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The Number of k Independent Sets of Graphs and The Mean Color Numbers of Graphs

Liming Yang* **Sihong Nian****

Department of Applied Mathematics,
Dalian University of Technology, Dalian 116024, P. R. China
e-mail: *yanglm65@yahoo.com.cn **niansihong@126.com

Abstract. In graph theory,enumeration of k Independent Sets of Graphs is NP-hard.In this paper, by means of $N(G,k)$, the authors present the formulas of the number $\alpha(G,k)$of exactly k Independent Sets of Graphs and $\mu(G)$(the mean color numbers), and derive classes of graphs $\alpha(G,k)$ and the mean color numbers $\mu(G)$of any $n-2$-regular graph,$n-3$-regular graph,tree, complete partite graph and any windgraph.Specially,$\alpha(G,k)$ of any tree is gained .

1. Introduction

In this paper, the authors discuss enumeration of the number $\alpha(G,k)$ of exactly k independent sets of graphs and the mean color number$\mu(G)$ by means of $N(G,k)$.

Definition 1.1. *For any n-colouring Γ of G, let $L(\Gamma)$ denote the actual number of colours used, the average of $L(\Gamma)^{,s}$ over all n-colouring Γ is called the mean color number. (see [1])*

Let $\mu(G)$ denote the mean color number of any graph G. In the paper[3], F. M. Dong gained bounds for mean color numbers of graphs.

Definition 1.2. *For $S^{(n)} = \{K_i : 1 \leq i \leq n\},\ n \geq 1,\ K_i$ is a complete graph with i vertices, if M is a subgraph of a any graph G, and each component of M is all isomorphic to some element of $S^{(n)} = \{K_i : 1 \leq i \leq n\}$, then M is called one $S^{(n)}$-subgraph, if M is a spanning subgraph of G, then M is called one $S^{(n)}$-factor of G.*

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Let $N(G,k)$ denote the number of $S^{(n)}$-factors with exactly k components.

In the paper[3], LiMin Yang has given the recurrence relation of $A(G)$. In the paper[4],LiMin Yang derived the recurrence formula of regular m-furcating tree. So far,we have solved counting problems of $N(G,k)$(see[6]),and gained the representing formula of $N(G,k)$, derived counting formulas of a great deal of graphs, for examples, any path, cycle, complete graph, $O\bigodot C_n$, wind graph K_n^d, complete d-partite graph, n-2-regular graph and n-3-regular graph. In this paper, the authors present the formulas of classes of graphs $\alpha(G,k)$ and $\mu(G)$by means of counting theory of $N(G,k)$. Specially, $\alpha(G,k)$ of any tree and $\mu(G)$ of any windgraph.

2. Lemmas

Here we will denote that $\alpha(G,k)$ is the number of partitions of $V(G)$ into exactly k non-empty independent sets of any graph G. $\alpha(G)$ is the number of all partitions of $V(G)$ into non-empty independent sets of any graph G, namely, $\alpha(G)=\sum_{k=1}^{n}\alpha(G,k)$.

Lemma 2.1 ([1]). *Let $\alpha(G,k)$ denote the number of partitions of $V(G)$ into exactly k non-empty independent sets, then the chromatic polynomial of G is $f(G,t)=\sum_{k=1}^{n}\alpha(G,k)(t)_k$, where $(t)_k=t(t-1)(t-2)\cdots(t-k+1)$.*

Proof. Let $N(G,k)$ denote the number of $S^{(n)}$-factors with exactly k components in G, then the chromatic polynomial of G is $f(G,t)=\sum_{k=1}^{n}N(\bar{G},k)(t)_k$, where $(t)_k=t(t-1)(t-2)\cdots(t-k+1)$. □

Lemma 2.2 ([5]). *If $\mu(G)$ is the mean color number of G, then*

$$\mu(G)=\frac{\sum\limits_{k=1}^{n}k(n)_k\alpha(G,k)}{\sum\limits_{k=1}^{n}(n)_k\alpha(G,k)},$$

where $\alpha(G,k)$ is the number of partitions of $V(G)$ into exactly k non-empty independent sets.

