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HOSOYA INDEX OF T LEAF CLOVER KEYCHAIN GRAPH AND ITS SEQUENCE

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Abstract. The Hosoya index of a graph is defined as the total number of matchings of the graph. The Hosoya index is a typical example of a graph invariants used in mathematical chemistry for quantifying relevant details of molecular structure. T leaf clover keychain graph is drawn throught the t leaf clover keychain.By analyzing method of components in graph theory, it is a method of counting $S^{(n)}$ - factors by using graph theory covering method, and recursive counting method with complete graph as branch. Computing the explicit formula of Hosoya index of four leaf grass keychain graph, and extended it to the general t leaf grass keychain graph, further, looking for laws of their Hosoya index sequences. Finally, solved the explicit formulas of Hosoya indexs of four leaf grass keychain graph, and t leaf grass keychain graph. Interestingly, found the second family of Fibonacci sequences with different initial values, which is of value for combinatorial chemistry and graph theory.

1. Introduction

Four leaf clover, also known as lucky grass, is a rare variety of clover or alfalfa. Among one hundred thousand alfalfa plants, only one four leaf clover may be found, because the probability is one-hundred-thousandth, the four leaf clover has become an internationally recognized symbol of luck. Four leaf clover, its four leaves represent love, health, reputation and wealth. Owning it will bring you good luck. Four leaf clover leads to four leaf tree, a tree with four leaves and n vertices is called a four leaf tree. In[1] , the rules of Hosoya indexs of four leaf trees and t leaf trees are discussed. Abstracting t leaf clover keychain graph from t leaf clover keychain, in this paper, the author will research Hosoya index of t leaf clover keychain graph and its sequence.

The Fibonacci numbers, commonly denoted by F_n forms a sequence, called Fibonacci sequence such that each number is the sum of the two preceding ones, starting from 0 and1. That is

$F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$[2−3].

Fibonacci sequence:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

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Each Lucas number L_n is defined to be the sum of the two immediate previous terms, starting from 2 and 1. That is

$L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$[4−5].

Lucas sequence:

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...

2. Preliminaries

2.1 Definitions

Definition 2.1. [6]. Let $S^{(n)} = \{K_i : 1 \leq i \leq n\}$, $n \geq 1$, and K_i be a complete graph with i vertices, if M is a subgraph of a graph G, and any component of M is isomorphic to some element of $S^{(n)} = \{K_i : 1 \leq i \leq n\}$, then M is called one $S^{(n)}$-subgraph, if M is a spanning subgraph of the graph G, then M is called one $S^{(n)} = \{K_i : 1 \leq i \leq n\}$-factor .

The number of all $S^{(n)}$ -factors of a graph G denoted by $A(G)$.

Definition 2.2. The number of all k-matchings of a graph is called Hosoya index. Hosoya index of a graph G denoted by Z (G).

2.2 Basic Lemmas

Lemma 2.3. [7] . *If G is a graph with n vertices, P is a fixed vertex of the graph G, all complete graphs through the vertex P are* $K_{i_1}, K_{i_2}, \ldots, K_{i_r}$*, then*

$A(G) = \sum_{j=1}^{r} A(G - V(K_{i_j}))$,

where $A(G - V(K_{i_j}))$ *is the graph that vertices*$V(K_{i_j})$ *and these edgs incident to* $V(K_{i_j})$ *are all deleted.*

Lemma 2.4. *If* $G \cap H = \phi$*, then* $A(G \cup H) = A(G) \cdot A(H)$*.*

Lemma 2.5. [7] *If* P_n *is any path with the length n, and has (n+1) vertices, then* $A(P_n) = F_{n+2}$*, where* F_{n+2} *is the (n+2)th Fibonacci number.*

Lemma 2.6. [7] *Let* C_n *be a cycle with n vertice, and* $n \geq 4$*. Then* $A(C_n) = L_n$*, where* L_n *is the nth Lucas number.*

Lemma 2.7. [8] *If G is a graph with n vertices, and without* K_3 *subgraph, then Hosoya index of G is equal to the number of all* S^n*-factors in G:* $Z(G) = A(G)$*.*

3. Main results

3.1 Explicit formula of Hosoya index of four leaf clover keychain graph

The explicit formula of Hosoya index of four leaf clover keychain graph is given and proved below. Let H_n^4 denote four leaf clover keychain graph.

