P), coincidence point, normed vector space, weakly compatible mappings, biased mappings, weakly biased mappings.

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1. INTRODUCTION

Let A and S be two self mappings of a metric space (X, d). The pair (A, S) is said to be compatible [3] if $d(ASx_n, SAx_n) \to 0$ as $n \to \infty$, whenever there exist a sequence $\{x_n\}$ in X such that $Ax_n \to t$ and $Sx_n \to t$ as $n \to \infty$, for some $t \in X$.

It may be remarked that compatible maps were introduced as a generalization of commuting mappings and weakly commuting mappings [13].

In 1993, Jungck, Murthy and Cho [4] defined a dierent type of compatibility called compatible mappings of type (A). Pair (A, S) of self mappings of a metric space (X, d) is said to be compatible of type (A) if $d(ASx_n, SSx_n) \to 0$ and $d(SAx_n, AAx_n) \to 0$ as $n \to \infty$, whenever there exist a sequence $\{x_n\}$ in X such that $Ax_n \to t$ and $Sx_n \to t$ as $n \to \infty$, for some $t \in X$. By Examples 2.1 and 2.2 in [4], it follows that the notion of compatibility and compatibility of type (A) are independent.

In 1995, Pathak and Khan [8] introduced the notion of compatible mappings of type (B), as a generalization of compatible mappings of type (A). The pair (A, S) is said to be compatible of type (B) if

$$\lim_n d\left(SAx_n, AAx_n\right) \le \frac{1}{2}\left[\lim_n d\left(SAx_n, St\right) + \lim_n d\left(St, SSx_n\right)\right],$$

$$\lim_n d\left(ASx_n, SSx_n\right) \le \frac{1}{2}\left[\lim_n d\left(ASx_n, At\right) + \lim_n d\left(At, AAx_n\right)\right],$$

whenever there exist a sequence $\{x_n\}$ in X such that $Ax_n \to t$ and $Sx_n \to t$ as $n \to \infty$, for some $t \in X$.

From the above definition it is clear that compatibility of type (A) implies compatibility of type (B). But by Example 2.4 of Pathak and Khan [8], we observe that the converse need not be true.

In 1995, Pathak *et al.* [9] introduced the notion of compatible mappings of type (P). The pair (A, S) is said to be compatible of type (P) if $d(AAx_n, SSx_n) \to 0$ as $n \to \infty$, whenever there exists a sequence $\{x_n\}$ in X such that $Ax_n \to t$ and $Sx_n \to t$ as $n \to \infty$, for some $t \in X$.

In 1996, the notion of weak compatibility was introduced by Jungck [5]. The pair (A, S) is said to be *weakly compatible* if they commute at their coincidence point. An equivalent notion called pointwise R-weakly commuting was given by Pant [6]. Pair (A, S) is said to be *pointwise R-weakly commuting* at $x \in X$ if there exist some positive real number R such that $d(ASx, SAx) \le Rd(Ax, Sx)$. Both notions ([5] and [6]) imply each other (see [12], p. 34).

2. DISCUSSION

The following lemma shows that weak compatibility is more general than compatibility or other types of compatibility (except pointwise R-weakly commuting) as defined above.

Lemma 2.1: ([3] , resp. [4], [8], [9]). Let A and S be compatible (resp. compatible of type (A), type (B), type (P)) pair of self mappings of a metric space (X, d) if $Ap = Sp$ for some $p \in X$, then $ASp = SAp$.

The following example shows that weak compatibility (pointwise R-weakly commuting) need not imply compatibility or other types of compatibility defined above.

Example 2.2: Let $X = \mathbb{R}$ with the usual metric d. Define $A, S : \mathbb{R} \to \mathbb{R}$ by $Ax = [x]$, the integral part of x, $\forall\, x \in \mathbb{R}$, and

$$Sx = \begin{cases} -1, & x \le 0 \\ 0, & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$

Then at $x = 1$ we have $AS1 = A1 = 1 = S1 = SA1$. Thus the pair (A, S) is weakly compatible. However, this pair of mappings is neither compatible nor compatible of type (A), (B) or (P). For, if we define a sequence $\{x_n\} = \left\{1 - \frac{1}{n}\right\}$ then we have $Ax_n = 0 = Sx_n$, as $n \to 1$, so that $t = 0 = At$ and $St = -1$. Further, $ASx_n = 0$, $SAx_n = -1$, $SSx_n = -1$, $AAx_n = 0$, $d(ASx_n, SAx_n) = 1$, $d(SSx_n, AAx_n) = 1$, $d(ASx_n, SSx_n) = 1 = d(SAx_n, AAx_n) \neq 0$ and

$$d\left(SAx_n, AAx_n\right) = 1 > \frac{1}{2}\left[d\left(SAx_n, St\right) + d\left(St, SSx_n\right)\right] = 0,$$

$$d\left(ASx_n, SSx_n\right) = 1 > \frac{1}{2}\left[d\left(ASx_n, St\right) + d\left(At, AAx_n\right)\right] = 0,$$

as $n \to \infty$. Hence we conclude that weak compatibility need not imply other types of compatibility (except pointwise R-weakly commuting). However by observation $x = 1$ is a common fixed point of the pair (A, S). Therefore weak compatibility of pair of mappings opens the possibility of having fixed point(s), which other types of compatibility need not have. Thus weak compatibility is a screen (net) for a pair of mappings having fixed point.

