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Hydrodynamic Model of Scale Relativity Theory in the Topological Dimension *D*=2

M. AGOP & P. NICA

The hydrodynamic model of the scale relativity theory in the topological dimension D=2 is built. The generalized Schrödinger equation results as an irrotational movement of Navier-Stokes type fluids having an imaginary viscosity coefficient and ψ simultaneously becomes wave-function and complex speed potential. Some abilities of the model on the wave-particle duality and tunneling effect are analyzed.

1. INTRODUCTION

The scale relativity theory (SRT) is a new approach to understanding quantum mechanics, and moreover physical domains involving scale laws, such as chaotic systems and cosmology [1,2]. It is based on a generalization of Einstein's principle of relativity to scale transformations. Namely, one redefines space-time resolutions as characterizing the state of scale of reference systems, in the same way as velocity characterizes their state of motion. Then one requires that the laws of physics apply whatever the state of the reference system, of motion (principle of motion-relativity) and of scale (principle of scale-relativity). The principle of scale-relativity is mathematically achieved by the principle of scale-covariance, requiring that the equations of physics keep their simplest form under transformations of resolution [1,2].

It is well known that the geometrical tool that implements Einstein's general motion-relativity is the concept of Riemannian, curved space-time. In a similar way, the concept of *fractal space-time* [1,2], also independently introduced by El Naschie [2-9], is the geometric tool adapted to construct the new theory. We use here the word 'fractal' in its general meaning [10], denoting a set that shows structures at all scales and is thus explicitly resolution-dependent. More precisely, one can demonstrate [10] that the D_τ– measure of a continuous, almost everywhere non-differentiable set of topological dimension D_τ is a function of resolution, $L=L(e)$, and diverges when resolution tends to zero, $L(e)\to\infty$ when $e\to 0$. In such a framework, resolutions are considered to be inherent to the description of the new, fractal, space-time. A new physical content may also be given to the concept of particles in this theory; various properties of 'particles' can be reduced to the geometric structures of the (fractal) geodesics of such a space-time [1].

Three levels of such a theory have been considered: (i) A 'Galileian' version corresponding to the standard fractals with constant fractal dimensions, and where dilatation laws are the usual ones [1,2]. This theory provides us a new foundation of quantum mechanics from first principles; (ii) A special scale-relativistic version that implements in a more general way the principle of scale-relativity. It yields new dilatation laws of a Lorentzian form, that imply to re-interpret the Planck length-scale as a lower, impassable scale, invariant under dilatations [1,2]. The predictions of such a theory depart from that of standard quantum mechanics at large energies [1-9,11,12]; (iii) The third level, 'general scale-relativistic' version of the theory deals with non-linear scale laws and accounts for the coupling between scale laws and motion laws [1]. It yields a new interpretation of gauge invariance and allows one to get new mass-charge relations that solve the scale-hierarchy problem [1]. Using this theory [1], both conceptual (the complex nature of wave function, the probabilistic nature of quantum theory, the principle of correspondence, the quantum-classical transition, the divergence of masses and charges, the nature of Planck scale, the nature and quantization of electric charge, the origin of mass discretization of elementary particles, the nature of cosmological constant [13], etc.) and quantized results (the mass-charge relations, the electro-weak scale, the electron scale, the elementary fermion mass spectrum etc.) are obtained.

In the present paper the hydrodynamic model of the SRT in topological dimension D=2 is given. The generalized Schrödinger equation results as an irrotational movement of Navier-Stokes type fluids having an imaginary viscosity coefficient and ψ simultaneously becomes wave-function and complex speed potential.

2. MATHEMATICAL MODEL

Let us suppose that the motion of the various physical objects takes on continuous but non-differentiable curves, hence on fractals. The "non-differentiable" nature of space-time implies a breaking of differential time reflection invariance. In such a context the usual definitions of the derivative of a given function with respect to time [1].

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t) - f(t-\Delta t)}{\Delta t} \tag{1}$$

are equivalent in the differentiable case. One passes from one to the other by the transformation $\Delta t \to -\Delta t$ (time reflection invariance at the infinitesimal level). In the non-differentiable case two functions (df_+ / dt) and (df_- / dt) are defined as explicit functions of t and of dt

$$\frac{df_+}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t, \Delta t) - f(t, \Delta t)}{\Delta t}, \frac{df_-}{dt} = \lim_{\Delta t \to 0} \frac{f(t, \Delta t) - f(t - \Delta t, \Delta t)}{\Delta t} \quad (2a,b)$$

The sign (+) corresponds to the forward process and (-) to the backward process.