Theorem 3.1. *Four leaf clover keychain graph is Fig.1 as follows, Hosoya index of four leaf clover keychain graph*

$Z(H_n^4) = 5L_{n-5} + F_{n-5}$; $n \geq 9$.

*Fig.*1

Proof. Using the branch analysis method of graph, analyze the given point V_{n-4}. All complete graphs passing through the vertex of V_{n-4} are V_{n-4}, $V_{n-4}V_{n-5}$, $V_{n-4}V_{n-3}$, $V_{n-4}V_{n-2}$, $V_{n-4}V_{n-1}$ and $V_{n-4}V_n$, one K_1 and five K_2, without K_i $(3 \leq i \leq n)$ subgraph, the discussion is divided into three situations:

Case 1 If the complete graph passing through V_{n-4} is K_1 , that is, piont V_{n-4}, V_{n-4} is a complete branch, then the number of $S^{(n)}$ - factors is as follows:

$$\begin{aligned} A(H_n^4 - K_1) &= A(V_{n-3} \cup V_{n-2} \cup V_{n-1} \cup V_n \cup C_{n-5}) \\ &= A(V_{n-3})A(V_{n-2})A(V_{n-1})A(V_n)A(C_{n-5}) \\ &= 1 \times 1 \times 1 \times 1 \times A(C_{n-5}) \\ &= A(C_{n-5}) \\ &(Lemma 2.4) \end{aligned}$$

According to Lemma 2.6,

$$A(H_n^4 - K_1) = L_{n-5}$$

Case 2 If the complete graphs passing through V_{n-4} are $V_{n-4}V_{n-3}$, $V_{n-4}V_{n-2}$, $V_{n-4}V_{n-1}$, $V_{n-4}V_n$, which are symmetric , K_2 is the complete branch of two vertices, then the number of $S^{(n)}$ - factors is as follows:

$$\begin{aligned} 4A(H_n^4 - K_2) &= 4A(V_{n-2} \cup V_{n-1} \cup V_n \cup C_{n-5}) \\ &= 4A(V_{n-2})A(V_{n-1})A(V_n)A(C_{n-5}) \\ &= 4 \times 1 \times 1 \times 1 \times A(C_{n-5}) \\ &= 4A(C_{n-5}) \\ &= 4L_{n-5} \end{aligned}$$

Case 3 If the complete graph passing through V_{n-4} is $V_{n-4}V_{n-5}$, K_2 is the complete branch of two vertices, then the number of $S^{(n)}$ - factors is as follows:

$$\begin{aligned} A(H_n^4 - K_2) &= A(V_{n-3} \cup V_{n-2} \cup V_{n-1} \cup V_n \cup P_{n-7}) \\ &= A(V_{n-3})A(V_{n-2})A(V_{n-1})A(V_n)A(P_{n-7}) \\ &= 1 \times 1 \times 1 \times 1 \times A(P_{n-7}) \\ &= A(P_{n-7}) \end{aligned}$$

According to Lemma 2.5,

$$A(H_n^4 - K_2) = F_{n-5}$$

In conclusion, according to Lemma 2.3,

$$A(H_n^4) = \sum_{j=1}^{r} A(H_n^4 - V(K_{i_j}))$$

$$\begin{aligned} &= A(H_n^4 - K_1) + 4A(H_n^4 - K_2) + A(H_n^4 - K_2) \\ &= L_{n-5} + 4L_{n-5} + F_{n-5} \\ &= 5L_{n-5} + F_{n-5}; \end{aligned}$$

$n \geq 9$.

