Journal of Nonlinear Functional Analysis and Differential Equations
Vol. 3 No. 1 (January-June, 2018)

SOME EXTENSIONS OF THE BROUWER-PETRYSHYN FIXED POINT THEOREM

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ABSTRACT: Petryshyn's extension of Brouwer's fixed point theorem states that a continuous mapping $T : B \to R^n$, of the n-dimensional closed unit ball to R^n has a fixed point if every point on the sphere S satisfies the Leray-Schauder condition, namely that $Tx \neq \lambda x$ for some $\lambda > 1$. The infinite dimensional version, extending Schauder's theorem for compact mappings in a Banach space, is also true. In this paper, we show that Petryshyn's theorem remains true under more general boundary conditions similar to that of Leray-Schauder's. Our second result extends the above theorem to any bounded closed subset that contains the origin in its interior. This result covers, in particular, domains that are not simply connected. Our third result, valid only for finite dimensional spaces, complements Petryshyn's Theorem: If the Brouwer degree at the origin of the restriction of $T : B \to R^n$ on the sphere is non-zero, then T has a fixed point if for all $x \in S$, $Tx \neq \lambda x$ for some $0 < \lambda < 1$.

1. INTRODUCTION

The celebrated Brouwer fixed point Theorem (1911) asserts that every continuous mapping of the n-dimensional closed unit ball into itself, $T : B \to B$, must have a fixed point.

See Stuckless [9] for a recent survey of different proofs and applications of this important result.

Rothe (1939) observed that it suffices to require that the image of the unit sphere S (instead of the entire ball) is inside B. More precisely, if $T : B \to R^n$ is continuous and $T(S) \subset B$, then T has a fixed point.

If some or all of $T(S)$ lies outside B, then in general there may not be a fixed point. It is interesting to ask whether, in this situation, additional conditions can be imposed on $T(S)$ to guarantee a fixed point. This problem has been studied by various authors, including Krasnoselskii, Petryshyn, and Amman (see Istratescu [4]). To date the best result is due to Petryshyn (1971) [8]:

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* Lucent Technologies, Inc. USA. The author would like to thank the Department of Mathematics of the University of Hong Kong for its financial support, as recipient of the Professor Y.C. Wong Lectureship in 2004. The results presented in this article come from a study started during that visit.

If every point $x \in S$ satisfies the Leray-Schauder condition

$$Tx \neq \lambda x, \text{ for some } \lambda > 1, \tag{1.1}$$

then T has a fixed point.

For applications to differential equations (see for example Amann [1], Cronin [2], and Guo and Lakshmikantham [3]), infinite dimensional versions of Brouwer's theorem and their extensions are needed. For this reason, Petryshyn's theorem is usually formulated for extensions of Brouwer's theorem to infinite dimensional spaces. In such cases the requirement of continuity on T alone is not sufficient. For instance, in the classical Schauder (1930) extension, T is required to be a continuous compact mapping (also called completely continuous) in a Banach space, namely, T maps bounded sets into pre-compact sets. Other concepts generalizing compact mappings include condensing map (see, for example, Jiménez-Melado and Morales [5]), k-set-contractions (Petryshyn [8]), and P-compactness (Petryshyn [7]).

To add some glamor to our terminology, let us call a point $x \in R^n$ T-LS-radiant (with respect to the origin O) if it violates Leray-Schauder's condition (1.1), namely, if

$$Tx = \lambda x, \text{ for some } \lambda > 1. \tag{1.2}$$

Geometrically, this means that T moves the point x to a point Tx along the path of a light ray radiating out from the origin O. We call a point T-LS-retracting (with respect to O) if it satisfies the dual condition

$$Tx = \lambda x, \text{ for some } 0 < \lambda < 1. \tag{1.3}$$

Petryshyn's theorem can thus be recast as: *If all points on S are non-T-LS-radiant, then T has a fixed point.*

Note that we can define T-LS-radiance with respect to a point π other than the origin:

$$Tx - \pi = \lambda(x - \pi), \text{ for some } \lambda > 1. \tag{1.4}$$

Petryshyn's theorem remains true when the condition of non-T-LS-radiance with respect to the original is replaced by non-T-LS-radiance with respect to any point in the interior of the ball B. In Section 2, we will give examples of more general types of T-radiance. All results in this paper remain true when T-radiance is interpreted in the general sense.

Known proofs of Petryshyn's theorem usually involve degree theory and start from first principle. In Section 2 we show how Petryshyn's theorem follows easily from Brouwer's theorem.