In the spaces coordinates $d\boldsymbol{X}$, we can write [1]

$$d\boldsymbol{X}_\pm = d\boldsymbol{x}_\pm + d\boldsymbol{\zeta}_\pm \quad (3)$$

where $d\mathbf{x}_\pm$ are the classical variables and $d\boldsymbol{\zeta}_\pm$ the fractal fluctuations induced by the fractal properties of the "trajectory". Particularly, by averaging, the Eq. (3) becomes

$$\langle d\boldsymbol{X}_\pm \rangle = \langle d\boldsymbol{x}_\pm \rangle \quad (4)$$

where

$$\langle d\boldsymbol{\xi}_\pm \rangle = 0 \quad (5)$$

From (3) we obtain the speed field:

$$\frac{d\boldsymbol{X}_\pm}{dt} = \frac{d\boldsymbol{x}_\pm}{dt} + \frac{d\boldsymbol{\zeta}_\pm}{dt} \quad (6)$$

We denoted by $(d\boldsymbol{x}_+/dt) = \boldsymbol{v}_+$ the "forward" speed and by $(d\boldsymbol{x}_-/dt) = \boldsymbol{v}_-$ the "backward" speed. If $(\boldsymbol{v}_+ + \boldsymbol{v}_-)/2$ may be considered as classical speed, the difference between them, i.e. $(\boldsymbol{v}_+ - \boldsymbol{v}_-)/2$ is the fractal speed, so that we can introduce the complex speed [1]

$$\boldsymbol{V} = \frac{\boldsymbol{v}_+ + \boldsymbol{v}_-}{2} - i\frac{\boldsymbol{v}_+ - \boldsymbol{v}_-}{2} = \frac{d\boldsymbol{x}_+ + d\boldsymbol{x}_-}{2dt} - i\frac{d\boldsymbol{x}_+ - d\boldsymbol{x}_-}{2dt} \quad (7)$$

Using the notations $d\mathbf{x}_\pm = d_\pm \mathbf{x}$, (7) becomes:

$$\boldsymbol{V} = \left(\frac{d_+ + d_-}{2dt} - i\frac{d_+ - d_-}{2dt} \right)\boldsymbol{x} \quad (8)$$

that allows defining the operator:

$$d = \frac{d_+ + d_-}{2dt} - i\frac{d_+ - d_-}{2dt} \tag{9}$$

To summarize, while the concept of velocity was classically a single concept, if space-time is non-differentiable, we must introduce two speeds instead of one, even when going back to the classical domain. Such a two-valuedness of the speed vector is a new, specific consequence of non-differentiability that has no standard counterpart (in the sense of differential physics), since it finds its origin in a breaking of the symmetry ($dt \to -dt$). Such a symmetry was considered self-evident up to now in physics (since the differential element *dt* disappears when passing to the limit), so that it has not been analyzed on the same footing as the other well-known symmetries. Note that it is actually different from the time reflection symmetry *T*, even though infinitesimal irreversibility implies global irreversibility [1].

Now, at the level of our description, we have no way to favor $\boldsymbol{v}_+$ rather than $\boldsymbol{v}_-$. Both choices are equally qualified for the description of the laws on nature. The only solution to this problem is to consider both the forward ($dt > 0$) and backward ($dt < 0$) processes together. The number of degrees of freedom is double with respect to the classical, differentiable description (6 velocity components instead of 3, see (7)).

Let us assume now that the fractal curve is immersed in a 3-dimensional space, and $\boldsymbol{X}$ of components X^i (i=$\overline{1,3}$) is the position vector of a point on the curve. Let us consider also a function $f(X,t)$ and the following Taylor series expansion up to the second order:

$$df = f(X^i + dX^i, t + dt) - f(X^i, dt) = \left(\frac{\partial}{\partial X^i} dX^i + \frac{\partial}{\partial t} dt\right) f(X^i, t) + \frac{1}{2}\left(\frac{\partial}{\partial X^i} dX^i + \frac{\partial}{\partial t} dt\right)^2 f(X^i, t) \tag{10}$$