For Fig. 1 without K_3 subgraph, according to Lemma 2.7,

$$Z(H_n^4) = A(H_n^4) = 5L_{n-5} + F_{n-5};$$

$n \geq 9$. □

Example 1 Compute Hosoya index of four leaf clover keychain graph when n=9, 10, 11, 12, 13, 14, 15, 16.

Solution According to Theorem 3.1, $Z(H_9^4) = 5L_4 + F_4 = 5 \times 7 + 3 = 38$,

$Z(H_{10}^4) = 5L_5 + F_5 = 5 \times 11 + 5 = 60$, $Z(H_{11}^4) = 5L_6 + F_6 = 5 \times 18 + 8 = 98$,

$Z(H_{12}^4) = 5L_7 + F_7 = 5 \times 29 + 13 = 158$, $Z(H_{13}^4) = 5L_8 + F_8 = 5 \times 47 + 21 = 256$,

$Z(H_{14}^4) = 5L_9 + F_9 = 5 \times 76 + 34 = 414$, $Z(H_{15}^4) = 5L_{10} + F_{10} = 5 \times 123 + 55 = 670$, $Z(H_{16}^4) = 5L_{11} + F_{11} = 5 \times 199 + 89 = 1084$.

Interestingly, the initial value $f_0 = 38, f_1 = 60$, the general term , $f_n = f_{n-1} + f_{n-2}$, it's like Fibonacci sequence, but the initial value is different. The following will prove that this law is correct.

Proof. For $f_n = 5L_{n-5} + F_{n-5}$, hence $f_{n-1} = 5L_{n-6} + F_{n-6}$ and $f_{n-2} = 5L_{n-7} + F_{n-7}$.

Because of $f_{n-1} + f_{n-2} = 5L_{n-6} + F_{n-6} + 5L_{n-7} + F_{n-7} = 5(L_{n-6} + L_{n-7}) + (F_{n-6} + F_{n-7}) = 5L_{n-5} + F_{n-5}$, then $f_n = f_{n-1} + f_{n-2}$. □

Table1 is the initial values of Hosoya index of four leaf clover keychain graph.

Table1

The Hosoya index sequence of four leaf clover keychain graph with n vertices and 4 leaves is obtained as follows:

38, 60, 98, 158, 256, 414, 670, 1 084, 1 754, 2 838, 4 592, 7 430, 12 022, 19 452 , 31 474, 50 926, 82 400, 133 326, 215 726, 349 052, 564 778, 913 830, 1 478 608, ...

3.2 Explicit formula of Hosoya index of t leaf clover keychain graph

Let H_n^t denote t leaf clover keychain graph with n vertices.

Theorem 3.2. *T leaf clover keychain graph is Fig.2 as follows, Hosoya index of t leaf clover keychain graph*

$Z(H_n^t) = (t+1)L_{n-t-1} + F_{n-t-1}$; $n \geq t+5$, $t \geq 2$.

Fig.2

Proof. Analogous to four leaf clover keychain graph, using the branch analysis method of graph, analyze the given point V_{n-t}. All complete graphs passing through the vertex of V_{n-t} are V_{n-t}, $V_{n-t}V_{n-t-1}$, $V_{n-t}V_{n-t+1}$, $V_{n-t}V_{n-t+2}$, ..., $V_{n-t}V_{n-1}$, and $V_{n-t}V_n$,, one K_1 and $(n+1)K_2$, without $K_i(3 \leq i \leq n)$ subgraph, the discussion is divided into three situations:

Case 1 If the complete graph passing through V_{n-t} is K_1 , that is, piont V_{n-t} , V_{n-t} is a complete branch, then the number of $S^{(n)}$ - factors is as follows:

$$A(H_n^t - K_1) = A(V_{n-t+1} \cup V_{n-t+2} \cup \ldots \cup V_{n-1} \cup V_n \cup C_{n-t-1})$$

$$= A(V_{n-t+1})A(V_{n-t+2})\dots A(V_{n-1})A(V_n)A(C_{n-t-1})$$
$$= 1 \times 1 \times \dots 1 \times 1 \times A(C_{n-t-1})$$
$$= A(C_{n-t-1})$$
$$(Lemma 2.4)$$