The interested reader may find another example of mappings which are weakly compatible (pointwise R-weakly commuting) but not compatible of type (A)[11], type (B)[12] or type (P)[12] is given in Pant [7].

In the light of above discussion we intend to remove the condition of compatibility of pairs (A, S) and (B, T) and put weak compatibility in the main theorem of Pathak and Fisher [10] (Theorem A as stated below). We also remove the gauge function (ϕ) without taking extra condition in the main theorem of Pathak and Fisher [10]. In fact, they proved the following theorem.

Theorem A [10]: Let (A, S) and (B, T) be two pairs of compatible mappings of a normed vector space X into itself. Let C be a closed, convex subset of X such that

$$(1 - k)A(C) + kS(C) \subseteq A(C),$$

$$(1 - k')B(C) + k'T(C) \subseteq B(C),$$

where $0 < k, k' < 1$ and suppose that

$$\|Sx - Ty\| \leq \phi\left(\frac{a\|Ax - By\|^{2p} + (1-a)\max\left\{\|Sx - Ax\|^{2p}, \|Ty - By\|^{2p}\right\}}{\max\left\{\|Sx - By\|^{p}, \|Ty - Ax\|^{p}\right\}}\right)$$

for all $x, y \in C$ for which $\max\{\|Sx - By\|, \|Ty - Ax\|\} \neq 0$, where $0 < a < 1$, $p > 0$ and ϕ is a function which is upper semi-continuous from the right of $\mathbb{R}^+$ into itself such that $\phi(t) < t$ for each $t > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively for $n = 0, 1, 2...$ by

$$Ax_{2n+1} = (1 - k)Ax_{2n} + kSx_{2n},$$
$$Bx_{2n+2} = (1 - k')Bx_{2n+1} + k'Tx_{2n+1},$$

converges to a point $z \in C$, and if A and B are continuous at z, then A, B, S and T have a unique common fixed point Tz in C. Further, if A and B are continuous at Tz, then S and T are continuous at Tz.

3. MAIN RESULTS

Before presenting our main result we prove the following Lemma.

Lemma 3.1: Let $A, B, S, T : X \to X$ be four mappings of a normed vector space X and let C be a nonempty closed, convex subset of X satisfying:

$$\|Sx - Ty\|^p \le \left(\frac{a\|Ax - By\|^{2p} + (1-a)\max\left\{\|Sx - Ax\|^{2p}, \|Ty - By\|^{2p}\right\}}{\max\left\{\|Sx - By\|^p, \|Ty - Ax\|^p\right\}} \right)$$

for all $x, y \in C$ for which $\max\{\|Sx - By\|, \|Ty - Ax\|\} \neq 0$ where $0 < a < 1$ and $p > 0$. If w is a common fixed point of A, B, S and T then it is unique. Moreover, if $A(B)$ is continuous at w then S(T) is continuous at w.

Proof: Suppose $\{y_n\}$ is an arbitrary sequence in C with the limit w where w is a common fixed point of A, B, S and T. Let the sequence $\{Sy_n\} \to u$, for some $u \neq Sw = w$ as $n \to \infty$. Then using above inequality, by putting $x = y_n$ and $y = w$, we have

$$\|Sy_n - w\|^p = \|Sy_n - Tw\|^p$$

$$\le \frac{a\|Ay_n - Bw\|^{2p} + (1-a)\max\left\{\|Sy_n - Ay_n\|^{2p}, \|Tw - Bw\|^{2p}\right\}}{\max\left\{\|Sy_n - Bw\|^p, \|Tw - Ay_n\|^p\right\}}$$

Letting $n \to \infty$, this yields

$$\lim_n \|Sy_n - w\|^p = \|u - w\|^p \le (1 - a)\|u - w\|^p,$$

a contradiction. This shows that S is continuous at w and w is unique.

Similarly, the continuity of B at w implies the continuity of T at w. Hence if $A(B)$ is continuous at w then $S(T)$ is continuous at w. This completes the proof.

We now state and prove our main result.