From here, the forward and backward average values of this relation using notations $dX_\pm^{\ i} = d_\pm X^i$ take the form:

$$\langle d_\pm f \rangle = \left\langle \frac{\partial f}{\partial t} dt \right\rangle + \langle \nabla f \cdot d_\pm \boldsymbol{X} \rangle + \frac{1}{2}\left\langle \frac{\partial^2 f}{\partial t^2}(dt)^2 \right\rangle + \left\langle \frac{\partial^2 f}{\partial X^i \partial t} d_\pm X^i dt \right\rangle + \frac{1}{2}\left\langle \frac{\partial^2 f}{\partial X^i \partial X^l} d_\pm X^i d_\pm X^l \right\rangle \tag{11}$$

We make the following stipulations: the mean values of the function *f* and its derivates coincide with themselves, and the differentials $d_\pm X^i$ and *dt* are independent, therefore the averages of their products coincide with the product of average. Then (11) becomes:

$$d_{\pm}f = \frac{\partial f}{\partial t}dt + \nabla f\left\langle d_{\pm}\boldsymbol{X}\right\rangle + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\left\langle (dt)^2\right\rangle + \frac{\partial^2 f}{\partial X^i \partial t}\left\langle d_{\pm}X^i dt\right\rangle + \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^l}\left\langle d_{\pm}X^i d_{\pm}X^l\right\rangle \qquad (12)$$

so that, using (3) in the form (4),

$$d_{\pm}f = \frac{\partial f}{\partial t}dt + \nabla f d_{\pm}\boldsymbol{x} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial X^i \partial t}d_{\pm}x^i dt + \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^l}\left(d_{\pm}x^i d_{\pm}x^l + \left\langle d\xi_{\pm}^i d\xi_{\pm}^l\right\rangle\right) \qquad (13)$$

Since $d\xi_{\pm}^i$ describes the fractal properties of the curve trajectory with the fractal dimension *D*, it is natural to impose $(d\xi_{\pm}^i)^D$ to be proportional with *dt*, i.e.

$$(d\xi_{\pm}^i)^D = \mathcal{D}_0 dt \qquad (14)$$

where D_0 is a coefficient of proportionality.

Let us focus now on the mean $\left\langle d\xi_{\pm}^i d\xi_{\pm}^l\right\rangle$. If $i \neq l$ this average is zero due the independence of $d\xi^i$ and $d\xi^l$. So, using (14) we can write:

$$\left\langle d\xi_{\pm}^i d\xi_{\pm}^l\right\rangle = \delta^{il}(\mathcal{D}_0 dt)^{2/D} \qquad (15)$$

with

$$\delta^{il} = \begin{cases} 1, & if \quad i = l \\ 0, & if \quad i \neq l \end{cases}$$

Through of a Peano type curves which covers a two-dimensional surface, i.e. *D*=2, (15) becomes:

$$\left\langle d\xi_{\pm}^i d\xi_{\pm}^l\right\rangle = \pm\delta^{il}(\mathcal{D}_0 dt) \qquad (16)$$

where we had considered that:

$$\begin{cases} \left\langle d\xi_{+}^i d\xi_{+}^l\right\rangle > 0 \; and \;\; dt > 0 \\ \left\langle d\xi_{-}^i d\xi_{-}^l\right\rangle > 0 \; and \; dt < 0 \end{cases}$$

Then (13) may be written under the form:

$$d_{\pm}f = \frac{\partial f}{\partial t}dt + \nabla f d_{\pm}\mathbf{x} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial X^i \partial t}d_{\pm}x^i dt + \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^l}d_{\pm}x^i d_{\pm}x^l + \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^l}\delta^{il}\mathcal{D}_0 dt \quad (17)$$

If we divide by dt and neglect the terms which contain differential factors, (17) is reduced to:

$$\frac{d_{\pm}f}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{v}_{\pm}\nabla f_{\pm} \pm \frac{\mathcal{D}_0}{2}\Delta f \quad (18)$$