Several examples are given in Section 2, and we discuss a connection between Petryshyn's theorem and the contractive form of the well known Krasnoselskii fixed point theorem for cone mappings in a Banach space (Krasnoselskii [6]).

In Section 3, we show that in the classical cases, Petryshyn's result is still true if the domain of T is any bounded closed set containing the origin in its interior. This, in particular, covers domains that have bubbles in it.

In Section 4, we confine ourselves to finite dimensional spaces and consider the situation "dual" to Rothe's condition. What happens if $T(S)$ lies entirely outside B and Leray-Schauder's condition is not satisfied? In general, T does not have a fixed point. For instance, when T translates the unit ball B by a distance greater than 1. It turns out that we can salvage the fixed point property of T if we add an additional assumption that the Brouwer degree at the origin of $T(S)$ is non-zero. We will show that this simple result also has a Petryshyn-type extension that can be used to handle cases in which part of $T(S)$ is inside B. More precisely, the above assumption on the degree at the origin plus the assumption that all points on S are non-T-retracting imply that T has a fixed point.

2. NON-RADIANT MAPPINGS

Let X denote our underlying space of points. It can be the finite dimensional R^n or a general Banach space.

Let $T : X \to X$ be a continuous nonlinear mapping that belongs to any class of mappings that has the Brouwer fixed point property, in the sense that if T maps a ball $B_a = \{|x| \le a\}$ into itself, then it has a fixed point. We require the class to be topologically invariant. We leave out the exact requirements on the class of such mappings. It should be clear from the proof how that can be formulated in each specific case.

In the classical case (Brouwer's theorem) of finite dimensional spaces, T is only required to be continuous. In the classical case of infinite dimensional spaces (Schauder's theorem), T is required to be completely continuous.

Brouwer's fixed point theorem is a topological property of the domain of the mapping T. Let G be a subset of X that is topologically equivalent to the unit ball B. Then $T : G \to G$ also has a fixed point. In particular, we can apply Brouwer's theorem to a simplex, a cube, or more generally to a bounded convex set. In the rest of this Section, we assume that G is a bounded closed convex set, possibly containing the origin in its interior.

Theorem 1: (Petryshyn) Let X, G, and $T : G \to X$ be as described above. If all points in ∂G are non-T-LS-radiant (with respect to an interior point of G), then T has a fixed point in G.

Proof: Let us first establish the result when G contains the origin in its interior and the center of radiance is chosen to be the origin. Since $T(G)$ is bounded, we can choose the number a large enough so that $G \cup T(G) \subset B_a$. We extend T to be defined on B_a:

$$T(x) = \begin{cases} T(x) \text{ if } x \in G \\ T(\bar{x}) \text{ if } x \in B_a \setminus G. \end{cases} \tag{2.1}$$

Here $\bar{x}$ is the point where the line segment joining x and the origin intersects ∂G. In the classical Brouwer and Schauder cases the extended mapping obviously preserves the Brouwer fixed point property. In the general case, this assertion should be included as one of the hypotheses of the theorem.

We conclude that the extended mapping T has a fixed point $T(x^*) = x^*$. This point cannot be outside G, for otherwise $x^* = \lambda \overline{x^*}$, for some $\lambda > 1$ and

$$T\left(\overline{x^*}\right) = T(x^*) = x^* = \lambda \overline{x^*}, \tag{2.2}$$

contradicting (1.1). Therefore $x^* \in G$ and is a fixed point of the original mapping T.

The above arguments rely on the simple observation that we can continuously collapse the set $B_a \backslash G$, radially towards the origin, onto the boundary ∂G of G. But this is not the only way. We can, for example, collapse the set radially towards any point in the interior of G and the same proof works for the general case.

Let us return to the classical Petryshyn extension for Brouwer and Schauder's theorems. It is easy to construct examples of mappings such that some points on S are T-LS-radiant with respect to the origin but none are T-LS-radiant with respect to a point different from the origin, and vice versa. The use of a general center of radiation is thus warranted.

