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BEST SIMULTANEOUS APPROXIMATION IN FUNCTION SPACES

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ABSTRACT: Let X be a Banach space and G be a closed subspace of X. We say G is 2-simultaneously proximinal in X if for any x_1, x_2 in X, there exists some $y \in G$ such that $\|x_1 - y\| + \|x_2 - y\| = \inf\{\|x_1 - z\| + \|x_2 - z\| : z \in G\} = d(\{x_1, x_2\}, G)$. In this paper, we give a formula for $d(\{x_1, x_2\}, G)$ in vector valued integrable functions. Results on simultaneous proximinality in such spaces will be presented.

Keywords: Simultaneous approximation, distance formula.

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1. INTRODUCTION

Let X be a Banach space and G be a closed subspace of X. For $E \subset X$, we write

$$d_1(E, G) = \inf\left\{\sum_{e\in E} \|e - y\| : y \in G\right\}.$$

Such infimum need not be attained. In case the infimum is attained for any subset $E \subset X$, we say that G is $|E|$-simultaneously proximinal in X, where $|E|$ is the cardinality of E. We say G is 2-simultaneously proximinal in X if for any x_1, x_2 in X, there exists some $y \in G$ such that $\|x_1 - y\| + \|x_2 - y\| = \inf\{\|x_1 - z\| + \|x_2 - z\| : z \in G\} = d(\{x_1, x_2\}, G)$. In case $|E| = 1$ then 1-simultaneous proximinality is just proximinality. The first result on 2-simultaneous approximation in $C(I, R)$, the space of continuous real valued functions on some compact interval I, is due to Dunham [2]. Many good results had appeared since then. We refer to [1], [4], [5], [6], [7], [8], and [9]. However, all these results except for [7], dealt with the space of continuous functions with d_∞ instead of d_1.

It is the object of this paper to study 2-simultaneous approximation in vector valued function spaces with d_1 distance. We present a formula for $d_1(E, G)$ when X is the space of Bochner integrable functions on some interval I. Many other results are presented.

Let I be a compact interval. With no loss of generality we assume $I = [0, 1]$. For a Banach space X, $L^1(I, X)$ denotes the space of strongly measurable functions f on I such that $\int\|f(t)\|dt < \infty$. For $f_1, f_2 \in L^1(I, X)$ and G a closed subspace of X, we set

$$d_1(\{f_1, f_2\}) = \inf(\int(\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\|)dt : g \in L^1(I, G)).$$

For any Banach space Y and closed subspace F of Y, we set:

$$J(Y) = Y \oplus_1 Y \text{ with } \|x + y\| = \|x\| + \|y\|$$

$$D(F) = \{(z, z) : z \in F\}, \text{ with } \|(z, z)\| = \|z\| + \|z\|.$$

Clearly, $D(F)$ is a closed subspace of $J(Y)$. Further, F is 2-simultaneously proximinal in Y if and only if $D(F)$ is proximinal in $J(Y)$.

2. THE DISTANCE FORMULA

In this section we deduce a distance formula for best simultaneous approximation in $L^1(I, X)$. We discuss only the distance for two functions.

Theorem 2.1: Let $f_1, f_2 \in L^1(I, X)$, and G be a closed subspace of X. Define

$$\varphi(s) = \inf\{\|f_1 - z\| + \|f_2 - z\| : z \in G\}.$$

Then φ is measurable and

$$\int \varphi(s)ds = \inf\{\|f_1 - h\| + \|f_2 - h\| : h \in L^1(I, G)\} = d(\{h_1, h_2\}, L^1(I, G)).$$

Proof: Let f_1, f_2 be any two elements in $L^1(I, X)$. Then there exist two sequences f_{1n}) and (f_{2n}),of simple functions, such that $\|f_1(t) - f_{1n}(t)\| \to 0$ and $\|f_2(t) - f_{2n}(t)\| \to 0$. Now, the function

$$d((x, y), D(G)) = \inf\{\|x - z\| + \|y - z\| : z \in G\}$$

is continuous. Consequently

$$\lim(d(f_{1n}(\mathrm{t}), f_{2n}(t)), D(G)) = d((f_1(t), f_2(t)), D(G)).$$

Define the sequence of functions $\varphi_n : I \to R$,

$$\varphi_n(s) = d((f_{1n}(s), f_{2n}(s)), D(G)) = \inf\{\|f_{1n}(s) - g\| + \|f_{2n}(s) - g\| : g \in D(G)\}.$$