Let us calculate, under the circumstances *(df/dt)*. Taking into account (18), we have:

$$\frac{df}{dt} = \frac{1}{2}\left[\frac{d_+f}{dt} + \frac{d_-f}{dt} - i\left(\frac{d_+f}{dt} - \frac{d_-f}{dt}\right)\right] = \frac{1}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_+\nabla f + \frac{\mathcal{D}_0}{2}\Delta f\right) + \frac{1}{2}\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_-\nabla f - \frac{\mathcal{D}_0}{2}\Delta f\right) -$$
$$-\frac{i}{2}\left[\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_+\nabla f + \frac{\mathcal{D}_0}{2}\Delta f\right) - \left(\frac{\partial f}{\partial t} + \boldsymbol{v}_-\nabla f - \frac{\mathcal{D}_0}{2}\Delta f\right)\right] = \frac{\partial f}{\partial t} + \left(\frac{\boldsymbol{v}_+ + \boldsymbol{v}_-}{2} - i\frac{\boldsymbol{v}_+ - \boldsymbol{v}_-}{2}\right)\nabla f - i\mathcal{D}_0\Delta f \quad (19)$$

or using (8):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{V}\cdot\nabla f - i\mathcal{D}\Delta f\ , \quad \mathcal{D} = 2\mathcal{D}_0 \quad (20a,b)$$

with $\mathcal{D}$ the Nottale's diffusion coefficient [1,2].

This relation also allows us to give the definition of the fractal operator [1]:

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla - i\mathcal{D}\Delta \quad (21)$$

We now apply the principle of scale covariance, and postulate that the passage from classical (differentiable) mechanics to the new non-differentiable mechanics that is considered here can be implemented by replacing the standard time derivative *d/dt* by the new complex operator $\delta/\delta t$. As a consequence, we are now able to write the equation of geodesics of the fractal space under its covariant form:

$$\frac{\delta \boldsymbol{V}}{\delta t} = \frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla \boldsymbol{V} - \eta\Delta\boldsymbol{V} = 0 \quad (22)$$

i.e. a generalized Navier-Stokes type equation having the imaginary viscosity coefficient $\eta = i\mathcal{D}$. This means that the time dependence of the speed field, $\partial_t \boldsymbol{V}$, is imposed both by the convective term, $\boldsymbol{V} \cdot \nabla \boldsymbol{V}$, and by the viscosity one, $\Delta \boldsymbol{V}$.

From here and from the operational relation

$$\boldsymbol{V} \cdot \nabla \boldsymbol{V} = \nabla(\boldsymbol{V}^2/2) - \boldsymbol{V} \times (\nabla \times \boldsymbol{V}) \tag{23}$$

we obtain the Eq.

$$\frac{\delta \boldsymbol{V}}{\delta t} = \frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left(\frac{\boldsymbol{V}^2}{2}\right) - \boldsymbol{V} \times (\nabla \times \boldsymbol{V}) - \eta \Delta \boldsymbol{V} = 0, \tag{24}$$

Let us admit that the “fluid” is irrotational, i.e.

$$\boldsymbol{\Omega} = \nabla \times \boldsymbol{V} = 0 \tag{25}$$

Then, we can choose $\boldsymbol{V}$ of the form:

$$\boldsymbol{V} = -2\eta \nabla \ln \psi = -2i\mathcal{D} \nabla \ln \psi \tag{26}$$

so that the Eq. (24) becomes:

$$\frac{\delta \boldsymbol{V}}{\delta t} = \frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left(\frac{\boldsymbol{V}^2}{2}\right) - \eta \Delta \boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left(\frac{\boldsymbol{V}^2}{2}\right) - i\mathcal{D} \Delta \boldsymbol{V} = 0 \tag{27}$$

with ψ the „complex speed potential”. $\mathcal{D}$ defines the fractal/nonfractal transition, i.e. the transition from the explicit scale dependence to scale independence. In the Nottale’s model of the scale relativity theory, $\mathcal{D}$ has the form [1,2].

$$\mathcal{D} = \frac{\lambda \cdot c}{2}$$

with λ a length scale and c the speed of light in the vacuum. This length scale is to be understood as a structure of scale space not of standard space (for example see the definition of the Compton length [1,2]).

Substituting (26) in (27) and considering the identities

$$(\nabla \ln f)^2 + \Delta \ln f = \frac{\Delta f}{f}, \; \nabla \Delta = \Delta \nabla, \; \nabla (\nabla f)^2 = 2(\nabla f \cdot \nabla)(\nabla f) \tag{28a-c}$$

which implies

$$\nabla(\frac{\Delta\psi}{\psi}) = \Delta(\nabla \ln \psi) + 2(\nabla \ln \psi \cdot \nabla)(\nabla \ln \psi) \tag{29}$$

it results

$$\frac{\delta \boldsymbol{V}}{\delta \boldsymbol{t}} = -2i\mathcal{D}\nabla\left[\frac{\partial_t \psi}{\psi} - i\mathcal{D}\frac{\Delta\psi}{\psi} + i\mathcal{D}\Delta \ln \psi\right] = 0 \tag{30}$$

The equation can now be integrated in a universal way which yields

$$\mathcal{D}^2 \Delta\psi + i\mathcal{D}\partial_t \psi = 0 \tag{31}$$

up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of ψ.