Theorem 3.2: Let (A, S) and (B, T) be two pairs of weakly compatible mappings of a normed vector space X into itself. Let C be a closed, convex subset of X such that

$$(1-k)A(C) + kS(C) \subseteq A(C), \tag{1}$$

$$(1-k')B(C) + k'T(C) \subseteq B(C), \tag{2}$$

where $0 < k, k' < 1$ and suppose that

$$\|Sx - Ty\|^p \leq \frac{a\|Ax - By\|^{2p} + (1-a)\max\left\{\|Sx - Ax\|^{2p}, \|Ty - By\|^{2p}\right\}}{\max\left\{\|Sx - By\|^p, \|Ty - Ax\|^p\right\}} \tag{3}$$

for all $x, y \in C$ for which $\max\{||Sx - By||, ||Ty - Ax||\} \neq 0$, where $0 < a < 1$ and $p > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively for $n = 0, 1, 2...$ by

$$Ax_{2n+1} = (1-k)Ax_{2n} + kSx_{2n}, \tag{4}$$

$$Bx_{2n+2} = (1-k')Bx_{2n+1} + k'Tx_{2n+1}, \tag{5}$$

converges to a point $z \in C$, and if A and B are continuous at z, then A, B, S and T have a unique common fixed point Tz in C. Further, if $A(B)$ is continuous at Tz then $S(T)$ is continuous at Tz.

Proof: First we prove that z is a coincidence point of A, B, S and T. From (4) and continuity of A at z, we have

$$\lim_n Ax_n = \lim_n Sx_{2n} = Az.$$

Similarly, from (5) and continuity of B at z we have

$$\lim_n Bx_n = \lim_n Tx_{2n+1} = Bz.$$

Now suppose that $Az \neq Bz$, so that for large enough n, $Sx_{2n} \neq Bx_{2n+1}$. Then using (3) we have

$$||Sy_{2n} - Tx_{2n+1}||^p$$

$$\leq \frac{a\|Ax_{2n} - By_{2n+1}\|^{2p} + (1-a)\max\left\{\|Sx_{2n} - Ax_{2n}\|^{2p}, \|Tx_{2n+1} - Bx_{2n+1}\|^{2p}\right\}}{\max\left\{\|Sx_{2n} - Bx_{2n+1}\|^p, \|Tx_{2n+1} - Ax_{2n}\|^p\right\}}.$$

Letting n tend to infinity it follows that

$$||Az - Bz||^p \leq a||Az - Bz||^p.$$

Thus $Az = Bz$. Now by putting $x = x_{2n}$ and $y = z$ in (3) and letting $n \to \infty$ we obtain $Tz = Az$. Similarly, we can obtain $Tz = Bz$. Thus z is a coincidence point of A, B, S and T, i.e.

$$Az = Bz = Sz = Tz \tag{6}$$

On the other hand, it follows from weak compatibility of the pairs (A, S) and (B, T) and the above equation that Tz is another coincidence point of A, B, S and T. Hence we have

$$S^2z = S(Tz) = SAz = ASz = A(Tz), \tag{7}$$

$$T^2z = TSz = TBz = BTz. \tag{8}$$

We claim that $S^2z = Tz$. If not, then from (3) we have

$$\left\|S^2x - Tz\right\|^p \leq \frac{a\left\|ASz - Bz\right\|^{2p} + (1-a)\max\left\{\left\|S^2z - ASz\right\|^{2p}, \left\|Tz - Bz\right\|^{2p}\right\}}{\max\left\{\left\|S^2z - Bz\right\|^p, \left\|Tz - ASz\right\|^p\right\}}$$

or, by (6) and (7) we have

$$||S^2z - Tz||^p \leq a||S^2z - Tz||^p,$$

a contradiction, since $0 < a < 1$. Thus $S^2z = Tz$. Similarly $T^2z = Sz$. Therefore Tz is a common fixed point of A, B, S and T. Uniqueness of the common fixed point and continuity of $S(T)$ at $Tz = w$ follows from Lemma 3.1. This completes the proof.

The following example shows the validity of Theorem 3.2.

Example 3.3: Let $X = \mathbb{R}$ with the Euclidean norm $||\cdot||$ and $C = [0, 1]$ be a closed, convex subset of X. Define the mappings A, B, S and T of X into itself by

$$Ax = \frac{1}{4}(3+x), \forall x \in X,$$

$$Sx = 1 + \left[\frac{x}{2}\right], \forall x \in X, \text{ where } [x] \text{ denotes the integral part of } x,$$

$$Bx = \frac{1}{2}(1+x), \forall x \in X,$$

$Tx = 1, x \in [-1, 2]$ and $Tx = 0$, otherwise.

Then, for some fixed $0 < k, k' < 1$, we have

(i) For some fixed $0 < k, k' < 1$,

$$(1-k)A(C) + kS(C) = (1-k)\left[\frac{3}{4}, 1\right] + k\{1\} = \left[\frac{1}{4}(3+k), 1\right] \subseteq A(C),$$

$$(1-k')B(C) + k'T(C) = (1-k')\left[\frac{1}{2}, 1\right] + k'\{1\} = \left[\frac{1}{2}(1+k'), 1\right] \subseteq B(C).$$

(ii) The point $x = 1$ is a weak compatible point in C. The point $x = -3 (\notin C)$ is a coincidence point of the pair (A, S) where they do not commute.