Lemma 2.3 ([6]). *If $N(G,k)$ is the number of $S^{(n)}$-factors with exactly k components in G, and the chromatic polynomial of the complementary graph $\bar{G}$ of G is $f(\bar{G},t)=\sum\limits_{p=1}^{n}Y_p t^p$, then the representing formula of $N(G,k)$ is the following*

$$N(G,k)=\sum_{p=k}^{n}N(K_p,k)Y_p\ ,$$

where n is the number of vertices of G.

Lemma 2.4 ([5])**.** *If* $G \cap H = \phi$, *then* $N(G \cup H, k) = \sum\limits_{l+m=k} N(G,l)N(H,m)$.

3. Basic Theorems

Theorem 3.1. *Suppose* $N(G,k)$ *is the number of* $S^{(n)}$*-factors with exactly* k *components in* G*, and the chromatic polynomial of graph* G *is* $f(G,t) = \sum\limits_{p=1}^{n} Y_p t^p$*, then the representing formula of* $\alpha(G,k)$ *is the following*

$$\alpha(G,k) = \sum_{p=k}^{n} N(K_p,k) Y_p \ ,$$

where

$$N(K_p,k) = \sum_{\sum_{i=1}^{p} ib_i = p \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!} \prod_{i \geq 2}^{p} \frac{1}{b_i!(i!)^{b_i}}.$$

Proof. By Lemma 2.1, $f(G,t) = \sum_{k=1}^{n} \alpha(G,k)(t)_k$, and by Lemma 2.2, $f(G,t) = \sum_{k=1}^{n} N(\bar{G},k)(t)_k$, so that $\sum_{k=1}^{n} \alpha(G,k)(t)_k = \sum_{k=1}^{n} N(\bar{G},k)(t)_k$, the equality $\alpha(G,k) = N(\bar{G},k)$. By Lemma 2.4, $N(G,k) = \sum_{p=k}^{n} N(K_p,k)Y_p$, where$Y_p($ $1 \leq p \leq n$ $)$ are coefficients of $f(G,t)$. Finally, we derive the representing formula of $\alpha(G,k)$ is the following

$$\alpha(G,k) = \sum_{p=k}^{n} N(K_p,k) Y_p \ .$$

□

Corollary 3.2. *There exists the equality* $\alpha(G,k) = N(\bar{G},k)$.

Proof. By the course of Theorem 2.1, the equality is derived $\alpha(G,k) = N(\bar{G},k)$. □

Theorem 3.3. *Suppose* $\mu(G)$ *is the mean color number of* G*, then*

$$\mu(G) = \frac{\sum\limits_{k=1}^{n} k(n)_k N(\bar{G},k)}{\sum\limits_{k=1}^{n} (n)_k N(\bar{G},k)} \ ,$$

where $N(G,k)$*is the number of* k*-components of* $S^{(n)}$*-factors of* G.

Proof. By Corollary 2.1 $\alpha(G,k) = N(\bar{G},k)$ and by Lemma 2.3

$$\mu(G) = \frac{\sum\limits_{k=1}^{n} k(n)_k \alpha(G,k)}{\sum\limits_{k=1}^{n} (n)_k \alpha(G,k)} \ ,$$

then we have the representing equality of $N(\bar{G}, k)$ with

$$\mu(G) = \frac{\sum_{k=1}^{n} k(n)_k N(\bar{G}, k)}{\sum_{k=1}^{n} (n)_k N(\bar{G}, k)} ,$$

where $N(G, k)$is the number of k-components of $S^{(n)}$-factors of G. □

4. Classes of Graphs $\alpha(G, k)$

In the section, we will obtain classes of graphs $\alpha(G, k)$. Specially, the number $\alpha(G, k)$ of k Independent Sets of any tree is give.