Therefore, the Schrödinger type equation is obtained as an irrotational movement of fluids having a "dispersive" coefficient depending on the length scale. Then, ψ simultaneously becomes wave-function and speed potential. This result scientifically proves the Nottale's theory [1,2].

New insight about the statistical meaning of the wave-function can also be gained from the very construction of the theory. Indeed, while in the standard quantum mechanics the existence of a complex wave-function is set as a founding axiom, in the scale relativity framework it is related to the twin speed field of the infinity family of geodesics with which the 'particle' is identified through Eq. (26) [1]. This opens the possibility of getting a derivation of Born's postulate in this context. Let us indeed derive the Born postulate using the complex speed field V

$$\boldsymbol{V} = \boldsymbol{v} - i\boldsymbol{u}, \boldsymbol{v} = 2\mathcal{D}\nabla S,\ \boldsymbol{u} = \mathcal{D}\nabla \ln \rho, \psi = \sqrt{\rho}e^{iS} \tag{32a-d}$$

with $\sqrt{\rho}$ the amplitude and S the phase of ψ. Substituting (32a-c) in (27) and separating the real and imaginary parts, we obtain:

$$\begin{aligned} &\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\left(\frac{\boldsymbol{v}^2 - \boldsymbol{u}^2}{2} - \mathcal{D}\nabla \cdot \boldsymbol{u}\right) = 0 \\ &\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\boldsymbol{v} \cdot \boldsymbol{u} + \mathcal{D}\nabla \cdot \boldsymbol{v}\right) = 0 \end{aligned} \tag{33a, b}$$

or up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of ψ ,

$$\begin{gathered}(m\partial_t \boldsymbol{v} + m(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) = -\nabla Q \\ \partial_t \rho + \nabla\cdot(\rho\boldsymbol{v}) = 0\end{gathered} \tag{34a, b}$$

with Q the quantum potential

$$Q = -2m\mathcal{D}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = -\frac{m\boldsymbol{u}^2}{2} - m\mathcal{D}\nabla\cdot\boldsymbol{u} \tag{35}$$

and m the rest mass of the test particle.

The wave function of $\Psi(\boldsymbol{r},t)$ is invariant when its phase changes by an integer multiple of 2p. Indeed, equation (32b) gives:

$$\oint m\boldsymbol{v}d\boldsymbol{r} = 2m\mathcal{D}\oint dS = 4\pi nm\mathcal{D}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{36}$$

a condition of compatibility between the SR hydrodynamic model and the wave mechanics.

For $\mathcal{D} = \hbar/2m^*$ with $m^* = 2m_e$ the mass of the Cooper type pair and $\hbar$ the reduced Planck's constant, the relation (36) becomes

$$\oint \boldsymbol{p}\cdot d\boldsymbol{r} = nh \tag{37}$$

This result can be identified with the quantification law of the gravitomagnetic flux

$$\phi_g = n\phi_{0g},\ \phi_{0g} = h/2m_e \tag{38a, b}$$

with ϕ_{0g} the gravitational fluxoid. Indeed, the generalized momentum of the Cooper type pair in the gravitomagnetic field $\boldsymbol{B}_g = \nabla\times\boldsymbol{A}_{\mathbf{g}}$ with $\boldsymbol{A}_{\mathbf{g}}$ the potential vector of the gravitomagnetic field,

$$\boldsymbol{P}_g = 2m_e\boldsymbol{v} + 2m_e\boldsymbol{A}_g = \hbar\nabla S + 2m_e\boldsymbol{A}_g \tag{39}$$

is null, i.e.

$$\boldsymbol{P}_g \equiv 0 \tag{40}$$

Through integration we obtain

$$\hbar\oint\nabla S = \pm 2\pi n\hbar = 2m_e\oint \boldsymbol{A}_g\cdot d\boldsymbol{r} = 2m_e\iint_{\partial\Sigma}\boldsymbol{B}_g\cdot d\boldsymbol{\Sigma} = 2m_e\phi_g \tag{41}$$

i.e. the (38a, b) results.

