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IMPACT OF SOME GRAPH OPERATIONS ON DOUBLE ROMAN DOMINATION NUMBER

ANU V AND APARNA LAKSHMANAN S

ABSTRACT. In this paper, we obtain bounds for the double Roman domination number, $\gamma_{dR}(G\Box H)$, in terms of $\gamma_{dR}(G)$ and $\gamma_{dR}(H)$, where $G\Box H$ denotes the cartesian product of G and H. The exact value of $\gamma_{dR}(G_{2,n})$ is obtained, where $G_{2,n} = P_2\Box P_n$. We also find the double Roman domination number of corona of G and H, $\gamma_{dR}(G \odot H)$, for $H \ncong K_1$, and obtain bounds for $\gamma_{dR}(G \odot K_1)$. The exact values of $\gamma_{dR}(G \odot K_1)$, where G is a path, a cycle, a complete graph or a complete bipartite graph are also obtained.

1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If there is no ambiguity in the choice of G, then we write $V(G)$ and $E(G)$ as V and E respectively. Let $f : V \to \{0, 1, 2, 3\}$ be a function defined on $V(G)$. Let $V_i^f = \{v \in V(G) : f(v) = i\}$. (If there is no ambiguity, V_i^f is written as V_i.) Then f is a double Roman dominating function (DRDF) on G if it satisfies the following conditions.
(i) If $v \in V_0$, then vertex v must have at least two neighbors in V_2 or at least one neighbor in V_3.
(ii) If $v \in V_1$, then vertex v must have at least one neighbor in $V_2 \cup V_3$.

The weight of a DRDF f is the sum $f(V) = \sum_{v\in V} f(v)$. The double Roman domination number, $\gamma_{dR}(G)$, is the minimum among the weights of DRDFs on G, and a DRDF on G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G [5].

The study of double Roman domination was initiated by R. A. Beeler, T. W. Haynes and S. T. Hedetniemi in [5]. They studied the relationship between double Roman domination and Roman domination and the bounds on the double Roman domination number of a graph G in terms of its domination number. They also determined a sharp upper bound on $\gamma_{dR}(G)$ in terms of the order of G and characterized the graphs attaining this bound. In [1], it is proved that the decision problem associated with $\gamma_{dR}(G)$ is NP-complete for bipartite and chordal graphs. Moreover, a characterization of graphs G with small $\gamma_{dR}(G)$ is provided. In [8], G. Hao et al. initiated the study of the double Roman domination of digraphs. L. Volkmann gave a sharp lower bound on $\gamma_{dR}(G)$ in [9]. In [3], it is proved that $\gamma_{dR}(G) + 2 \leqslant \gamma_{dR}(M(G)) \leqslant \gamma_{dR}(G) + 3$, where $M(G)$ is the Mycielskian graph of G. It is also proved that there is no relation between the double Roman domination number of a graph and its induced subgraphs. In [2], J. Amjadi et al. improved an upper bound on $\gamma_{dR}(G)$ given in [5] by showing that for any connected graph G of order n with minimum degree at least two, $\gamma_{dR}(G) \leqslant \frac{8n}{7}$.

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1.1. Basic Definitions and Preliminaries. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u : uv \in E\}$, and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. The vertices in $N(v)$ are called the neighbors of v. For a set $D \subseteq V$, the open neighborhood is $N(D) = \cup_{v \in D} N(v)$ and the closed neighborhood is $N[D] = N(D) \cup D$. A set D is a dominating set if $N[D] = V$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G.

If $f : A \to B$ is a function from A to B, and C is a subset of A, then the restriction of f to C is the function which is defined by the same rule as f but with a smaller domain set C and is denoted by $f|_C$.

A complete graph on n vertices, denoted by K_n, is the graph in which any two vertices are adjacent. A trivial graph is a graph with no edges. A path on n vertices P_n is the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. If in addition, v_n is adjacent to v_1 and $n \geq 3$, it is called a cycle of length n, denoted by C_n. A universal vertex is a vertex adjacent to all the other vertices of the graph. A pendant (or leaf) vertex of G is a vertex adjacent to only one vertex of G. The unique vertex adjacent to a pendant vetrtex is called its support vertex. A graph G is bipartite if the vertex set can be partitioned into two non-empty subsets X and Y such that every edge of G has one end vertex in X and the other in Y. A bipartite graph in which each vertex of X is adjacent to every vertex of Y is called a complete bipartite graph. If $|X| = p$ and $|Y| = q$, then the complete bipartite graph is denoted by $K_{p,q}$.