We reiterate the fact that a crucial argument in the proof of Theorem 1 is the ability to collapse $B_a \backslash G$ continuously onto ∂G. There are other ways of collapsing besides radially towards a given point. For instance, we can contain $G \cup T(G)$ inside a large enough cube and collapse it in directions parallel to the axes. Let us use the two-dimensional case to illustrate how this can be done. Take G to be the square with the vertices $A(-1, 1)$, $B(1, 1)$, $C(1, -1)$, and $D(-1, -1)$. Any point $E = (a, b)$

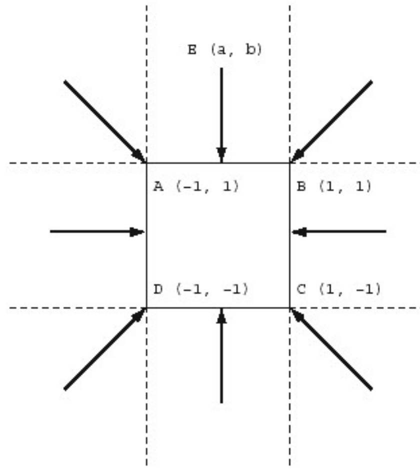

Fig. 1: Collapsing Exterior Points onto a Square

outside the square can be collapsed to a point on the boundary of the square according to the following rules (which are visualized geometrically in Fig. 1):

A point above the edge *AB* is moved vertically downwards until it hits *AB*. A point to the right of *BC* is moved horizontally leftwards until it hits the line and so on. All points in one of the four corner regions are mapped to the respective corner.

More precisely,

If $|a| \leq 1$, and $|b| > 1$, the point is mapped to $(a, \text{sign}(b))$.

If $|a| > 1$, and $|b| \leq 1$, the point is mapped to $(\text{sign}(a), b)$.

If $|a| > 1$, and $|b| > 1$, the point is mapped to $(\text{sign}(a), \text{sign}(b))$.

In fact, the inner region (i.e. the square *ABCD* in the above Fig. 1) need not be a square. Fig. 2 illustrates how points exterior to a circle can be collapsed onto the boundary of the circle in the above "square" manner. Note that different choices of the positions of the "corner" points *A*, *B*, *C*, and *D* correspond to different ways of collapsing onto the circle.

The square-collapsing algorithm can be described in a slightly different way. Let $\lambda > 1$ be any given real number. We define the mapping $P_\lambda : R^2 \to R^2$ as follows. Let $E = (a, b) \in R^2$. If $|a| \geq |b|$, then we define

$$P_\lambda(E) = \begin{cases} (a/\lambda, b) & \text{if } |a/\lambda| > |b| \\ (a/\lambda, \text{sign}(b)|a/\lambda|) & \text{if } |a/\lambda| \leq |b|. \end{cases} \tag{2.3}$$

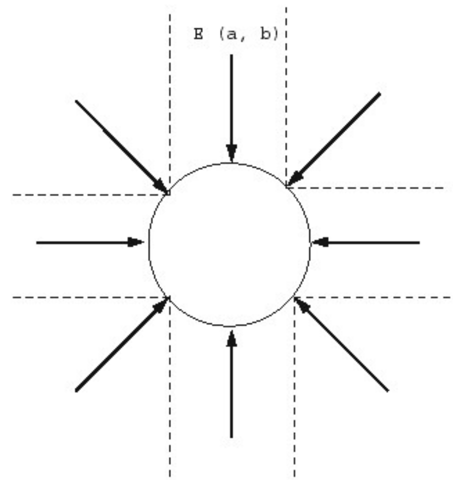

Fig. 2: Collapsing Exterior Points "Squarely" onto a Circle

If $|a| < |b|$, then

$$P_\lambda(E) = \begin{cases} (a, b/\lambda) & \text{if } |b/\lambda| > |a| \\ (\operatorname{sign}(a)|b/\lambda|, b/\lambda) & \text{if } |b/\lambda| \le |a|. \end{cases} \tag{2.4}$$

A point y can be squarely-collapsed to another point x if there exists a $\lambda > 1$ such that $P_\lambda(y) = x$.

It is obvious how the above example can be generalized to the n-dimensional case involving the corresponding n-dimensional cube.

Let G be a closed convex neighborhood of the origin O and $T : G \to R^n$ be a continuous mapping. A point on the boundary of G is said to be T-cube-radiant if $T(x)$ can be collapsed back to x by the above rules. In other words, there exists a $\lambda > 1$, such that $P_\lambda(T(x)) = x$.

It is easy to see how the proof used to establish Theorem 1 can be adapted to prove.

Theorem 2: If all points in ∂G are non-T-cube-radiant (can be replaced by any general sense of non-T-radiance as described below), then T has a fixed point in G.