(iii) Since $\|Sx - By\| = \frac{1}{2}\|1-y\|$ and $\|Ty - Ax\| = \frac{1}{4}\|1-x\|$, for all $x, y \in C$, so that $\max\{\|Sx - By\|, \|Ty - Ax\|\} \neq 0$, for all $x, y \in C$ (except at $x = y = 1$, where $\|Sx - Ty\|^p = 0$). We also have from condition (3) that for all $x, y \in C$, where $\|Sx - Ty\|^p = 0$. Hence condition (3) satisfies.

(iv) Now for an arbitrary $x_0 \in C$, we will show that the sequence $\{x_n\}$, converges to the point $z = 1$ in C.

Since $Ax_1 = (1-k)Ax_0 + kSx_0 = \frac{1}{4}(1-k)(3+x_0) + k.1 \in \left[\frac{1}{4}(3+x_0), 1\right] \subseteq C$. we have $Ax_1 = \frac{1}{4}(3+x_1) = \frac{1}{4}(1-k)(3+x_0) + k$. This gives $x_1 = (1-k)x_0 + k.1 \in [x_0, 1] = I_0$ (say).

Similarly for this value of x_1 we have

$$Bx_2 = (1-k')Bx_1 + k'Tx_1 = \frac{1}{2}(1-k')(1+x_1) + k'.1 \in \left[\frac{1}{2}(1+x_1), 1\right],$$

therefore $Bx_2 = \frac{1}{2}(1+x_2) = \frac{1}{2}(1-k')(1+x_1) + k'$. This gives $x_2 = (1-k')x_1 + k'.1 \in [x_1, 1] = I_1$.

Similarly, for this value of x_2 we calculate

$$x_3 = (1-k)x_2 + k.1 \in [x_2, 1] = I_2$$
$$x_4 = (1-k')x_3 + k'.1 \in [x_3, 1] = I_3$$

Proceeding in this manner we have, for all $n = 0, 1, 2...$

$$x_{2n+1} = (1-k)x_{2\mathrm{n}} + k.1 \in [x_{2n}, 1] = I_{2n}. \tag{9}$$
$$x_{2n+2} = (1-k')x_{2n+1} + k'.1 \in [x_{2n+1}, 1] = I_{2n+1}. \tag{10}$$

That is,

$$x_{2n+1} - x_{2n} = k(1 - x_{2n}), \tag{11}$$
$$x_{2n+2} - x_{2n+1} = k'(1 - x_{2n+1}), \tag{12}$$

where $x_0 \le x_1 \le x_2 \le ... \le 1$.

Let us denote the length of the bounded interval I_{2n+2} by $|I_{2n+2}|$, then from (10) we have,

$$|I_{2n+2}| = ||1 - x_{2n+2}|| = (1-k')\ ||1 - x_{2n+1}|| \le \lambda ||1 - x_{2n+1}|| = |I_{2n+1}|,$$

where $= \max\{1-k, 1-k'\} < 1$. Similarly for the length of the interval I_{2n+1}, we have from (9),

$$|I_{2n+1}| = ||1 - x_{2n+1}|| = (1-k)\ ||1 - x_{2n}|| \le \lambda ||1 - x_{2n}|| = |I_{2n}|.$$

Hence we obtain a sequence of numbers $\{|I_n|\}_{n=1}^{n=\infty}$ such that

$$|I_{2n+2}| \le \lambda |I_{2n+1}| \le \lambda^2 |I_{2\mathrm{I}}| \le ... \le \lambda^{2n+2} |I_0|$$

where $0 < \lambda < 1$. It follows that $|I_n| \to 0$ as $n \to \infty$.

Thus a sequence of nested closed intervals of diminishing diameters $\{|I_n|\}_{n=1}^{n=\infty}$ is obtained such that $I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$, so by Cantor's intersection theorem $\bigcap_{n=1}^{n=\infty} I_n \ne \phi$ and $1 \in \bigcap_{n=1}^{n=\infty} I_n$ for all $n \in \mathbb{N}$. Hence by the Bolzano Weierstrass theorem, $z = 1$ is a cluster point of infinite closed bounded subset C of $\mathbb{R}$. The uniqueness of cluster point follows from (11) and (12). Hence $\{x_n\}$ converges to $z = 1$.

(v) A is continuous in $[0, 1]$, where S is also continuous. Further S is discontinuous at $x = \pm 2, \pm 4$ ... where A is continuous, but these points are not in C. Similarly, B is continuous in $[0, 1]$, where T is also continuous. Further, $x = -1$ and $x = 2$ are two points of discontinuity of T, where B is continuous but these points are not in C.