The cartesian product of two graphs G and H, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \Box H$ if *(i)* $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or *(ii)* $u_1 u_2 \in E(G)$ and $v_1 = v_2$. If $G = P_m$ and $H = P_n$, then the cartesian product $G \Box H$ is called the $m \times n$ grid graph and is denoted by $G_{m,n}$.

The corona of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \odot G_2$, is the graph obtained by taking one copy of G_1 and $|V_1|$ copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

A rooted graph is a graph in which one vertex is labelled in a special way so as to distinguish it from other vertices. The special vertex is called the root of the graph. Let G be a labelled graph on n vertices. Let H be a sequence of n rooted graphs $H_1, H_2, \ldots, H_n$. Then by $G(H)$ we denote the graph obtained by identifying the root of H_i with the i^{th} vertex of G. We call $G(H)$ the rooted product of G by H [7].

A Roman dominating function (RDF) on a graph $G = (V, E)$ is defined as a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The weight of a RDF is the value $f(V) = \sum_{v \in V} f(v)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of all possible RDFs on G.

Let (V_0, V_1, V_2, V_3) be the ordered partition of V induced by a DRDF f, where $V_i = \{v \in V : f(v) = i\}$. Note that there exists a $1-1$ correspondence between the functions f and the ordered partitions (V_0, V_1, V_2, V_3) of V. Thus we will write $f = (V_0, V_1, V_2, V_3)$.

For any graph theoretic terminology and notations not mentioned here, the readers may refer to [4]. The following propositions are useful in this paper.

Proposition 1.1. [5] *In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value* 1.

Hence, without loss of generality, in determining the value $\gamma_{dR}(G)$ we can assume that $V_1 = \phi$ for all double Roman dominating functions under consideration.

Proposition 1.2. [5] *For any graph G, $2\gamma(G) \leqslant \gamma_{dR}(G) \leqslant 3\gamma(G)$.*

Proposition 1.3. [1] *For $n \geqslant 3$,*

$$\gamma_{dR}(C_n) = \begin{cases} n, & \text{if } n \equiv 0,\, 2, 3, 4 \,(mod\, 6), \\ n+1, & \text{if } n \equiv 1,\, 5 \,(mod\, 6). \end{cases}$$

Proposition 1.4. [5] *For any nontrivial connected graph G, $\gamma_R(G) < \gamma_{dR}(G) < 2\gamma_R(G)$.*

2. Cartesian Product

The Roman domination number of cartesian product graphs is studied in [12]. As far as we know, there are no results on the double Roman domination number of cartesian product graphs. In [5], it is proved that for every graph G, $\gamma_R(G) < \gamma_{dR}(G)$. Also it is proved in [10] that $\gamma_R(G \Box H) \geqslant \gamma(G)\gamma(H)$. Hence we can deduce a general relationship between the double Roman domination number of cartesian product graphs and the domination number of its factors as follows:

$$\gamma_{dR}(G \Box H) > \gamma(G)\gamma(H)$$

Proposition 2.1. *Let G be a graph. For any γ_{dR}-function $f = (V_0, V_2, V_3)$ of G,*
(i) $|V_3| \leqslant \gamma_{dR}(G) - 2\gamma(G)$ and
(ii) $|V_2| \geqslant 3\gamma(G) - \gamma_{dR}(G)$.

Proof. Since $V_2 \cup V_3$ is a dominating set for G, we have $\gamma(G) \leqslant |V_2| + |V_3|$. So, $2\gamma(G) \leqslant 2|V_2| + 2|V_3| = \gamma_{dR}(G) - |V_3|$, and hence (i) is deduced. Also, $3\gamma(G) \leqslant 3|V_2| + 3|V_3| = \gamma_{dR}(G) + |V_2|$, and hence (ii) is obtained. □

Theorem 2.2. *For any graphs G and H, $\gamma_{dR}(G \Box H) \geqslant \frac{\gamma(G)\gamma_{dR}(H)}{2}$.*