The set of Eqs. (34a,b) represents a complete system of differential Eqs. for the fields $\rho(\boldsymbol{r},t)$ and $\boldsymbol{v}(\boldsymbol{r},t)$; relation (36) relates each solution $(\rho,\boldsymbol{v})_n$ with the wave solution Y in a unique way.

The field $\rho(\boldsymbol{r},t)$ is a probability distribution, namely the probability of finding the particle in the vicinity $d\boldsymbol{r}$ of the point $\boldsymbol{r}$ at time *t*,

$$dP = \rho\,\mathrm{d}\boldsymbol{r},\quad \iiint\rho\,\mathrm{d}\boldsymbol{r} = 1, \tag{42a, b}$$

the space integral being extended over the entire area of the system. Any time variation of the probability density $\rho(\boldsymbol{r},t)$ is accompanied by a probability current $\rho\boldsymbol{v}$ pointing towards or outwards, the corresponding field point $\boldsymbol{r}$ (Eq. (34b)). Therefore, the Eq. (34b) by means of Eq. (42a,b) corresponds to the Born's postulate.

The position probability of the real velocity field $\boldsymbol{v}(\boldsymbol{r},t)$ (Eq. (34a)), varies with space and time similar to a hydrodynamic fluid placed in the quantum potential (35). The fluid (in the sense of a statistical particles ensemble) exhibits, however, an essential difference compared to an ordinary fluid: in a rotation motion $\boldsymbol{v}(\boldsymbol{r},t)$ increases (decreases) with the decreasing (increasing) distance $\boldsymbol{r}$ from the center (Eq. (36)).

The expectation values for the real velocity field and the velocity operator $\hat{\boldsymbol{v}} = -2iD\nabla$ of wave mechanics are equal

$$\langle\boldsymbol{v}\rangle = \iiint\rho\boldsymbol{v}d\boldsymbol{r} = \iiint\Psi^*\hat{\boldsymbol{v}}\Psi d\boldsymbol{r} = \langle\hat{\boldsymbol{v}}\rangle_{WM} \tag{43}$$

but in the higher-order, $|n|>2$, similar identities are invalid, namely $\langle\boldsymbol{v}^n\rangle \neq \langle\boldsymbol{v}^n\rangle_{WM}$. The expectation for the 'quantum force' vanishes at all times (theorem of Ehrenfest [14]), i.e.,

$$\langle -\nabla Q\rangle = \iiint \rho(-\nabla Q)d\boldsymbol{r} = 0 \tag{44}$$

or explicitly

$$2m\mathcal{D}^2 \iiint \rho\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)d\boldsymbol{r} = m\mathcal{D}^2 \oint (\rho\nabla\nabla \ln \rho)\cdot d\boldsymbol{\sigma} = 0 \tag{45}$$

Two types of stationary states are to be distinguished:

(i) Dynamic states. For $\partial/\partial t = 0$ and $\boldsymbol{v} \neq 0$, Eqs. (34a, b) give

$$\nabla\left(\frac{1}{2}m\boldsymbol{v}^2 - 2m\mathcal{D}^2\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) = 0, \quad \nabla(\rho\boldsymbol{v}) = 0 \tag{46a, b}$$

namely

$$\frac{1}{2}m\boldsymbol{v}^2 - 2m\mathcal{D}^2\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = E, \quad \rho\boldsymbol{v} = \nabla\times\boldsymbol{F} \tag{47a, b}$$

Consequently, inertia $m\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ and quantum forces ($-\nabla Q$) are in balance at every field point (Eq. (46a)). The sum of the kinetic energy $m\boldsymbol{v}^2/2$ and quantum potential energy (Q) is invariant, i.e., equal to the integration constant $E \neq E(\boldsymbol{r})$ (Eq. (47a)). $E \equiv \langle E\rangle$ represents the total energy of the dynamic system. The probability flow density $\rho\boldsymbol{v}$ has no sources (Eq. (46b)), *i.e.* its streamlines are closed (Eq.(47b)).

