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Some identities involving generalized Fibonacci and Lucas polynomials

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Abstract. In this paper, some relations between generalized Fibonacci polynomials and Bernoulli polynomials, generalized Lucas polynomials and Euler numbers, and generalized Lucas polynomials and Euler polynomials are established.

1. Introduction

We consider a polynomial sequence $\{W_n(x)\}$ defined by

$$W_{n+2}(x) = P(x)W_{n+1}(x) + Q(x)W_n(x), \qquad n = 0, 1, 2, \dots. \tag{1.1}$$

Obviously, when $W_0(x) = 0$, $W_1(x) = 1$, $\{W_n(x)\}$ reduces to the generalized Fibonacci polynomial sequence $\{U_n(x)\}$, and $U_n(x) = \frac{\alpha^n-\beta^n}{\alpha-\beta}$, where $\alpha = \frac{P(x)+\sqrt{\triangle x}}{2}$, $\beta = \frac{P(x)-\sqrt{\triangle x}}{2}$ and $\triangle x = P(x)^2+4Q(x)$; when $W_0(x) = 2$, $W_1(x) = P(x)$, $\{W_n(x)\}$ reduces to the generalized Lucas polynomial sequence $\{V_n(x)\}$, and $V_n(x) = \alpha^n + \beta^n$. Moreover if $P(x) = p$, $Q(x) = q$, p and q are integers and $p > 0$, sequences $\{U_n(x)\}$ and $\{V_n(x)\}$ are just the generalized Fibonacci and Lucas numbers sequences $\{U_n\}$ and $\{V_n\}$. About the above mentioned polynomials and numbers, there were many results obtained in [3]- [6].

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Recently, T. Zhang and Y. Ma [2] established the following identity:

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{U_{a_1}(x)}{a_1!}\cdots\frac{U_{a_k}(x)}{a_k!}\frac{B_{b_1}}{b_1!}\cdots\frac{B_{b_k}}{b_k!}\big(\sqrt{\triangle x}\big)^{b_1+\cdots+b_k} = \frac{(k\beta)^{n-k}}{(n-k)!}, \tag{1.2}$$

where $B_i (1 \le i \le k)$ are Bernoulli numbers, which are defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.3}$$

Furthermore, Euler numbers E_n, Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by [1]:

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \tag{1.4}$$

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.5}$$

$$\frac{2e^{tx}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.6}$$

And the following formulae hold: $B_n = B_n(0)$, $E_n = 2^n E_n(\frac{1}{2})$.

Identity (1.2) established the relation between the generalized Fibonacci polynomials and the Bernoulli numbers. In this paper, we use elementary methods to establish the relations between generalized Fibonacci polynomials and Bernoulli polynomials, generalized Lucas polynomials and Euler numbers, and generalized Lucas polynomials and Euler polynomials.

2. Main results

Theorem 2.1. *For positive integers k and n, there exists the following identity:*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{V_{a_1}(x)}{a_1!}\cdots\frac{V_{a_k}(x)}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\Big(\frac{\sqrt{\triangle x}}{2}\Big)^{b_1+\cdots+b_k} = \frac{2^{k-n}(kP(x))^n}{n!}. \tag{2.1}$$

Theorem 2.2. *For positive integers k and n, we have the identities:*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{U_{a_1}(x)}{a_1!}\cdots\frac{U_{a_k}(x)}{a_k!}\frac{B_{b_1}(\frac{-\beta}{\sqrt{\triangle x}})}{b_1!}\cdots\frac{B_{b_k}(\frac{-\beta}{\sqrt{\triangle x}})}{b_k!}\big(\sqrt{\triangle x}\big)^{b_1+\cdots+b_k} = \delta_{n,k}, \tag{2.2}$$

where δ is the Kronecker delta.

Theorem 2.3. *For positive integers k and n, we have the identity:*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n>0} \frac{V_{a_1}(x)}{a_1!}\cdots\frac{V_{a_k}(x)}{a_k!}\frac{E_{b_1}(\frac{-\beta}{\sqrt{\triangle x}})}{b_1!}\cdots\frac{E_{b_k}(\frac{-\beta}{\sqrt{\triangle x}})}{b_k!}\big(\sqrt{\triangle x}\big)^{b_1+\cdots+b_k} = 0. \tag{2.3}$$

Now we give their proofs.