Proof. Let $V(G)$ and $V(H)$ be the vertex sets of G and H respectively. Let $f = (V_0, V_2, V_3)$ be a γ_{dR}-function of $G \Box H$. Let $S = \{u_1, u_2, \ldots, u_{\gamma(G)}\}$ be a dominating set for G. Let $\{A_1, A_2, \ldots, A_{\gamma(G)}\}$ be a vertex partition of G such that $u_i \in A_i$ and $A_i \subseteq N[u_i]$ (Note that this partition always exists and it may not be unique). Let $\{\Pi_1, \Pi_2, \ldots, \Pi_{\gamma(G)}\}$ be the vertex partition of $G \Box H$ such that $\Pi_i = A_i \times V(H)$ for every $i \in \{1, 2, \ldots, \gamma(G)\}$.

For every $i \in \{1, 2, \ldots, \gamma(G)\}$, let $f_i : V(H) \to \{0, 2, 3\}$ be a function such that $f_i(v) = max\{f(u, v) : u \in A_i\}$. For every $j \in \{0, 2, 3\}$, let $X_j^{(i)} = \{v \in V(H) : f_i(v) = j\}$. Let $Y_0^{(i)} = \{x \in X_0^{(i)} : |N(x) \cap X_2^{(i)}| \leqslant 1$ and $N(x) \cap X_3^{(i)} = \phi\}$. Hence, we have that $f'_i = (X_0^{(i)} - Y_0^{(i)}, X_2^{(i)} + Y_0^{(i)}, X_3^{(i)})$ is a double Roman dominating function on H. Thus,

$$\begin{aligned} \gamma_{dR}(H) &\leqslant 3|X_3^{(i)}| + 2|X_2^{(i)}| + 2|Y_0^{(i)}| \\ &\leqslant 3|V_3 \cap \Pi_i| + 2|V_2 \cap \Pi_i| + 2|Y_0^{(i)}|. \end{aligned}$$

Hence,

$$\begin{aligned} \gamma_{dR}(G \Box H) &= 3|V_3| + 2|V_2| \\ &= \sum_{i=1}^{\gamma(G)} [3|V_3 \cap \Pi_i| + 2|V_2 \cap \Pi_i|] \\ &\geqslant \sum_{i=1}^{\gamma(G)} [\gamma_{dR}(H) - 2|Y_0^{(i)}|] \\ &= \gamma(G)\gamma_{dR}(H) - 2\sum_{i=1}^{\gamma(G)} |Y_0^{(i)}|. \end{aligned}$$

So,

$$\sum_{i=1}^{\gamma(G)} |Y_0^{(i)}| \geqslant \frac{1}{2}[\gamma(G)\gamma_{dR}(H) - \gamma_{dR}(G\Box H)]. \tag{2.1}$$

Now, for every $v \in V(H)$, let $Z^v \in \{0,1\}^{\gamma(G)}$ be a binary vector associated to v as follows:

$$Z_i^v = \begin{cases} 1, & if \quad v \in Y_0^{(i)}, \\ 0, & otherwise. \end{cases}$$

Let t_v be the number of components of Z^v equal to one. Hence,

$$\sum_{v \in V(H)} t_v = \sum_{i=1}^{\gamma(G)} |Y_0^{(i)}|. \tag{2.2}$$

Note that, if $Z_i^v = 1$ and $u \in A_i$, then vertex (u,v) belongs to V_0. Moreover (u,v) is not adjacent to any vertex of $V_3 \cap \Pi_i$ and is adjacent to at most one vertex of $V_2 \cap \Pi_i$. So, since V_0 is double Roman dominated by $V_2 \cup V_3$, there exists $u' \in X_v = \{x \in V(G) : (x,v) \in V_2 \cup V_3\}$ which is adjacent to u. Hence, $S_v = (S - \{u_i \in S : Z_i^v = 1\}) \cup X_v$ is a dominating set for G.

Now, if $t_v > |X_v|$, then we have

$$\begin{aligned} |S_v| &\leqslant |S| - t_v + |X_v| \\ &= \gamma(G) - t_v + |X_v| \\ &< \gamma(G) - t_v + t_v = \gamma(G), \end{aligned}$$

which is a contradiction. So, we have $t_v \leq |X_v|$ and we obtain

$$\sum_{v \in V(H)} t_v \leq \sum_{v \in V(H)} |X_v| = |V_2 \cup V_3|$$

which leads to

$$2 \sum_{v \in V(H)} t_v \leq 2|V_2| + 2|V_3| \leq \gamma_{dR}(G\Box H). \tag{2.3}$$

Thus, by (1), (2) and (3), we deduce $\gamma_{dR}(G\Box H) \geqslant \frac{\gamma(G)\gamma_{dR}(H)}{2}$. □

Proposition 1.2 and Theorem 2.2 lead to the following result.