According to Lemma 2.6,

$$A(H_n^t - K_1) = L_{n-t-1}$$

Case 2 If the complete graphs passing through V_{n-t} are $V_{n-t}V_{n-t+1}$, $V_{n-t}V_{n-t+2}$, ..., $V_{n-t}V_{n-1}$, and $V_{n-t}V_n$, which are symmetric , K_2 is the complete branch of two vertices, then the number of $S^{(n)}$ - factors is as follows:

$$tA(H_n^t - K_2) = tA(V_{n-t+2} \cup V_{n-t+3} \cup \dots \cup V_{n-1} \cup V_n \cup C_{n-t-1})$$
$$= tA(V_{n-t+2})A(V_{n-t+3})\dots A(V_{n-1})A(V_n)A(C_{n-t-1})$$
$$= t \times 1 \times 1 \times \dots 1 \times 1 \times A(C_{n-t-1})$$
$$= tA(C_{n-t-1}) = tL_{n-t-1}$$

Case 3 If the complete graph passing through V_{n-t} is $V_{n-t}V_{n-t-1}$, K_2 is the complete branch of two vertices, then the number of $S^{(n)}$ - factors is as follows:

$$A(H_n^t - K_2) = A(V_{n-t+1} \cup V_{n-t+2} \cup \dots \cup V_{n-1} \cup V_n \cup P_{n-t-3})$$
$$= A(V_{n-t+1})A(V_{n-t+2})\dots A(V_{n-1})A(V_n)A(P_{n-t-3})$$
$$= 1 \times 1 \times \dots 1 \times 1 \times A(P_{n-t-3})$$
$$= A(P_{n-t-3}) = F_{n-t-1}$$
$$(Lemma 2.5)$$

Summrize above, according to Lemma 2.3,

$$A(H_n^t) = \sum_{j=1}^{r} A(H_n^t - V(K_{i_j}))$$

$$= A(H_n^t - K_1) + tA(H_n^t - K_2) + A(H_n^t - K_2)$$
$$= L_{n-t-1} + tL_{n-t-1} + F_{n-t-1}$$
$$= (t+1)L_{n-t-1} + F_{n-t-1}$$

For Fig.2 without K3 subgraph, according to Lemma 2.7,

$$Z(H_n^t) = A(H_n^t) = (t+1)L_{n-t-1} + F_{n-t-1};$$

$n \geq t+5$, $t \geq 2$ □

Corollary 3.3. *As shown in Fig. 3, the Hosoya index of the two leaf clover keychain graph is as follows*

$$Z(H_n^2) = 3L_{n-3} + F_{n-3};$$

$n \geq 7$

$Fig.3$

Proof. In Theorem 3.2, put t=2, then

$$Z(H_n^2) = 3L_{n-3} + F_{n-3};$$

$n \geq 7$ □

The same interesting is that, the initial values $f_0 = 24, f_1 = 38$, the general term: $f_n = f_{n-1} + f_{n-2}$, , it's like Fibonacci sequence, but the initial value is different. The following will prove that this law is correct.

Proof. For $f_n = 3L_{n-3} + F_{n-3}$, hence $f_{n-1} = 3L_{n-4} + F_{n-4}$, and $f_{n-2} = 3L_{n-5} + F_{n-5}$. Because of

$$f_{n-1} + f_{n-2} = 3L_{n-4} + F_{n-4} + 3L_{n-5} + F_{n-5} = 3(L_{n-4} + L_{n-5}) + (F_{n-4} + F_{n-5})$$
$$= 3L_{n-3} + F_{n-3},$$

then
$f_n = f_{n-1} + f_{n-2}$. □

The Hosoya index sequence of two leaf clover keychain graph with n vertices and 2 leaves is obtained as follows:
24, 38, 62, 100, 162, 262, 424, 686, 1 110, 1 796, 2 906, 4 702, 7 608, 12 310 , 19 918, 32 228, ...