Hence $Tz = 1$ is the unique common fixed point of A, B, S and T and $A(B)$ and $S(T)$ are continuous at $Tz = 1$. This validates Theorem 3.2.

In case when $S = T$ and $A = B$, we have the following corollary.

Corollary 3.4: Let (A, S) be a pair of weakly compatible mappings of a normed vector space X. Let C be a closed convex subset of X such that

$$(1-k)A(C) + kS(C) \subseteq A(C), \tag{13}$$

where $0 < k < 1$, and suppose that

$$\|Sx - Ty\|^p \leq \frac{a\|Ax - Ay\|^{2p} + (1-a)\max\left\{\|Sx - Ax\|^{2p}, \|Sy - Ay\|^{2p}\right\}}{\max\left\{\|Sx - Ay\|^p, \|Sy - Ax\|^p\right\}} \tag{14}$$

for all $x, y \in C$ for which $\max\{||Sx - Ay||, ||Sy - Ax||\} \neq 0$, where $0 < a < 1$ and $p > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively for $n = 0, 1, 2 \ldots$ by

$$Ax_{n+1} = (1-k)Ax_n + kSx_n, \tag{15}$$

converges to a point $z \in C$, and if A is continuous at z, then A and S have a unique common fixed point Sz in C. Further, if A is continuous at Sz then S is continuous at Sz.

In case when $A = B = I_X$, the identity mapping on X, we have the following corollary.

Corollary 3.5: Let S and T be two self mappings of a normed vector space X. Let C be a closed convex subset of X such that

$$(1-k)C + kS(C) \subseteq C, \tag{16}$$

$$(1-k')C + k'T(C) \subseteq C, \tag{17}$$

where $0 < k, k' < 1$, and suppose that

$$\|Sx - Ty\|^p \leq \frac{a\|x - y\|^{2p} + (1-a)\max\left\{\|Sx - x\|^{2p}, \|Ty - y\|^{2p}\right\}}{\max\left\{\|Sx - y\|^p, \|Ty - Ax\|^p\right\}} \tag{18}$$

for all $x, y \in C$ for which $\max\{||Sx - y||, ||Ty - x||\} \neq 0$, where $0 < a < 1$ and $p > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively for $n = 0, 1, 2\ldots$ by

$$x_{2n+1} = (1-k)x_{2n} + kSx_{2n}, \tag{19}$$

$$x_{2n+2} = (1-k')x_{2n+1} + k'Tx_{2n+1}, \tag{20}$$

converges to a point $z \in C$, then S and T have a unique common fixed point Tz in C. Further, S and T are continuous at Tz.

In case when $A = B = I_X$ and $S = T$, we have the following corollary.

Corollary 3.6: Let T be a self mapping of a normed vector space X. Let C be a closed convex subset of X such that

$$(1-k)C + kT(C) \subseteq C, \tag{21}$$

where $0 < k < 1$ and suppose that

$$\|Tx - Ty\|^p \le \frac{a\|x-y\|^{2p} + (1-a)\max\left\{\|Tx-x\|^{2p}, \|Ty-y\|^{2p}\right\}}{\max\left\{\|Tx-y\|^p, \|Ty-x\|^p\right\}} \tag{22}$$

for all $x, y \in C$ for which $\max\{\|Tx-y\|, \|Ty-x\|\} \neq 0$, where $0 < a < 1$ and $p > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively for $n = 0, 1, 2...$ by

$$x_{n+1} = (1-k)x_n + kTx_n, \tag{23}$$

converges to a point $z \in C$, then T have a unique common fixed point Tz in C. Further, T is continuous at Tz.

In case when $A = B$, we have the following corollary.

Corollary 3.7: Let A, S and T be three self mappings of a normed vector space X and let C be a closed convex subset of X such that

$$(1-k)A(C) + kS(C) \subseteq A(C), \tag{24}$$

$$(1-k')A(C) + k'T(C) \subseteq A(C), \tag{25}$$

where $0 < k, k' < 1$, and suppose that

$$\|Sx - Ty\|^p \le \frac{a\|Ax-Ay\|^{2p} + (1-a)\max\left\{\|Sx-Ax\|^{2p}, \|Ty-Ay\|^{2p}\right\}}{\max\left\{\|Sx-y\|^p, \|Ty-Ax\|^p\right\}} \tag{26}$$

for all $x, y \in C$ for which $\max\{\|Sx-Ay\|, \|Ty-Ax\|\} \neq 0$, where $0 < a < 1$ and $p > 0$. If for some $x_0 \in C$, the sequence $\{x_n\}$ in X defined inductively for $n = 0, 1, 2...$ by

$$Ax_{2n+1} = (1-k)Ax_{2n} + kSx_{2n}, \tag{27}$$

$$Ax_{2n+2} = (1-k')Ax_{2n+1} + k'Tx_{2n+1}, \tag{28}$$

converges to a point $z \in C$, and if A is continuous at z and A, S and T commutes mutually at the coincidence point z, then A, S and T have a unique common fixed point Tz in C. Further, if A is continuous at Tz then S and T are continuous at Tz.