Corollary 2.3. *For any graphs G and H, $\gamma_{dR}(G\Box H) \geqslant \frac{\gamma_{dR}(G)\gamma_{dR}(H)}{6}$.*

Theorem 2.4. *For any graphs G and H of orders n_1 and n_2 respectively, $\gamma_{dR}(G\Box H) \leqslant min\{n_2\gamma_{dR}(G), n_1\gamma_{dR}(H)\}$.*

Proof. Let f_1 be a γ_{dR}-function of G. Let $f : V(G) \times V(H) \to \{0,2,3\}$ be a function defined by $f(u,v) = f_1(u)$. If there exists a vertex $(u,v) \in V(G) \times V(H)$ such that $f(u,v) = 0$, then $f_1(u) = 0$. Since f_1 is a γ_{dR}-function of G, there exists either $u_1 \in N_G(u)$ such that $f_1(u_1) = 3$ or $u_2, u_3 \in N_G(u)$ such that $f_1(u_2) = f_1(u_3) = 2$. Hence, we obtain that there exists either $(u_1,v) \in N_{G\Box H}((u,v))$ with $f((u_1,v)) = 3$ or $(u_2,v),(u_3,v) \in N_{G\Box H}((u,v))$ with

$f((u_2, v)) = f((u_3, v)) = 2$. So, f is a DRDF of $G \Box H$. Therefore,

$$\begin{aligned}\gamma_{dR}(G \Box H) &\leqslant \sum_{(u,v) \in V(G) \times V(H)} f((u,v)) \\ &= \sum_{v \in V(H)} \sum_{u \in V(G)} f_1(u) \\ &= \sum_{v \in V(H)} \gamma_{dR}(G) = n_2 \gamma_{dR}(G).\end{aligned}$$

Similarly, we can prove that $\gamma_{dR}(G \Box H) \leqslant n_1 \gamma_{dR}(H)$ and hence the result is true. □

In [6], it is proved that for the $2 \times n$ grid graph $G_{2,n}$, $\gamma_R(G_{2,n}) = n + 1$. Hence it is natural to study the double Roman domination number of grid graphs. For $n = 2$, $G_{2,n}$ is C_4 and by proposition 1.3, $\gamma_{dR}(C_4) = 4$. So, in the next theorem, we omit the case when $n = 2$.

Theorem 2.5. *For the $2 \times n$ grid graph $G_{2,n}, n \neq 2, \gamma_{dR}(G_{2,n}) = \lfloor \frac{3n+4}{2} \rfloor$.*

Proof. Let the vertices of $G_{2,n}$ be denoted by $(u_1, v_1), \ldots, (u_1, v_n), (u_2, v_1), \ldots, (u_2, v_n)$ and define a DRDF f as follows: If n is odd,

$$f(u_i, v_j) = \begin{cases} 3, & for \;\; i = 1 \; and j = 3 + 4k; \; i = 2 \; and \; j = 1 + 4k \; for \; k \geqslant 0 \; and \; j \leqslant n, \\ 0, & otherwise. \end{cases}$$

If n is even,

$$f(u_i, v_j) = \begin{cases} 3, & for \;\; i = 1 \; and \; j = 3 + 4k; \; i = 2 \; and \; j = 1 + 4k \; for \; k \geqslant 0 \; and \; j < n, \\ 2, & for \; i = 1 \; \& \; j = n, \; if \;\; n \equiv 2 \; (mod \; 4); \; i = 2 \; \& \; j = n, \; if \; n \equiv 0 \; (mod \; 4), \\ 0, & otherwise. \end{cases}$$

It can be easily verified that f is a DRDF and

$$f(V) = \begin{cases} \frac{3(n+1)}{2}, & if \; n \; is \; odd, \\ \frac{3n}{2} + 2, & if \; n \; is \; even. \end{cases}$$

i.e., $f(V) = \lfloor \frac{3n+4}{2} \rfloor$ and hence $\gamma_{dR}(G_{2,n}) \leq \lfloor \frac{3n+4}{2} \rfloor$.