Proof of Theorem 2.1: By means of $V_n(x) = \alpha^n + \beta^n$, we can easily obtain the generating function of $V_n(x)$:

$$V(x,t) = \sum_{n=0}^{\infty} \frac{V_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{\alpha^n + \beta^n}{n!} t^n = e^{\alpha t} + e^{\beta t} = e^{\beta t}(e^{t\sqrt{\triangle x}} + 1). \tag{2.4}$$

So, we have

$$V(x,t)\frac{2e^{\frac{\sqrt{\triangle x}}{2}t}}{e^{t\sqrt{\triangle x}} + 1} = 2e^{\beta t + \frac{\sqrt{\triangle x}}{2}t}.$$

From (1.4) and (2.4), we have

$$\Big(\sum_{m=0}^{\infty} \frac{V_m(x)}{m!} t^m\Big)\Big(\sum_{n=0}^{\infty} \frac{E_n}{n!} (\frac{\sqrt{\triangle x}}{2} t)^n\Big) = 2e^{\frac{P(x)}{2}t},$$

then k times on the both sides of the above identity, we get

$$\sum_{n=0}^{\infty} \sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{V_{a_1}(x)}{a_1!} \cdots \frac{V_{a_k}(x)}{a_k!} \frac{E_{b_1}}{b_1!} \cdots \frac{E_{b_k}}{b_k!} \Big(\frac{\sqrt{\triangle x}}{2}\Big)^{b_1+\cdots+b_k} t^n$$
$$= 2^k \sum_{n=0}^{\infty} \frac{1}{n!} (\frac{kP(x)}{2})^n t^n.$$

Comparing the coefficients of t^n on the above, we immediately obtain the identity (2.1). □

Proof of Theorem 2.2: In view of $U_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, the generating function of $U_n(x)$ is

$$U(x,t) = \sum_{n=0}^{\infty} \frac{U_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{\alpha^n - \beta^n}{(\alpha - \beta)n!} t^n = \frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta} = \frac{e^{\beta t}(e^{t\sqrt{\triangle x}} - 1)}{\sqrt{\triangle x}}. \tag{2.5}$$

Therefore, we have

$$\frac{U(x,t)}{t} \frac{t\sqrt{\triangle x} \cdot e^{t\sqrt{\triangle x} \cdot \frac{-\beta}{\sqrt{\triangle x}}}}{e^{t\sqrt{\triangle x}} - 1} = 1.$$

From (2.5) and (1.5), we get

$$\Big(\sum_{m=0}^{\infty} \frac{U_m(x)}{m!} t^{m-1}\Big)\Big(\sum_{n=0}^{\infty} \frac{B_n(\frac{-\beta}{\sqrt{\triangle x}})}{n!} (t\sqrt{\triangle x})^n\Big) = 1.$$

Then k times on the both sides of the identity, we have

$$\sum_{n=0}^{\infty} \sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{U_{a_1}(x)}{a_1!} \cdots \frac{U_{a_k}(x)}{a_k!} \frac{B_{b_1}(\frac{-\beta}{\sqrt{\triangle x}})}{b_1!} \cdots \frac{B_{b_k}(\frac{-\beta}{\sqrt{\triangle x}})}{b_k!}$$
$$\times\big(\sqrt{\triangle x}\big)^{b_1+\cdots+b_k} t^{n-k} = 1.$$

Comparing the coefficients of t^{n-k}, we easily obtain the identities (2.2). □

Proof of Theorem 2.3: By (2.4), we also have

$$V(x,t)\frac{2e^{t\sqrt{\triangle x}\cdot\frac{-\beta}{\sqrt{\triangle x}}}}{e^{t\sqrt{\triangle x}}+1}=2.$$

From (2.4) and (1.6), we get

$$\Big(\sum_{m=0}^{\infty}\frac{V_m(x)}{m!}t^m\Big)\Big(\sum_{n=0}^{\infty}\frac{E_n(\frac{-\beta}{\sqrt{\triangle x}})}{n!}\big(t\sqrt{\triangle x}\big)^n\Big)=2.$$

Similarly to the proof of Theorem 2.2, we immediately obtain the identity (2.3). □

In the following, we will give some special cases of the Theorem 2.1, 2.2 and 2.3.