Note that the classical radial-collapsing notion in Leray-Schauder's condition corresponds to a mapping $\overline{P}_\lambda(y) = y/\lambda$, which is similar to the cube-collapsing P_λ defined above.

In fact, one can study more general collapsing algorithms that can be used to establish Petryshyn-type theorems. Each algorithm corresponds to a family of projection-like mappings similar to P_λ above and the corresponding concept of T-radiant and T-retracting can be defined. When we say that all points in a set are non-T-radiant, we mean that they are non-T-radiant with respect to the same collapsing algorithm. But if we have two non-overlapping sets, then we allow points in one set to be non-T-radiant in one sense while points in the other set are allowed to be non-T-radiant in possibly a different sense.

We repeat our earlier remark that in the statement of all the theorems in this paper, the condition of being T-radiant can be interpreted in the sense of any of these more general conditions.

To illustrate the wide variety of general collapsing algorithms, we can construct the hybrid projection mapping $\hat{P}_\lambda : R^2 \to R^2$, in which we define $\hat{P}_\lambda$, when restricted to the half-plane $\{(a, b) : a \geq 0\}$, to be the radial-collapsing projection, and when restricted to the half-plane, $\{(a, b) : a < 0\}$ to be the square-collapsing projection. However, we shall not pursue such generalizations any further.

To conclude this section we look at a few examples.

Example 1: Fig. 3 is an example of a mapping that has a T-LS-radiant point N on the boundary of G, but no T-cube-radiant point. More specifically: M is the mid-point of the edge AB; O is the center of the square; N' lies on the extension of the line ON; P is vertically under N'; T maps the line segment MN onto NN', NP onto $N'P'$, and then the rest of the boundary $PBCDAN$ onto the curve $P'B'C'D'A'N'$

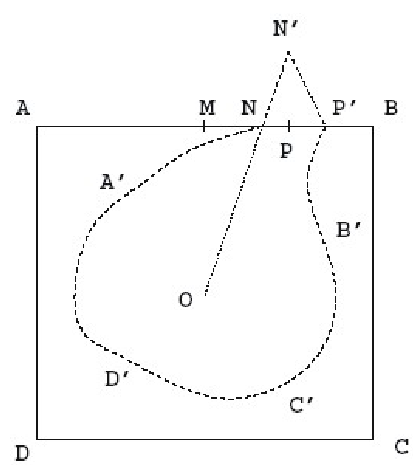

Fig. 3: An LS-Radiant Mapping which is Non-Cube-Radiant

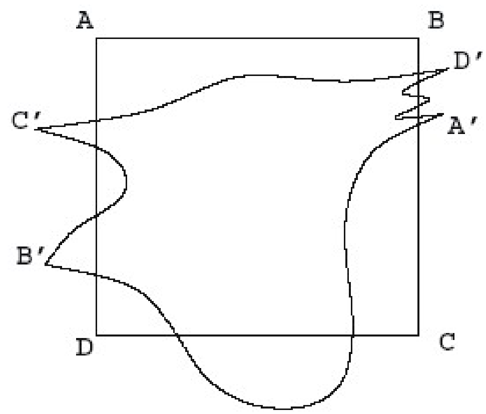

Fig. 4: An Application of Petryshyn's Theorem

which lies inside the square. Certainly, this example looks somewhat contrived, but it serves to show that *T*-*LS*-radiance does not necessarily imply *T*-cube-radiance. Similar examples can be easily constructed to show that implication in the other direction is also not true. In other words, the two notions of radiance are independent of each other.

Example 2: Fig. 4 is the visualization of a particular application of Petryshyn's Theorem. The mapping *T* maps the upper edge *AB* of a square to some curve that lies below *AB*. Similarly, the image of the edge *CD* lies above *CD*; the image of *AD* lies to the right of *AD*; and the image of *BC* lies to the left of *BC*. Note that there is no requirement that the image of these edges must lie completely inside the square. It follows from either Theorem 1 or Theorem 2 that *T* has a fixed point.

Example 3: We mention Example 2 because it has the following interesting corollary. Let us topologically deform the square into the annular area with the same labels as shown in Fig. 5.

We now have a wedge-shaped region, which is called a cone in the higher-dimensional case. Instead of assuming that *T* is defined only for the area *ABCD*, we require that *T* be defined for all points in the (infinite) cone region, and we require that *T* maps the cone into itself. Under this more stringent assumption, it is obviously true that the line *AB* is mapped "below" itself (but the image is not allowed to get below *CD*) and the line *CD* is mapped above itself. The condition that the image of *BC* lies to the left of *BC*, now becomes $|Tx| < |x|$ for all x on the arc *BC*. Likewise, $|Tx| > |x|$ for all x on the arc *AD*. Then *T* has a fixed point.