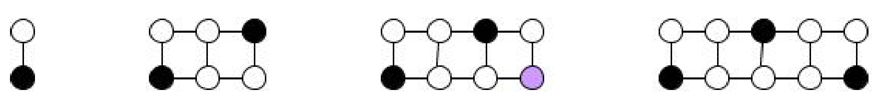

FIGURE 1. DRDF f for $G_{2,n}, n = 1, 3, 4, 5$. Black circles denote vertices in V_3, grey circle denote vertex in V_2 and empty circles denote vertices in V_0.

For the reverse inequality, let $\{x_1, x_2, \ldots, x_\gamma\}$ be any dominating set for $G_{2,n}$. If n is odd, $\{N[x_1], N[x_2], \ldots, N[x_\gamma]\}$ is a partition of vertex set of $G_{2,n}$ and $|N[x_i]| \geqslant 3$, for $i = 1, 2, \ldots, \gamma$. So we have to give 3 to each x_i, $i = 1, 2, \ldots, \gamma$, under any DRDF and hence $\gamma_{dR}(G_{2,n}) \geqslant \frac{3(n+1)}{2}$. (Note that $\gamma(G_{2,n}) = \lceil \frac{n+1}{2} \rceil$). If n is even, let $\{A_1, A_2, \ldots, A_\gamma\}$ be any partition of vertex set of $G_{2,n}$ such that $x_i \in A_i$ and $A_i \subseteq N[x_i]$, $i = 1, 2, \ldots, \gamma$. Then $|A_i| = 1$ for at most one i, say k, and $|A_i| \geq 3$, $i \neq k$. So we have to give 3 to each x_i, $i = 1, 2, \ldots, \gamma$; $i \neq k$ and 2 to x_k under any DRDF. Hence $\gamma_{dR}(G_{2,n}) \geqslant \frac{3n}{2} + 2$, if n is even. Hence the result follows. □

3. Corona Operator

In this section, first we find the double Roman domination number of $G \odot H$, where $H \ncong K_1$, and obtain bounds for $\gamma_{dR}(G \odot K_1)$. We also prove that these bounds are strict and obtain a realization for every value in the range of the bounds obtained. The exact values of $\gamma_{dR}(G \odot K_1)$ where G is a path, a cycle, a complete graph or a complete bipartite graph are also obtained. Also we prove that the value of $\gamma_{dR}((G \odot K_1) \odot K_1)$ depends only on the number of vertices in G.

Proposition 3.1. *For every graph G and every $H \ncong K_1$, $\gamma_{dR}(G \odot H) = 3n$, where $n = |V(G)|$.*

Proof. The function which assigns 3 to all vertices of G and 0 to all other vertices is a DRDF of $G \odot H$ so that $\gamma_{dR}(G \odot H) \leqslant 3n$. Also, there are n mutually exclusive copies of H each of which requires at least weight 3 in a DRDF. Hence the result is true. □

Proposition 3.2. *For any graph G, $2n + 1 \leqslant \gamma_{dR}(G \odot K_1) \leqslant 3n$, where $n = |V(G)|$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let u_i' be the leaf neighbor of u_i in $G \odot K_1$. We get a DRDF of $G \odot K_1$ by simply assigning the value 3 to each u_i' so that $\gamma_{dR}(G \odot K_1) \leqslant 3n$.

To prove the left inequality, let f be any DRDF of $G \odot K_1$. Being a pendant vertex, each u_i' must be either in $V_2^f \cup V_3^f$ or adjacent to a vertex in V_3^f. Also, if $u_i' \in V_2^f$, for all $i = 1, 2, \ldots, n$, none of the vertices u_i can be double Roman dominated by u_i' alone. Therefore, $f(V) \geqslant 2n+1$ and hence $\gamma_{dR}(G \odot K_1) \geqslant 2n + 1$. □

Proposition 3.3. *Any positive integer a is realizable as the double Roman domination number of $G \odot K_1$ for some graph G if and only if $2n + 1 \leqslant a \leqslant 3n$, where $n = |V(G)|$.*

Proof. Let G be a graph with $|V(G)| = n$. If $\gamma_{dR}(G \odot K_1) = a$, then by Proposition 3.2, $2n + 1 \leqslant a \leqslant 3n$.