Since f_{1n} and f_{2n} are simple functions, then we can assume: $f_{1n} = \sum_{i=1}^{n} 1_{A_i} \otimes x_i$ and $f_{2n} = \sum_{i=1}^{n} 1_{A_i} \otimes y_i$ with A_i are disjoint. But then one can see that

$$\begin{aligned} \varphi_n(s) &= \inf\{\Sigma 1A_i(s)(\|xi - g\| + \|y_i - g\|) \\ &= \Sigma 1A_i(s) \inf(\|x_i - g\| + \|y_i - g\|), \end{aligned}$$

since the $A_i^{,s}$ are disjoined. Hence, each φ_n is a simple function, and consequently the function φ is measurable. Now, let $g \in L^1(I, G)$. Then

$$\begin{aligned} \|f_1 - g\| + \|f_2 - g\| &= \int_I (\|f_1(s) - g(s)\| + \|f_2(s) - g(s)\|) ds \\ &\geq \int_I d((f_{1n}(s), f_{2n}(s)), D(G)) ds = \|\varphi\|_1. \end{aligned}$$

Taking the infimum over all $g \in L^1(I, G)$, we get

$$d((f_1, f_2), L^1(I, G)) \geq \|\varphi\|_1 \tag{1}$$

For the reverse inequality: Let $\epsilon > 0$ be arbitrary. Choose P and Q, simple functions such that

$$P = \sum_{i=1}^{m} 1_{B_i} \oplus z_i, Q = \sum_{i=1}^{m} 1_{B_i} \oplus w_i,$$

$\|f_1 - P\| < \epsilon$, and $\|f_2 - Q\| < \epsilon$, where the $B_i^{,s}$ are disjoint and $\mu(B_i) > 0$ for all i.

From the definition of the distance there exists $h_i \in G$ such that

$$\|z_i - h_i\| + \|w_i - h_i\| < d((z_i, w_i), D(G)) + \epsilon.$$

Now

$$\begin{aligned} d((f_1, f_2), D(L^1(I, G))) &= \inf\{\|f_1 - h\| + \|f_2 - h\| : h \in L^1(I, G)\} \\ &\leq \inf\{\|f_1 - P\| + \|f_2 - Q\| + \|h - P\| + \|h - Q\| : h \in L^1(I, G)\} \\ &\leq 2\epsilon + \inf\{\|h - P\| + \|h - Q\| : h \in L^1(I, G)\} \\ &\leq 2\epsilon + \inf\{\|g - P\| + \|g - Q\| : g \in L^1(I, G), \end{aligned}$$

with g of the form $\left. g = \sum_{i=1}^{m} 1_{B_i} \otimes h_i \right\}$

$$\leq 2\epsilon + \inf \sum_{i=1}^{m} \mu(B_i)(\|z_i - h_i\| + \|w_i - h_i\|)$$

$$\leq 3\epsilon + \sum_{i=1}^{m} \int_{B_i} d((z_i, w_i), D(G)) ds \qquad \left(\text{since} \sum \mu(A_i) = 1\right)$$

$$\leq 2\epsilon + \inf \sum_{i=1}^{m} \mu(B_i)\left[d((z_i, w_i), D(G)) + \epsilon\right]$$

$$= 3\epsilon\int_I d(P(s), Q(s)), D(G))ds$$
$$= 3\epsilon\int_I \inf\{\|P(s) - z\| + \|Q(s) - z\| : z \in G\}ds$$
$$\leq 3\epsilon + \int_I \inf\{\|f_1(s) - z\| + \|f_2(s) - z\| + \|f_1(s) - P(s)\| + \|f_2(s) - Q(s)\| : z \in G\}ds$$
$$\leq 3\epsilon + \int_I \inf\{\|f_1(s) - z\| + \| f_2(s) - z\| : z \in G\}ds + (\| f_1 - P\| + \| f_2 - Q\|)$$
$$\leq 5\epsilon + \int_I \inf\{\|f_1(s) - h(s)\| + \|f_2(s) - h(s)\| : h \in L^1(I, G)\}ds$$
$$\leq 5\epsilon + \|\varphi\|_1 \quad (2)$$

Since ϵ was arbitrary, equations (1) and (2) ends the proof.