Remark 3.8: The conclusion of Theorem 3.2 remains valid if we take pairs (A, S) and (B, T) to be pointwise R-weakly commuting instead of weakly compatible.

Remark 3.9: The condition of weak compatibility of the pair of mappings (A, S) in Corollary 3.3 is necessary as shown in the following example.

Example 3.10: Let $X = [0, 1)$ with the Euclidean norm $||\cdot||$ and $C = [0, 1]$. Define $A, S : X \to X$ by

$$Ax = \begin{cases} 1 + x, x \in [0, 1] \\ 2x, x \in (1, \infty), \text{ and} \end{cases}$$

$$Sx = 1, \forall x \in X.$$

Here we observe that $||Sx - Sy||^p = 0$, for all $x, y \in C$, where $p > 0$ and $0 < a < 1$. Also $\max\{||Sx - Ay||^p, ||Sy - Ax||^p\} = \max\{||y||^p, ||x||^p\} \neq 0$. For some $0 < k < 1$ we have $(1-k)A(C) + kS(C) = [1, 2-k] \subseteq [1, 2] = A(C)$. Further, we see that the pair (A, S) is not weakly compatible as they do not commute at their coincidence point $x = 0$. On the other hand A and S have no common fixed point in C. Thus the condition of weak compatibility of the pair mappings (A, S) is necessary. Similar conclusion arises for the pair (B, T).

4. APPLICATIONS

We now apply our main result in Section 3 to some problems in dynamic programming and product spaces.

(I) An Application to Dynamic Programming

We will use our main theorem to study the solution of functional equations arising in dynamic programming. A dynamic programming problem is a decision-making problem in n variables in which the problem being subdivided into n sub-problems (*stages*), each being a decision-making problem in one variable only. The decision is the "goodness" of a selected alternatives depend on satisfying optimal policy of the problem. The *state* of the system at any stage is regarded as the information that links the stages together, such that the optimal decisions for the remaining stages can be made. The state allows us to consider each stage separately and guarantees that the solution is feasible for all the stages.

Throughout this section we assume that X and Y are Banach spaces, $S \subseteq X$ is a state space, $D \subseteq Y$ a decision space and $\mathbb{R} = (-\infty, +\infty)$. We denote by $B(S)$ the set of all bounded real-valued functions defined on S.

The basic form of the functional equations of dynamic programming as suggested by Bellman and Lee [1] is

$$\mathrm{f(x) = Opt_y H(x, y, f(T(x, y)))}, \tag{29}$$

where x is a state vector, y is a decision vector, T represents the transformation of the process, $f(x)$ represents the optimal return function and "*Opt*" denotes '*max*' or '*min*' of the problem.

In this section, we shall study the existence and uniqueness of a common solution of the following functional equations:

$$f(x) = \sup_{y\in D} H_1(x, y, f(T(x, y))), \ x \in S \tag{30}$$

$$g(x) = \sup_{y\in D} H_2(x, y, g(T(x, y))), \ x \in S \tag{31}$$

$$p(x) = \sup_{\mathrm{y\in D}} \mathrm{F}_1(x, y, p(T(x, y))), \ x \in S \tag{32}$$

$$q(x) = \sup_{\mathrm{y\in D}} \mathrm{F}_2(x, y, q(T(x, y))), \ x \in S \tag{33}$$

where $T : S \times D \to S$ and $H_i, F_i : S \times D \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ which in turn respectively given by equations (39) and (40) in the following theorem.

Theorem 4.1: Suppose that the following conditions are satisfied:

(i) H_i and F_i are bounded, $i = 1, 2$.

(ii) For any positive real number p such that $0 < p \le 1$,

$$|H_1(x, y, h(t)) - H_2(x, y, k(t))|^p \le M^{-1}(a|T_1h(t) - T_2k(t)|^{2p} + (1-a)\max\{|A_1h(t) - T_1h(t)|^{2p}, |A_2k(t) - T_2k(t)|^{2p}\}) \tag{34}$$

where

$$M = \max\{\sup_{y\in D}|A_1h(t) - T_2k(t)|^p, \sup_{y\in D}|A_2k(t) - T_1h(t)|^p\} \neq 0, \tag{35}$$

for all $(x, y) \in S \times D$, where $t \in S$, $h(t), k(t) \in B(S)$, $0 < a < 1$, A_i and $T_i : S \times D \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$.

(iii) For a given sequence $\{k_n\} \subseteq B(S)$ and $k \in S$ with

$$\lim_{n\to\infty}\sup_{x\in S}|k_n(x) - k(x)| = 0 \Rightarrow \lim_{n\to\infty}\sup_{x\in S}|T_ik_n(x) - T_ik(x)| = 0, \tag{36}$$

(iv) For any $h \in B(S)$ there exist $k_1, k_2 \in B(S)$ such that for all $x \in S$,

$$(1 - \lambda_i)T_i h(x) + \lambda_i A_i h(x) = T_i k_i(x) \tag{37}$$

where $0 < \lambda i < 1$, $i = 1, 2$.