Theorem 4.1. *If G is a n-2-regular graph with n(even 2m)vertices, then*

$$\alpha(G, k) = \begin{cases} 0\,, & 1 \le k < m\,, \\ \binom{m}{k-m}, & m \le k \le 2m\,. \end{cases}$$

Proof. Let G be n-2-regular graph with n(even 2m) vertices, then $\bar{G}$ is a 1-regular graph, namely,$\bar{G} = K_2 \bigcup K_2 \bigcup \ldots \bigcup K_2$, and the number m of K_2.

$$N(\bar{G}, k) = \begin{cases} 0\,, & 1 \le k < \frac{n}{2}\,, \\ \binom{\frac{n}{2}}{k-\frac{n}{2}}, & \frac{n}{2} \le k \le n\,, \end{cases}$$

(see [5]). Finally, by means of Corollary 3.1, we give the result

$$\alpha(G, k) = \begin{cases} 0\,, & 1 \le k < m\,, \\ \binom{m}{k-m}, & m \le k \le 2m\,. \end{cases}$$

□

Theorem 4.2. *If G is a n-3-regular graph with n vertices,$n \ge 6$ and $\bar{G} \cong C_n$, then*

$$\alpha(G, k) = \begin{cases} 0\,, & 1 \le k < \frac{n}{2}\,, \\ \frac{n}{k}\binom{k}{n-k}, & \frac{n}{2} \le k \le n\,. \end{cases}$$

Proof. Suppose G is a n-3-regular graph with n vertices,$n \ge 6$ and $\bar{G} \cong C_n$, because $\bar{G}$ is a 2-regular graph, the graph would be able to join the disjoint cycles, thus assume that C_n, say. Then we have

$$N(\bar{G},k) = N(C_n,k) = \begin{cases} 0\,, & 1 \le k < \frac{n}{2}\,, \\ \frac{n}{k}\binom{k}{n-k}, & \frac{n}{2} \le k \le n\,. \end{cases}$$

(see [5]). By Corollary 1 $\alpha(G,k) = N(\bar{G},k)$, then the result is given

$$\alpha(G,k) = \begin{cases} 0\,, & 1 \le k < \frac{n}{2}\,, \\ \frac{n}{k}\binom{k}{n-k}, & \frac{n}{2} \le k \le n\,. \end{cases}$$

□

Corollary 4.3. *If G is a n-3-regular graph with n vertices,*

$$\bar{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q},$$

with $n_1 + n_2 + \cdots + n_q = n$*,*$3 \le n_j \le n, 1 \le j \le q, q \ge 1,$ *then*

$$\alpha(G,k) = \sum_{l_1+l_2+\cdots+l_q=k} \prod_{j=1}^{q} N(C_{n_j}, l_j)$$

where

$$when\ n_j = 3, N(C_3,l) = \begin{cases} 1\,, & l = 1\,, \\ 3\,, & l = 2\,, \\ 1\,, & l = 3\,, \end{cases}$$

$$when\ n_j \ge 4, N(C_{n_j},l_j) = \begin{cases} 0\,, & 1 \le l_j < \frac{n_j}{2}\,, \\ \frac{n_j}{l_j}\binom{l_j}{n_j-l_j}, & \frac{n_j}{2} \le l_j \le n_j\,. \end{cases}$$

Proof. Let G be a n-3-regular graph with n vertices, $n \ge 6$ and

$$\bar{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q},$$

$n_1 + n_2 + \cdots + n_q = n$,$C_{n_i} \cap C_{n_j} = \phi$ for any i and j, $i \neq j$,$3 \leq n_j \leq n, 1 \leq j \leq q, q \geq 1$, then

$$\begin{aligned} N(\overline{G}, k) &= N(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_q}, k) \\ &= \sum_{l_1+l_2+\cdots+l_q=k} N(C_{n_1}, l_1) N(C_{n_2}, l_2) \cdots N(C_{n_q}, l_q) \\ &= \sum_{l_1+l_2+\cdots+l_q=k} \prod_{j=1}^{q} N(C_{n_j}, l_j) \end{aligned}$$