Corollary 3.4. *As shown in Fig. 4, the Hosoya index of the three leaf clover keychain graph is as follows*

$$Z(H_n^3) = 4L_{n-4} + F_{n-4};$$

$n \geq 8$.

$Fig.4$

Proof. In Theorem 3.2, let t=3, then

$$Z(H_n^3) = 4L_{n-4} + F_{n-4};$$

$n \geq 8$. □

For the Hosoya index sequence, the initial values $f_0 = 31, f_1 = 49$, the general term :$f_n = f_{n-1} + f_{n-2}$, , it's like Fibonacci sequence, but the initial value is different. The following will prove that this law is correct.

Proof. With $f_n = 4L_{n-4} + F_{n-4}$, so $f_{n-1} = 4L_{n-5} + F_{n-5}$, and $f_{n-2} = 4L_{n-6} + F_{n-6}$,

$$f_{n-1} + f_{n-2} = 4L_{n-5} + F_{n-5} + 4L_{n-6} + F_{n-6} = 4(L_{n-5} + L_{n-6}) + (F_{n-5} + F_{n-6})$$
$$= 4L_{n-4} + F_{n-4},$$

then
$f_n = f_{n-1} + f_{n-2}$. □

The Hosoya index sequence of three leaf clover keychain graph with n vertices and 3 leaves is obtained as follows:

31, 49, 80, 129, 209, 338, 547, 885, 1 432, 2 317, 3 749, 6 066, 9 815, 1 5881, 25 696, 41 477, 67 173, 108 650, ...

For the general t leaf grass keychain graph, it is found that its Hosoya index has the same rule. The initial values $f_0 = 7t + 10, f_1 = 11t + 16$, the general term $:f_n = f_{n-1} + f_{n-2}$.

Proof. For $f_n = (t+1)L_{n-t-1} + F_{n-t-1}$, so $f_{n-1} = (t+1)L_{n-t-2} + F_{n-t-2}$, and $f_{n-2} = (t+1)L_{n-t-3} + F_{n-t-3}$,

$$f_{n-1} + f_{n-2} = (t+1)L_{n-t-2} + F_{n-t-2} + (t+1)L_{n-t-3} + F_{n-t-3}$$

$$= (t+1)(L_{n-t-2} + L_{n-t-3}) + (F_{n-t-2} + F_{n-t-3})$$

$$= (t+1)L_{n-t-1} + F_{n-t-1},$$

then

$f_n = f_{n-1} + f_{n-2}$. □

The Hosoya index sequence of t leaf clover keychain graph with n vertices and t leaves is obtained as follows:

7t+10, 11t+16, 18t+26, 29t+42, 47t+68, 76t+110, 123t+178, 199t+288, 322t+466, 521t+754, ...

Corollary 3.5. *As shown in Fig. 5, the Hosoya index of the 5 leaf grass keychain graph is as follows*

$$Z(H_n^5) = 6L_{n-6} + F_{n-6};$$

$n \geq 10$.

$Fig.5$

Proof. In Theorem 3.2, let t=5, then

$$Z(H_n^5) = 6L_{n-6} + F_{n-6};$$

$n \geq 10$.

□

The keychain diagram of the five leaf grass has the same rule as the Hosoya index of the t leaf grass keychain diagram. The Hosoya index sequence of 5 leaf clover keychain graph with n vertices and 5 leaves is obtained as follows:

45, 71, 116, 187, 303, 490, 793, 1 283, 2 076, 3 359, 5 435, 8 794, 14 229, 23 023, 37 252, 60 275, 97 527, 157 802, ...

4. Conclusions

In this paper, we have solved the explicit formula of Hosoya index of four leaf grass keychain graph, and extended it to the general t leaf grass keychain graph. Interestingly, we found the second family of Fibonacci sequences with different initial values, which are valuable in science.

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