(v) For any $h \in B(S)$ with $A_i h = T_i h$, $i = 1, 2$ we have

$$T_i A_i h = A_i T_i h. \tag{38}$$

Then the system of functional equations

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), x \in S \tag{39}$$

$$T_i h(x) = \sup_{y \in D} F_i(x, y, k(T(x, y))), x \in S \tag{40}$$

has a unique solution in B(I)

Proof: Let us define a norm $||\cdot||$ in $B(S)$ for any $h, k \in B(S)$ by

$$||h - k|| = \sup\{|h(x) - k(x)| : x \in S\} \tag{41}$$

then $(B(S), ||\cdot||)$ is a normed linear space.

From condition (iii) T_i is continuous, by (iv) equations (1) and (2) follows and by (v) weak compatibility of T_i and A_i, $(i = 1, 2)$ satisfies. Hence A_i and T_i are self mappings of $B(S)$ with the norm $||\cdot||$.

Let h_1, h_2 be any two points of $B(S)$, let $x \in S$ and η be any positive number, there exist y_1 and y_2 in D such that

$$A_i h_i(x) < H_i(x, y_i, h_i(x_i)) + \eta, (i = 1, 2). \tag{42}$$

where $x_i = T(x, y_i)$.

Also we have

$$A_1 h_1(x) \geq H_1(x, y_2, h_1(x_2)) \tag{43}$$

$$A_2 h_2(x) \geq H_2(x, y_1, h_2(x_1)) \tag{44}$$

From (42) for $i = 1$, (44) and (34) we have

$$A_1 h_1(x) - A_2 h_2(x) < H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \eta$$

$$= \left[\left\{\left|H_1\left(x, y_1, h_1\left(x_1\right)\right) - H_2\left(x, y_1, h_2\left(x_1\right)\right)\right| + \eta\right\}^p\right]^{\frac{1}{p}}$$

$$\leq \left[\left\{\left|H_1\left(x, y_1, h_1\left(x_1\right)\right) - H_2\left(x, y_1, h_2\left(x_1\right)\right)\right|^p + \eta^p\right\}\right]^{\frac{1}{p}}$$

$$\leq \left\{M^{-1}a\left|T_1h_1\left(x1\right)-T_2h_2\left(x_1\right)\right|^{2p}\right.$$

$$\left.+\left(1-a\right)\max\left\{\left|A_1h_1\left(x_1\right)-T_1h_1\left(x_1\right)\right|^{2p},\left|A_2h_2\left(x_1\right)-T_2h_2\left(x_1\right)\right|^{2p}\right\}+\eta^p\right\}^{\frac{1}{p}}$$

$$\leq \left\{M^{-1}\left(a\left\|T_1h_1-T_2h_2\right\|^{2p}\right.\right.$$

$$\left.+\left(1-a\right)\max\left\{\left\|A_1h_1-T_1h_1\right\|^{2p},\left\|A_2h_2-T_2h_2\right\|^{2p}\right\}+\eta^p\right\}^{\frac{1}{p}} \tag{45}$$

Similarly from (42) for $i = 2$, (43) and (34) we have

$$A_1h_1(x) - A_2h_2(x) \geq |H_1(x, y_2, h_1(x_2)) - H_2(x, y_2, h_2(x_2))| - \eta$$

$$\geq -\left\{M^{-1}\left(a\left\|T_1h_1-T_2h_2\right\|^{2p}\right.\right.$$

$$\left.\geq +\left(1-a\right)\max\left\{\left\|A_1h_1-T_1h_1\right\|^{2p},\left\|A_2h_2-T_2h_2\right\|^{2p}\right\}+\eta^p\right\}^{\frac{1}{p}} \tag{46}$$

Hence from (45) and (46) we have

$$|A_1h_1(x) - A_2h_2(x)| \leq \{M^{-1}(a\|T_1h_1 - T_2\,h_2\|^{2p}$$

$$\left.+\left(1-a\right)\max\left\{\left\|A_1h_1-T_1h_1\right\|^{2p},\left\|A_2h_2-T_2h_2\right\|^{2p}\right\}+\eta^p\right\}^{\frac{1}{p}} \tag{47}$$

Since (47) is true for any $x \in S$ and is any positive number, by taking supremum over all $x \in S$, $\eta \to 0$ and then taking p power's on both sides we have

$$\|A_1h_1 - A_2h_2\|^p$$

$$\leq \frac{a\left\|T_2h_1-T_2h_2\right\|^{2p}+\left(1-a\right)\max\left\{\left\|A_1h_1-T_2h_2\right\|^{2p},\left\|A_2h_2-T_2h_2\right\|^{2p}\right\}}{\max\left\{\left\|A_2h_1-T_2h_2\right\|^{p},\left\|A_2h_2-T_1h_1\right\|^{p}\right\}}. \tag{48}$$

Therefore by Theorem 1, the mappings A_l, A_2, T_1 and T_2 have a unique common fixed point $h^* \in B(S)$, i.e. $h^*(x)$ is a unique solution of functional equations (39)-(40). This completes the proof.