By Corollary 3.1, so that we have

$$\alpha(G, k) = \sum_{l_1+l_2+\cdots+l_q=k} \prod_{j=1}^{q} N(C_{n_j}, l_j)$$

where

$$when\ n_j = 3, N(C_3, l) = \begin{cases} 1\,, & l = 1\,, \\ 3\,, & l = 2\,, \\ 1\,, & l = 3\,. \end{cases}$$

$$when\ n_j \geq 4, N(C_{n_j}, l_j) = \begin{cases} 0\,, & 1 \leq l_j < \frac{n_j}{2}\,, \\ \frac{n_j}{l_j}\binom{l_j}{n_j - l_j}, & \frac{n_j}{2} \leq l_j \leq n_j\,. \end{cases}$$

□

Theorem 4.4. *If G is a tree with n vertices, then*

$$\alpha(T, k) = \sum_{p=k}^{n} (-1)^{n-p} \binom{n-1}{p-1} N(K_p\,, k)\ ,$$

where

$$N(K_p\,, k) = \sum_{\sum_{i=1}^{p} ib_i = p,\ \sum_{i=1}^{p} b_i = k} \frac{p!}{b_1!} \prod_{i \geq 2} \frac{1}{b_i!(i!)^{b_i}}\ ,\ 2 \leq p\ \ k \leq n\ .$$

Proof. If G is a tree with n vertices, then the chromatic polynomial of G is

$$\begin{aligned} f(T,t) &= t(t-1)^{n-1} \\ &= t\sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}t^k = \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}t^{k+1} \\ &= (-1)^{n-1}t + (-1)^{n-2}\binom{n-1}{1}t^2 + \cdots + (-1)^{n-p}\binom{n-1}{p-1}t^p \\ &\quad + \cdots + (-1)\binom{n-1}{n-2}t^{n-1} + t^n \ . \end{aligned}$$

Coefficients of the chromatic polynomial of G are $Y_p = (-1)^{n-p}\binom{n-1}{p-1}, 1 \le p \le n$.

By Theorem 4.1 $N(\bar{T},k) = \sum_{p=k}^{n} N(K_p\,,k)Y_p$ and by Corollary 3.1 $\alpha(G,k) = N(\bar{G},k)$, then we have

$$\alpha(G,k) = N(\bar{T},k) = \sum_{p=k}^{n}(-1)^{n-p}\binom{n-1}{p-1}N(K_p\,,k)\ ,$$

where

$$N(K_p\,,k) = \sum_{\sum\limits_{i=1}^{p} ib_i=p,\ \sum\limits_{i=1}^{p} b_i=k} \frac{p!}{b_1!}\prod_{i\ge 2}\frac{1}{b_i!(i!)^{b_i}}\ ,\ 2 \le p\ \ k \le n\ .$$

□

Theorem 4.5. *If G is a complete d-partite graph*$K_{n_1,n_2,\cdots,n_d}$*, and* $n_1+n_2+\cdots+n_d = n$*, then*

$$\alpha(G,k) = \sum_{l_1+\cdots+l_d=k}\prod_{j=1}^{d} S(n_j,l_j)\ ,$$

where $S(n,k)$ *is the Stiring number of the second kind.*

Proof. Suppose $G = K_{n_1,n_2,\cdots,n_d}$ and $n_1+n_2+\cdots+n_d = n$,then $\bar{G} = K_{n_1}\bigcup K_{n_2}\bigcup\cdots\bigcup K_{n_d}$, $n_1+n_2+\cdots+n_d = n$,$K_{n_i}\cap K_{n_j} = \phi$ for any i and j,$i \ne j$,$3 \le n_j \le n, 1 \le j \le d, d \ge 1$, by Lemma 5,

$$N(\bar{G},k) = \sum_{l_1+l_2+\cdots+l_d=k} N(K_{n_1},l_1)N(K_{n_2},l_2)\cdots N(K_{n_d},l_d).$$