This is a special case of the well known contractive form of Krasnoselskii's fixed point theorem for cone mappings. In other words, we have demonstrated that

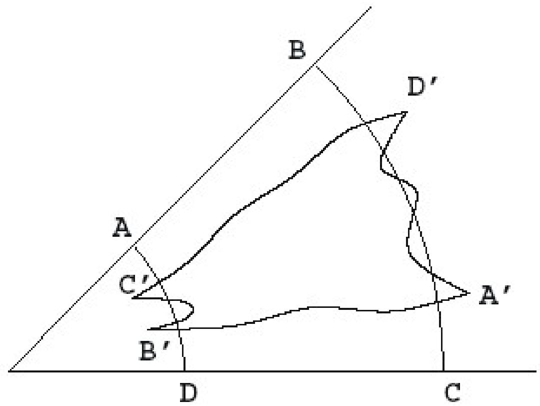

Fig. 5. Example of a Cone Mapping Having a Fixed Point

the contractive form of Krasnoselskii's theorem follows from Petryshyn's theorem. The above example can be easily extended to higher dimensions (and even infinite dimensions). Furthermore, the norm-style boundary condition in the above example can be relaxed to a Leray-Schauder-type condition, namely that all points on the arc BC are non-T-radiant and all points on the arc AD are non-T-retracting.

3 DOMAINS WITH BUBBLES

It is well known that Brouwer's theorem is no longer true if G, the domain of T, is not simply-connected.

In this section and the next one, the term "*domain*" is loosely used to described a closed set that can be used as the domain of the mapping T, rather than the customary meaning of an open connected set.

Example 4: To take an example, let K be the unit circle in R^2 minus the interior of the circle H with center at (0, 1/2) and radius 1/4. Then there are obvious examples of continuous mappings $T : K \to K$ that have no fixed point.

However, if we add an additional condition that $T(\partial H) \subset B_{1/5}$, where $B_{1/5}$ denotes the ball with radius 1/5, then we can still conclude that T has a fixed point. The easiest way to prove this is to observe that T can be extended to be defined on the whole of the unit ball in the following way: suppose $x \in H$ is of distance $\rho < 1/4$ from the center of H and let $\overline{x} \in \partial H$ be the point that the line extending from the center of H towards x meets the boundary of H, then we define $T(x) = 4\rho T(\overline{x})/5$. It is easy to see that the extended mapping T is continuous and maps the unit ball into itself. Therefore T has a fixed point. But since H does not intersect $T(H) \subset B_{1/5}$, this fixed point must be in K, and so the original mapping T has a fixed point in K.

The fact that the continuous mapping T on K can be extended to a continuous mapping defined on the whole unit ball is a special case of the well known Dugundji extension theorem.

The above simple example motivates the next result, which states that Petryshyn's theorem holds for domains formed by removing an interior open subset from a closed convex set. The Dugundji extension theorem could have been used to slightly simplify the proof, but we choose not to rely on this theorem.

Theorem 3: Let H be the closure of an open set H^o contained in the interior of a closed convex set G, and $H \cap \partial G = \emptyset$. Assume also that the origin $0 \notin H$. Let $K = G \backslash H^o$ be the set obtained by removing H^o from G and $T : K \to X$ is a compact mapping, such that all points on ∂G are non-T-radiant (in one sense), and all points on ∂H are non-T-radiant (in possibly a dierent sense). Then T has a fixed point in K.

Proof: The boundary ∂K is the union of ∂G and ∂H, the boundaries of G and H, respectively.

Using the device in the proof of Theorem 1, we can reduce the situation in which T maps some parts of K (in particular, some parts of ∂G) outside G to the case that T maps K strictly inside G. In this way, we have taken care of condition (1.1) for those points on ∂G, and we can concentrate on dealing with the boundary of H. We first consider the case that all points on ∂H are non-T-LS-radiant.

For any $s \geq 0$, we define

$$J_s = \{y \in K : d(y, H) = s\}, \tag{3.1}$$

where $d(y, H)$ denotes the well known distance function from the point y to the closed set H. Let $\epsilon > 0$ be a chosen small number, and define

$$H_\epsilon = \bigcup_{0 \leq s \leq \epsilon} J_s. \tag{3.2}$$

Geometrically, H_ϵ is a thin shell of points exterior to H^o.