(ii) Static states. For $\partial/\partial t = 0$ and $\boldsymbol{v} = 0$, Eqs. (34a,b) give

$$\nabla\left(2m\mathcal{D}^2\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \tag{48}$$

i.e.

$$2m\mathcal{D}^2\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} = E \tag{49}$$

The quantum force ($-\nabla Q$) at any field point has the same value (Eq. (49)). The quantum potential energy (Q) is invariant, i.e., equal to the integration constant $E \neq E(\boldsymbol{r})$ (Eq. (49)). $E \equiv \langle E \rangle$ represents the total energy of the static system.

The Eq. (49) describes a specific state of matter. Interpreting Eq. (49) as the geodesics of the fractal space-time – see also Eq. (31), it results that the particles, moving along these geodesics, are simultaneously source and test particles for their own internal field. Such environment we shall name fractal superconductor.

For E>0 and denoting $\boldsymbol{\Pi} = \sqrt{\rho}\boldsymbol{u}$, the Eq. (49) by applying the ∇ operator becomes

$$\Delta \boldsymbol{\Pi} + \frac{1}{\Lambda^2}\boldsymbol{\Pi} = 0 \tag{50}$$

which describes the space oscillations of Π. Hence, the fractal space-time associated with the motion of the test particle in a fractal superconductor is endowed with regular properties (non-homogeneities of Π) like a crystal (the 'World Crystal').

Thus:

(i) the lattice constant Λ of the 'World Crystal' is

$$\Lambda = \left(\frac{2m\mathcal{D}^2}{E}\right)^{1/2} \tag{51}$$

Since

$$E = \frac{p^2}{2m} \tag{52}$$

from Eq. (51) it results

$$p = \frac{2m\mathcal{D}}{\Lambda} \tag{53}$$

In the particular case $\mathcal{D} = \hbar/2m$ (53) reduces to the de Broglie type relation

$$p = \frac{h}{\Lambda} \tag{54}$$

Consequently, the wave length associated to a moving particle may be identified with the lattice constant of the 'World Crystal', or in other words, the dual character of the particle is not its intrinsic characteristic, but a space-time property;

(ii) The lattice constant of the 'World Crystal' could be related to a 'length scale' through the relation

$$\Lambda = \left(\frac{mc^2}{2E}\right)^{1/2} \lambda = \frac{mc}{p}\lambda \tag{55}$$

The followings scale dependencies result

$$E = \frac{mc^2}{2}\frac{\lambda}{\Lambda}, \qquad p = mc\frac{\lambda}{\Lambda} \tag{56a,b}$$

For the classical case $\lambda \equiv \Lambda$, the relations (56a,b) in the form

$$E = \frac{mc^2}{2}, \qquad p = mc \tag{57a,b}$$

become scale-independent.

(iii) By means of Π, it appears that the fractal field crystallizes the space-time, while the inertial speed field induces its defects.

For E<0 and using the same expression for Π, the Eq. (49) takes the form

$$\Delta \boldsymbol{\Pi} - \frac{1}{\Lambda^2}\boldsymbol{\Pi} = 0 \tag{58}$$

This means that the 'lines' of the field Π are expelled from the considered space. By an analogy with the Meissner effect for a superconductor [15], we will name this effect the fractal Meissner type effect. From a quantum point of view, the solution of the Eq. (49) for the one-dimensional case, *i.e.* $\rho = \rho_0\,\mathrm{Exp}[-2x/\Lambda]$ with $\rho_0 = \mathrm{const.}$, defines the probability density in the tunneling effect. In this case, in order to observe the particle in the region x>0, it has to be localized inside a 'length' of order Λ where the probability is maximal.

3. CONCLUSIONS

The main conclusions of the present paper are as follows:

(i) In the topological dimension D=2 of the scale relativity theory, the generalized Schrödinger equation as an irrotational movement of Navier-Stokes type fluids having an imaginary viscosity coefficient is obtained. Then, ψ simultaneously becomes wave-function and complex speed potential;

(ii) A hydrodynamic model of the scale relativity theory is built. In this framework, the fluid equations are firstly established for the general case, and then for the stationary and static cases;

(iii) One can stress out that the quantum potential introduced in the hydrodynamic model of scale relativity comes from the non-differentiability of the quantum space-time;

(iv) In the static case the wave-particle duality and the tunneling effect are analyzed.

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M. Agop & P. Nica
Department of Physics
Technical "Gh. Asachi" University
Blvd. Mangeron, Iasi, 700029, Romania