To prove the converse part, take G as $K_{1,m} \cup (n - m - 1)K_1$. For definiteness, let $u_1, u_2, \ldots, u_{m+1}$ be the vertices of $K_{1,m}$ in which u_1 is the universal vertex and $u_{m+2}, \ldots, u_n$ be the isolated vertices in G. Let u_i' be the leaf neighbor of u_i in $G \odot K_1$. Define f on $V(G \odot K_1)$ as follows:

$$f(v) = \begin{cases} 3, & if \quad v = u_i \ for \ i = 1, m+2, m+3, \ldots, n, \\ 2, & if \quad v = u_i' \ for \ i = 2, 3, \ldots, m+1, \\ 0, & otherwise. \end{cases}$$

Clearly, f is a γ_{dR}-function with weight $3(n-m) + 2m = 3n - m$. As m varies from 0 to $n-1$, we get G with $\gamma_{dR}(G \odot K_1)$ varies from $3n$ to $2n + 1$. (Note that $K_{1,0}$ is considered as K_1.) Hence, the result is true. □

Proposition 3.4.

$$\gamma_{dR}(P_n \odot K_1) = \begin{cases} \frac{7n}{3}, & if \quad n = 3k, \\ \frac{7n+2}{3}, & if \quad n = 3k+1, \\ \frac{7n+1}{3}, & if \quad n = 3k+2. \end{cases}$$

Proof. Let $P_n : u_1 u_2 \ldots u_n$ be a path and let u_i' be the vertex adjacent to u_i in $P_n \odot K_1$. In a γ_{dR}-function, a pendant vertex must be either in V_2 or adjacent to a vertex in V_3. If $n = 3k$ or $3k + 2$, we have a γ_{dR}-function of P_n with $V_2 = \phi$. If $n = 3k + 1$, let f be a DRDF with minimal weight such that $V_2 = \phi$. Define g on $V(P_n \odot K_1)$ as follows:

$$g(v) = \begin{cases} 3, & if \quad v = u_i \in V_3^f, \\ 2, & if \quad v = u_i' \ with \ u_i \in V_0^f, \\ 0 & otherwise. \end{cases}$$

Clearly, g is a γ_{dR}-function. Hence if $n = 3k$, $\gamma_{dR}(P_n \odot K_1) = 3.\frac{n}{3} + 2.\frac{2n}{3} = \frac{7n}{3}$. If $n = 3k+1$, $\gamma_{dR}(P_n \odot K_1) = 3(\frac{n-1}{3} + 1) + 2\frac{2(n-1)}{3} = \frac{7n+2}{3}$. If $n = 3k+2$, $\gamma_{dR}(P_n \odot K_1) = 3(\frac{n-2}{3} + 1) + 2(\frac{2(n-2)}{3} + 1) = \frac{7n+1}{3}$. □

Proposition 3.5.

$$\gamma_{dR}(C_n \odot K_1) = \begin{cases} \frac{7n}{3}, & if \quad n = 3k, \\ \frac{7n+2}{3}, & if \quad n = 3k+1, \\ \frac{7n+1}{3}, & if \quad n = 3k+2. \end{cases}$$

Proof. The proof is similar to that of P_n. □

Proposition 3.6. $\gamma_{dR}(K_n \odot K_1) = 2n + 1$.

Proof. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and let u_i' be the leaf neighbor of u_i in $K_n \odot K_1$. A γ_{dR}-function can be obtained for $K_n \odot K_1$ by assigning 3 to any one vertex, say u_1 of K_n, 2 to u_i' with $i \neq 1$, and 0 to all other vertices. Hence $\gamma_{dR}(K_n \odot K_1) = 2n + 1$. □

Proposition 3.7.

$$\gamma_{dR}(K_{p,q} \odot K_1) = \begin{cases} 2(p+q)+1, & if \quad p = 1 \ or \ q = 1, \\ 2(p+q+1), & otherwise. \end{cases}$$

Proof. Let $V(K_{p,q}) = \{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q\}$ and let u_i' be the leaf neighbor of u_i, for $i = 1, 2, \ldots, p$ and v_j' be the vertex adjacent to v_j, for $j = 1, 2, \ldots, q$ in $K_{p,q} \odot K_1$. By the left inequality of Proposition 3.2, $\gamma_{dR}(K_{p,q} \odot K_1) \geqslant 2(p+q) + 1$.