Taking $P(x)=x$ and $Q(x)=1$, (2.1)-(2.3) reduce to the following

Corollary 2.4. *For positive integers k and n, we have the formulae:*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n}\frac{L_{a_1}(x)}{a_1!}\cdots\frac{L_{a_k}(x)}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\Big(\frac{\sqrt{\triangle x}}{2}\Big)^{b_1+\cdots+b_k}=\frac{2^{k-n}(kx)^n}{n!},$$

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n}\frac{F_{a_1}(x)}{a_1!}\cdots\frac{F_{a_k}(x)}{a_k!}\frac{B_{b_1}(\frac{-\beta}{\sqrt{\triangle x}})}{b_1!}\cdots\frac{B_{b_k}(\frac{-\beta}{\sqrt{\triangle x}})}{b_k!}\big(\sqrt{\triangle x}\big)^{b_1+\cdots+b_k}=\delta_{n,k},$$

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n>0}\frac{L_{a_1}(x)}{a_1!}\cdots\frac{L_{a_k}(x)}{a_k!}\frac{E_{b_1}(\frac{-\beta}{\sqrt{\triangle x}})}{b_1!}\cdots\frac{E_{b_k}(\frac{-\beta}{\sqrt{\triangle x}})}{b_k!}\big(\sqrt{\triangle x}\big)^{b_1+\cdots+b_k}=0,$$

where $\beta=\frac{x-\sqrt{\triangle x}}{2}$, $\triangle x=x^2+4$.

In Corollary 2.4, let $x=1$, we get

Corollary 2.5. *For positive integers k and n, we have the formulae:*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n}\frac{L_{a_1}}{a_1!}\cdots\frac{L_{a_k}}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\Big(\frac{\sqrt{5}}{2}\Big)^{b_1+\cdots+b_k}=\frac{2^{k-n}k^n}{n!},$$

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n}\frac{F_{a_1}}{a_1!}\cdots\frac{F_{a_k}}{a_k!}\frac{B_{b_1}(\frac{\sqrt{5}-1}{2\sqrt{5}})}{b_1!}\cdots\frac{B_{b_k}(\frac{\sqrt{5}-1}{2\sqrt{5}})}{b_k!}\big(\sqrt{5}\big)^{b_1+\cdots+b_k}=\delta_{n,k},$$

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n>0}\frac{L_{a_1}}{a_1!}\cdots\frac{L_{a_k}}{a_k!}\frac{E_{b_1}(\frac{\sqrt{5}-1}{2\sqrt{5}})}{b_1!}\cdots\frac{E_{b_k}(\frac{\sqrt{5}-1}{2\sqrt{5}})}{b_k!}\big(\sqrt{5}\big)^{b_1+\cdots+b_k}=0,$$

where F_n and L_n are classical Fibonacci and Lucas numbers.

If we let $P(x)=p$ and $Q(x)=q$ in (2.1)-(2.3), we have

Corollary 2.6. *For positive integers k and n, we have the formulae:*

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{V_{a_1}}{a_1!}\cdots\frac{V_{a_k}}{a_k!}\frac{E_{b_1}}{b_1!}\cdots\frac{E_{b_k}}{b_k!}\left(\frac{\sqrt{\triangle x}}{2}\right)^{b_1+\cdots+b_k} = \frac{2^{k-n}(kp)^n}{n!},$$

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n} \frac{U_{a_1}}{a_1!}\cdots\frac{U_{a_k}}{a_k!}\frac{B_{b_1}(\frac{-\beta}{\sqrt{\triangle x}})}{b_1!}\cdots\frac{B_{b_k}(\frac{-\beta}{\sqrt{\triangle x}})}{b_k!}\left(\sqrt{\triangle x}\right)^{b_1+\cdots+b_k} = \delta_{n,k},$$

$$\sum_{a_1+\cdots+a_k+b_1+\cdots+b_k=n>0} \frac{V_{a_1}}{a_1!}\cdots\frac{V_{a_k}}{a_k!}\frac{E_{b_1}(\frac{-\beta}{\sqrt{\triangle x}})}{b_1!}\cdots\frac{E_{b_k}(\frac{-\beta}{\sqrt{\triangle x}})}{b_k!}\left(\sqrt{\triangle x}\right)^{b_1+\cdots+b_k} = 0,$$

where $\beta = \frac{p-\sqrt{\triangle x}}{2}$ *and* $\triangle x = p^2 + 4q$.

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