The assumption $0 \notin H$ implies that 0 is an interior point of K. We can, therefore, find a ball $B_{2\sigma}$ of radius $2\sigma > 0$, such that $B_{2\sigma} \subset K$ and $B_{2\sigma} \cap H = \emptyset$. This implies that

$$B_\sigma \cap H_\sigma = \emptyset, \tag{3.3}$$

which in turn implies that

$$|x| \geq \sigma, \text{ for all } x \in H_\sigma. \tag{3.4}$$

Next, because $T(H_\sigma)$ is bounded, we can find a positive number $\rho < 1$ small enough so that $\rho T(H_\sigma) \subset B_\sigma$. Hence,

$$\rho|T(x)| \leq \sigma, \text{ for all } x \in H_\sigma. \tag{3.5}$$

Combining (3.4) and (3.5), we see that

$$\frac{|x|}{|T(x)|} \geq \rho, \text{ for all } x \in H_\sigma. \tag{3.6}$$

Let us now pick an $\epsilon \in (0, \sigma)$, and modify the given mapping T as follows. For x outside of H_ϵ, we preserve the image,

$$T_\epsilon(x) = T(x), \text{ for all } x \in (K \backslash H_\epsilon). \tag{3.7}$$

For $x \in J_s$, $0 \leq s\ \epsilon$, we define

$$T_\epsilon(x) = \frac{sT(x)}{\epsilon}. \tag{3.8}$$

From this definition, we see that when x is at J_ϵ, the outer boundary of H_ϵ, $s = \epsilon$. and so $T_\epsilon(x) = T(x)$, and when $x \epsilon \partial H$, $s = 0$ and so $T_\epsilon(x) = 0$. For points in the interior of H_ϵ, $T_\epsilon(x)$ is an appropriate retracting of $T(x)$ towards the origin 0.

Geometrically, we can think of H_ϵ being decomposed into "concentric" layers J_s (each layer contains points of the same s). The given mapping T on H_ϵ is modified in such a way that T_ϵ keep the outermost layer intact, pulls the image of the innermost layer into the origin and pulls the image of the intervening layers towards the origin with a scaling factor that lies between 0 and 1.

If we want to prove the theorem for non-cube-radiant mappings, we simply replace the radial shrinking of $T(J_s)$ by an appropriate cube-collapsing of $T(J_s)$ using the family of functions P_λ defined in Section 2.

Obviously we can now extend T_ϵ to be defined on the whole of the unit ball by letting

$$T_\epsilon(x) = 0, x \in H. \tag{3.9}$$

Continuity of T_ϵ follows from that of T. By Brouwer's theorem, To has a fixed point, say y. If $y \notin H_\epsilon$, then it is a fixed point of T and the theorem is proved. So we assume that $y \notin H_\epsilon$. By definition, $T_\epsilon(y)$ is a diminished scaling of $T(y)$. Thus

$$y = T_\epsilon(y) = \mu T(y) \text{ for some } \mu \in [0, 1). \tag{3.10}$$

Since $y \in H_\epsilon \subset H_\sigma$, (3.6) implies that $\mu \geq \rho$.

Taking $\lambda = 1/\mu$, we have

$$T(y) = \lambda y \text{ for some } \lambda \in (1, 1/\rho). \tag{3.11}$$

We now take a sequence of numbers $\epsilon_n \to 0$, and for each n, we find using the above procedure a point y_n such that

$$T(y_n) = \lambda_n y_n, \lambda_n \in (1, 1/\rho). \tag{3.12}$$

Since both $[1, 1/\rho]$ and the image of K under T are compact, we may assume without loss of generality (by passing to a subsequence if necessary) that both sequences $\lambda_n \to \lambda_\infty$ and $\lambda_n y_n$ converge. It then follows that $y_n \to y_\infty \in K$, converges, and by the continuity of T, we also have

$$T(y_\infty) = \lambda_\infty y_\infty, \lambda_\infty \in [1, 1/\rho]. \tag{3.13}$$

If $\lambda_\infty = 1$, then y_∞ is a fixed point of T, otherwise the above equation contradicts Leray-Schauder's condition. The proof of the theorem is therefore complete.

In the proof above, the compactness of the image of K under T is required to guarantee the convergence of a subsequence of $\{y_n\}$.