Case 1 : $p = 1$ or $q = 1$.

For definiteness, let $p = 1$. Then the function f defined by

$$f(u) = \begin{cases} 3, & for \ u = u_1, \\ 2, & for \ all \ u = u_i', i = 2, 3, \ldots, p \ and \ u = v_j', j = 1, 2, \ldots, q, \\ 0, & otherwise, \end{cases}$$

is a DRDF of $K_{p,q} \odot K_1$ with weight $2(p+q)+1$. Therefore, $\gamma_{dR}(K_{p,q} \odot K_1) = 2(p+q)+1$.

Case 2 : $p, q \geqslant 2$.

Define f as follows:

$$f(u) = \begin{cases} 3, & for \ u = u_1 \ and \ u = v_1, \\ 2, & for \ all \ u = u_i', \ i = 2, 3, \ldots, p \ and \ u = v_j', \ j = 2, 3, \ldots, q, \\ 0, & otherwise. \end{cases}$$

f is a DRDF of $K_{p,q} \odot K_1$ with weight $6 + 2(p+q-2) = 2(p+q+1)$ and hence $\gamma_{dR}(K_{p,q} \odot K_1) \leqslant 2(p+q+1)$. For the reverse inequality, if possible suppose that there exists a DRDF g of $K_{p,q} \odot K_1$ with weight $2(p+q)+1$. Out of $p+q$ pendant vertices in $K_{p,q} \odot K_1$, let k vertices be in V_2^g. Then the remaining $p+q-k$ pendant vertices are either in V_3^g or adjacent to vertices in V_3^g. Hence the weight of g, $g(V) = 2(p+q)+1 \geqslant 2k + 3(p+q-k)$ which implies $k \geqslant p+q-1$. If $k > p+q-1$, then $k = p+q$ so that all the pendant vertices are in V_2^g and none of them can double Roman dominate any of the non pendant vertices. Therefore, we need more vertices having non zero values under g which contradicts the fact that $g(V) = 2(p+q+1)$. If $k = p+q-1$, then one pendant vertex, say x, is either in V_3^g or adjacent to a vertex in V_3^g. If x is in V_3^g, then x can double Roman dominate only its support vertex. If x is adjacent to a vertex in V_3^g, then its support vertex, say y, is in V_3^g and y cannot double Roman dominate any of the remaining vertices in the partite set of $K_{p,q}$ containing y. In either case, we need more vertices having non zero values under g, which again leads to a contradiction as above. Hence the result is true.

□

Proposition 3.8. *For any graph G, $\gamma_{dR}((G \odot K_1) \odot K_1) = 5n$, where $n = |V(G)|$.*

Proof. Let G be a graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let v_i be the leaf neighbor of u_i in $G \odot K_1$. Let u_i' and v_i' be the leaf neighbors of u_i and v_i respectively in $(G \odot K_1) \odot K_1$. Then $(G \odot K_1) \odot K_1$ contains n vertex disjoint P_4's, $u_i'u_iv_iv_i'$, for $i = 1, 2, \ldots, n$. Let f be any $DRDF$ on $(G \odot K_1) \odot K_1$. Then the two pendant vertices, u_i' and v_i', in each P_4 should be either in $V_2^f \cup V_3^f$ or adjacent to a vertex in V_3^f. If all the pendant vertices are in V_2^f, then to double Roman dominate non pendant vertices, u_i and v_i, we need more vertices with non zero values in each P_4. Also note that pendant vertices have no common neighbors. Hence, under f, the sum of the values of vertices in each of the above mentioned P_4's must be at least 5. Therefore, $f(V) \geqslant 5n$.

To prove the reverse inequality, define g as follows:

$$g(u) = \begin{cases} 3, & for\ u = u_i, \\ 2, & for\ \ u = v_i', \\ 0, & otherwise. \end{cases}$$

Clearly g is a DRDF on $(G \odot K_1) \odot K_1$ with $g(V) = 5n$. Hence, the result is true. □

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Department of Mathematics, St. Peter's College, Kolenchery - 682 311, Kerala, India
E-mail address: anusaji1980@gmail.com

Department of Mathematics, Cochin University of Science and Technology, Cochin - 22, Kerala, India
E-mail address: aparnals@cusat.ac.in, aparnaren@gmail.com