As an application to Theorem 2.1 we have:

Theorem 2.2: Let G be a closed subspace of the Banach space X, and $f_1, f_2 \in L^1(I, X)$. Then for any $g \in L^1(I, G)$, the following are equivalent.

(i) g is a best simultaneous approximant for f_1, f_2 in $L^1(I, G)$.

(ii) $g(t)$ is a best simultaneous approximant for $f_1(t), f_2(t)$ in G.

Another nice application of Theorem 2.1 is

Theorem 2.3: Let G be a closed subspace of X. If $L^1(I, G)$ is simultaneously proximinal in $L^1(I, X)$, then G is simultaneously proximinal in X.

Proof: Let $x, y \in X$. Define $f_1 = 1 \otimes x$ and $f_2 = 1 \otimes y$, where 1 is the constant function 1. Clearly f_1 and f_2 are in $L^1(I, X)$. By assumption, there exists $g \in L^1(I, G)$ such that

$$\|f_1 - g\| + \|f_2 - g\| \leq \|f_1 - h\| + \|f_2 - h\| \text{ for all } h \in L^1(I, G).$$

By Theorem 2.1,

$$\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \leq \|f_1(t) - h(t)\| + \|f_2(t) - h(t)\| \text{ for all } h \in L^1(I, G).$$

Thus

$$\|x - g(t)\| + \|x - g(t)\| \leq \|x - h(t)\| + \|x - h(t)\| \text{ for all } h \in L^1(I, G).$$

Let h runs over all functions of the form $1 \otimes z$, for $z \in G$, the result follows.

Now, we give a very simple proof of one of the main results in [7].

Theorem 2.4: Let G be a reflexive subspace of the Banach space X. Then $L^1(I, G)$ is simultaneously proximinal in $L^1(I, X)$.

Proof: Since G is reflexive then $D(G)$ is a reflexive subspace of $G \underset{1}{\oplus} G \subseteq X \underset{1}{\oplus} X$.

Now $L^1(I, X) \underset{1}{\oplus} L^1(I, X)$ is isometrically isomorphic to $L^1\left(1, X \underset{1}{\oplus} X\right)$, and $D(L^1(I, G))$ is isometrically isomorphic to $L^1(I, D(G))$. The result now follows from the main result in [3]. That ends the proof.

3. FURTHER RESULTS

A closed subspace G is called 1-summand in X if there exists a subspace(closed) Y such that $X = G \underset{1}{\oplus} Y$. It is known that [3], that a 1-summand subspace G of X is proximinal, and $L^1(I, G)$ is proximinal in $L^1(I, X)$. Now we prove:

Theorem 3.1: Let G be a 1-summand subspace of X. Then G is simultaneously proximinal.

Proof: Let x, y be any two elements in X. Since X is 1-summand, then $X = G \underset{1}{\oplus} M$.

So $x = x_1 + x_2$ and $y = y_1 + y_2$. Let $z = \dfrac{x_1 + y_1}{2}$. Then $x - z = \dfrac{x_1 - y_1}{2} + x_2$, and $y - z = \dfrac{y_1 - x_1}{2} + y_2$. Hence

$$
\begin{aligned}
\|x - z\| + \|y - z\| &= \left\|\frac{x_1 - y_1}{2}\right\| + \|x_2\| + \left\|\frac{y_1 - x_1}{2}\right\| + \|y_2\| \\
&= \|x_2\| + \|y_2\| + \|x_1 - y_1\| \\
&\leq \|x_2\| + \|y_2\| + \|x_1 - w\| + \|y_1 - w\|, \text{ (for any } w \in G) \\
&= (\|x_2\| + \|x_1 - w\|) + (\|y_2\| + \|y_1 - w\|) \\
&= \|x - w\| + \|y - w\|.
\end{aligned}
$$

Hence z is a best simultaneous approximation in G for x and y, and G is simultaneously proximinal.

As a corollary, we get the following:

Theorem 3.2: If G is 1-summand in X, then $L^1(I, G)$ is simultaneously proximinal in $L^1(I, X)$.

Proof: $L^1(I,X) = L^1\left(I, G \underset{1}{\oplus} M\right) = L^1(I,G) \underset{1}{\oplus} L^1(I,M)$. Hence $L^1(I, G)$ is 1-summand in $L^1(I, X)$. Hence by Theorem 3.1, $L^1(I, G)$ is Simultaneous Proximinal.

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