We allow the possibility of different types of non-radiance being used for the non-overlapping boundaries ∂G and ∂H. See Example 6 below. In general, if the boundary of ∂K can be decomposed into several non-overlapping closed sets, then a different type of non-radiance may be assumed on each set.

Example 5: Let K be the same set as in Example 4. Suppose that $T : K \to R^2$ maps the unit circle S (the outer boundary of K) into the unit ball B, and rotates the circle H about the origin 0 (not about the center of H) by an angle less than 360°. Then T has a fixed point.

Example 6: Let K be the set defined by removing the circle H in Example 4 from the square *ABCD* in Example 1. Let T be given such that T maps the boundary of the square *ABCD* as in Example 1, and T moves the boundary of the circle H vertically upwards by a small distance, say 0.1. Then no points on the boundary of *ABCD* is T-square-radiant (but the point N is T-LS-radiant), and no points on the boundary of H is T-LS-radiant (but some points on the upper half of the circle are T-square-radiant). The hypotheses of Theorem 3 are satisfied and so T has a fixed point.

Example 7: The hypothesis that $0 \notin H$ is essential for Theorem 3 to hold, as shown by the simple counter-example of rotating an annulus by a small angle.

The following example indicates an alternative way to handle domains with a bubble. Let $K \subset R^n$ be the annular region between two given spheres S_a and S_b, $a < b$, and $T : K \to K$ be a mapping of K into itself, such that $T(S_a)$ can be homotopically shrunk to a point, while remaining always inside K. Then T has a fixed point. This is because the homotopy used to shrink $T(S_a)$ to a point can be used to extend T to be defined on the entire B_b and the extended mapping must have a fixed point, which can be easily shown to be a fixed point of the original T.

In the two-dimensional case, if the winding number of $T(S_a)$ with respect to the origin is 0, then $T(S_a)$ can be homotopically shrunk to a point and we can conclude that T has a fixed point.

The same proof of Theorem 3 can be extended to treat the case when K is any bounded closed set. We omit the details.

Theorem 4: Let K be a closed bounded set with 0 being an interior point, and $T : K \to X$ is a compact mapping, such that all points on ∂K are non-T-radiant. Then T has a fixed point in K.

We, however, believe that most interesting applications of the theorem will involve K of the form as stated in Theorem 3.

4. NON-RETRACTING MAPPINGS

The result in this section only applies to finite dimensional spaces, because for infinite dimensional spaces the required assumption on the Brouwer degree of T conflicts with the usual requirement that T be completely continuous.

We assume that the concept of the degree of a mapping is known. See the references Cronin [2], Guo and Lakshmikantham [3], or Istratescu [4] for an exposition of the theory. Let $n = \deg(T, G, O)$ denote the usual degree of the mapping T on the closed convex set G at the origin O. It is well known that this number only depends on the action of T on ∂G. In the two-dimensional case, if O is not in $T(\partial G)$, the degree has the interpretation of the winding number. As we trace the point x one complete round along the perimeter of G, the image $T(x)$ winds around the origin exactly n times.

The following homotopy property of the degree gives a simple proof of our next result. Let T be homotopic to T_1, i.e. there is a continuous mapping $h : G \times [0, 1] \to R^n$ such that $h(x, 0) = T((x)$, $h(x, 1) = T_1(x)$, and $h(x, s) \neq O$ for all $x \in \partial G$ and $s \in [0, 1]$. Then $\deg(T, G, 0) = \deg(T_1, G, 0)$.

Theorem 5: Let $T : G \to R^n$ be a continuous mapping such that $\deg(T, G, O) \neq 0$. If all points on ∂G are non-T-retracting, then T has a fixed point.

Proof: We define the homotopy

$$h(x, s) = T(x) - sx, x \in G, s \in [0, 1]. \tag{4.1}$$

Then

$$h(x, 0) = T(x) \tag{4.2}$$

and

$$h(x, 1) = T(x) - x. \tag{4.3}$$

For all points $x \in \partial G$, and $s \in [0, 1)$, $h(x, s) \neq O$. For $s = 1$, there may be an x such that $h(x, 1) = T(x) - x = 0$. But if so, we have a fixed point and we are done.

Hence, we may assume that $h(x, 1) \neq 0$ for all $x \in \partial G$. Then by the homotopy property, $\deg(T(x) - x, G, O) = \deg(T, G, O) \neq 0$. This implies that $T(x) - x = 0$ has a solution and we also get a fixed point.