(II) An Application to Product Space

Before giving an application in product space, we need to define the projection mapping:

The mappings $\pi_x : X \times Y \to X$ such that $\pi_x((x, y)) = x$, $\forall(x, y) \in X \times Y$ and $\pi_y : X \times Y \to Y$ such that $\pi_y((x, y)) = y$, $\forall(x, y) \in X \times Y$ are called *projections* of $X \times Y$ on X and Y, respectively.

Now we establish the following result.

Theorem 4.2: Let C be a closed, convex subset of a normed vector space X. Let A, S and T be three mappings of $X \times X$ into X such that

$$(1-k)A(C \times C) + kS(C \times C) \subseteq A(C \times C), \tag{49}$$

$$(1-k')A(C \times C) + k'T(C \times C) \subseteq A(C \times C), \tag{50}$$

$$A\xi = A((x, y)) = x \tag{51}$$

for all $x, y \in C$, where $0 < k, k' < 1$ and suppose that for $\xi = (x, y)$, $= (u, v)$ the condition:

$$\|S\xi - T\|^p \leq b\|y - v\|^p + \frac{a\|A\xi - A\eta\|^{2p} + (1-a)\max\left\{\|S\xi - A\xi\|^{2p}, \|T\eta - A\eta\|^{2p}\right\}}{\max\left\{\|S\xi - A\eta\|^{p}, \|T\eta - A\xi\|^{p}\right\}}. \tag{52}$$

for all $x, y, u, v \in C$, for which $\max\{\|S\xi - A\eta\|, \|T - A\xi\|\} \neq 0$, where $0 < a < 1$, $p > 0$ and $0 < b < 1 - a$ satisfies.

If for each fixed $y \in C$ and some $x_0(y) \in C$, the sequence $\{x_n(y)\}$ in C defined by:

$$x_{2n+1}(y) = (1-k)A(x_{2n}(y), y) + kS(x_{2n}(y), y) \tag{53}$$

$$x_{2n+2}(y) = (1-k')A(x_{2n+1}(y), y) + k'T(x_{2n+1}(y), y) \tag{54}$$

converges to a point $z \in C$, then there exists a unique point $w \in C$ such that

$$A(w, w) = w = S(w, w) = T(w, w). \tag{55}$$

Proof: Put $\xi = (x, y)$ and $\eta = (u, y)$ then from (29) and (30) we have

$$\|S(x, y) - T(u, y)\|^p \leq \frac{a\|x-u\|^{2p} + (1-a)\max\left\{\|S(x, y) - x\|^{2p}, \|T(u, y) - u\|^{2p}\right\}}{\max\left\{\|S(x, y) - u\|^{p}, \|T(u, y) - x\|^{p}\right\}}. \tag{56}$$

Therefore as in Corollary 4, for each $y \in C$, there exists a unique $z(y) \in C$ such that

$$S(z(y), y) = z(y) = T(z(y), y) = A(z(y), y) \tag{57}$$

Now for any $y, y' \in C$ we obtain from (29) and (30),

$$\|S(z(y), y) - T(z(y'), y')\|^p \leq b\|y - y'\|^p +$$

$$\leq \frac{a\|z(y)-z(y')\|^{2p} + (1-a)\max\left\{\|S(z(y), y)-z(y)\|^{2p}, \|T(z(y'), y')-z(y')\|^{2p}\right\}}{\max\left\{\|S(z(y), y)-z(y')\|^{p}, \|T(z(y'), y')-z(y')\|^{p}\right\}}. \tag{58}$$

Using (35), this yields

$$\|x(y)-x(y')\| \leq \left\{\frac{b}{1-a}\right\}^{\frac{1}{p}} \|y-y'\|. \tag{59}$$

From the Banach contraction principle, the mapping $z : C \to C$ has a unique fixed point $w \in C$, i.e. $z(w) = w$. Thus we have from (35)

$$w = z(w) = A(w, w) = S(w, w) = T(w, w).$$

Uniqueness of $w \in C$ satisfying above relation for A, S and T is easy to establish. This completes the proof.

5. SOME REMARKS

Recently, Shahzad and Sahar [14] establihed some common fixed point theorems for biased mappings. Cirić and Ume [2] obtained some common fixed points via weakly biased Greguš type mappings. We hope that the results of our paper can be extended to obtain a common fixed point theorem for a quadruple of weakly biased mappings.

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