Example 8: Every point on the two-dimensional unit circle S has the complex-number representation $e^{i\theta}$. Let $T : B \to R^2$ be a continuous function such that each $x = e^{i\theta} \in S$, is mapped to $T(x) = \mathrm{r}(\theta)\mathrm{e}^{\mathrm{i}\theta}$, where $\mathrm{r}(\theta)$ is an arbitrary continuous positive function $\mathrm{r} : [0, 2\pi] \to (0, \infty)$, such that $r(\theta) > 1$. Then the winding number of $T(S)$ at the origin is 1. Note that the hypotheses of Theorem 1 are not satisfied because every point on the circle is T-LS-radiant, but Theorem 5 implies that there exists a fixed point.

For this example, the existence of a fixed point can still be deduced using Theorem 1. Let us perturb the mapping T by rotating the image $T(B)$ by a small angle ϕ. Then every point on S now becomes non-T-LS-radiant, and so the perturbed mapping T_ϕ has a fixed point x_ϕ. Now take a sequence $_{\phi_n} \to 0$, and consider the corresponding sequence of fixed points x_{ϕ_n}. By passing to a sub-sequence if necessary, x_{ϕ_n} will then converge to a fixed point of T.

Example 9: We modify the previous example by assuming that each $x = e^{i\theta} \in S$, is mapped to $T(x) = r(\theta)e^{2i\theta}$, where $\mathrm{r}(\theta)$ is an arbitrary continuous positive function $r : [0, 2\pi] \to (0, \infty)$, such that $r(0) = r(2\pi) > 1$. Then the winding number of $T(S)$ at the origin is 2. The hypotheses of Theorem 5 are thus satisfied and T has a fixed point. Note that the hypotheses of Theorem 1 are not satisfied, because (1, 0) is T-LS-radiant. Furthermore, the trick used in the previous example cannot be used to perturb T to satisfy the hypotheses of Theorem 1.

Another example is to require that $e^{i\theta}$ be mapped to $T(x) = r(\theta)e^{-i\theta}$. Then the winding number of $T(S)$ is -1 and by Theorem 5 T has a fixed point.

These examples shows that Theorem 5 complements Theorem 1. Indeed, it can be seen easily, at least in the two-dimensional case $T : B \to R^2$, that if T satisfies the hypotheses of Theorem 5, it cannot satisfy the hypotheses of Theorem 1. In other words, Theorem 5 truly complements Theorem 1.

Example 10: The following example is "dual" to the example in Fig. 4. We assume that T is a mapping on the square $ABCD$, with the property that it maps the edge AB to a curve above AB, the edge BC to a curve to the right of BC, the edge CD to a curve below CD, and the edge DA to a curve to the left of DA. Then T has a fixed point.

Example 11: In the same manner with which we derived Krasnoselskii's contractive cone fixed point result from Example 4, we can deduce the following "dual" result. Let G be the annular region of a cone C in R^n (this result holds only for finite dimensional spaces) between two spheres S_a and S_b, $a < b$. Let ∂G_1 denote the boundary of G that lies on the boundary of the cone. If $T : C \to R^n$ is a continuous mapping such that $T(\partial G_1)$ is outside the cone C, $T(C \cap S_a) \subset B_a \backslash 0$, $T(C \cap S_b)$ is outside the ball B_b, and the Brouwer degree of T at an interior point of G is non-zero, then T has a fixed point.

Finally, we can adopt the technique used to prove Theorem 3 to extend Theorem 5 to domains with bubbles inside them. Let K be a domain as described in Theorem 5, namely, a closed convex set G with an interior bubble H^o, and the origin as an interior point. Let $T : K \to R^n$ be a continuous mapping.

By Dugundji's theorem we can always extend T to be defined on G, so we can talk about $\deg(T, G, O)$. As mentioned before this number depends only on the action of T on ∂G and in the two-dimensional situation, it can be interpreted as the winding number of T, when restricted to ∂G, about the origin O.

Theorem 6: Let K be given as in Theorem 3 and $T : K \to R^n$ be a continuous mapping such that $\deg(T, G, O) \neq 0$. If all points on ∂G are non-T-retracting, and all points on ∂H are non-T-radiant, then T has a fixed point.

Again, the requirement that 0 is in the interior of K is important. Therefore, if we need to apply Theorem 3 or Theorem 6 to an annulus $A = \{a < |x| < b\}$, we have to take a point π such that $a < |\pi| < b$ and use, for instance, T-LS-radiance with respect to π.

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