With $N(K_n, k) = S(n, k)$ (see [3]), then

$$N(\bar{G}, k) = \sum_{l_1+\cdots+l_d=k} \prod_{j=1}^{d} S(n_j, l_j)\ .$$

By Corollary 4.1, then we have

$$\alpha(G, k) = \sum_{l_1+\cdots+l_d=k} \prod_{j=1}^{d} S(n_j, l_j)\ .$$

□

Corollary 4.6. *If G is a complete d-partite graph $G = K_{n,n,\cdots,n}$, then*

$$\alpha(G, k) = \sum_{l_1+\cdots+l_d=k} \prod_{j=1}^{d} S(n, l_j)\ .$$

Proof. Omitted. □

Corollary 4.7. *If G is a complete tri- partite graph $G = K_{n,n,n}$, then*

$$\alpha(G, k) = \sum_{l_1+l_2+l_3=k} \prod_{j=1}^{3} S(n, l_j)\ .$$

Proof. Omitted. □

Theorem 4.8. *If G is a windgrpah K_n^d, $K_1$is a meet vertice of K_n with the number d, then $\alpha(G, k) = detM_{k-1}$, where M_{k-1} is*

$$\begin{pmatrix} s(1,1) & s(2,1) & \cdots & \sum\limits_{\sum\limits_{1\le j\le d} l_j=1} \prod\limits_{j=1}^{d} s(n-1,l_j) & \cdots & s(d(n-1),1) \\ 0 & s(2,2) & \cdots & \sum\limits_{\sum\limits_{1\le j\le d} l_j=2} \prod\limits_{j=1}^{d} s(n-1,l_j) & \cdots & s(d(n-1),2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sum\limits_{\sum\limits_{1\le j\le d} l_j=d(n-1)-1} \prod\limits_{j=1}^{d} s(n-1,l_j) & \cdots & s(d(n-1),d(n-1)-1) \\ 0 & 0 & \cdots & \sum\limits_{\sum\limits_{1\le j\le d} l_j=d(n-1)} \prod\limits_{j=1}^{d} s(n-1,l_j) & \cdots & s(d(n-1),d(n-1)) \end{pmatrix}_{r\times r},$$

(k-1 column)

here $r = d(n-1)$ and $s(n, k)$ is the Stirling number of the first kind, $2 \le k \le d(n-1)$.

Proof. Omitted.(see LiMin Yang and TianMing Wang[12],2004)

5. Classes of Graphs $\mu(G)$

Theorem 5.1. *If G is a n-2-regular graph with n(even 2m) vertices, then the explicit formula*

$$\mu(G) = \frac{\sum_{k=m}^{2m} k(2m)_k \binom{m}{k-m}}{\sum_{k=m}^{2m} (2m)_k \binom{m}{k-m}} .$$

Proof. By Theorem 4.2 $\mu(G) = \frac{\sum_{k=1}^{n} k(n)_k N(\bar{G},k)}{\sum_{k=1}^{n} (n)_k N(\bar{G},k)}$, and by the proving course of Theorem 4.3, G is a n-2-regular graph with n (even 2m) vertices, then

$$N(\bar{G},k) = \begin{cases} 0\,, & 1 \le k < m\,, \\ \binom{m}{k-m}, & m \le k \le 2m\,. \end{cases}$$

so that we derive the result

$$\mu(G) = \frac{\sum_{k=m}^{2m} k(2m)_k \binom{m}{k-m}}{\sum_{k=m}^{2m} (2m)_k \binom{m}{k-m}} .$$

□

Theorem 5.2. *If G is a n-3-regular graph with n vertices, $n \ge 6$ and $\bar{G} \cong C_n$, then the explicit formula*

$$\mu(G) = \frac{\sum_{k=m}^{n} k(n)_k \binom{k}{n-k}}{\sum_{k=m}^{n} (n)_k \binom{k}{n-k}} ,$$

where $m = [\frac{n}{2}]$.

Proof. Let G be a n-3-regular graph with n vertices,$n \ge 6$ and $\bar{G} \cong C_n$,by Theorem 5 then

$$N(\bar{G},k) = \begin{cases} 0\,, & 1 \le k < \frac{n}{2}\,, \\ \binom{\frac{n}{2}}{k-\frac{n}{2}}, & \frac{n}{2} \le k \le n\,. \end{cases}$$

By Theorem 4.2, we have the result when n is even,

$$\mu(G)=\frac{\sum\limits_{k=\frac{n}{2}}^{n} k(n)_k\binom{k}{n-k}}{\sum\limits_{k=\frac{n}{2}}^{n} (n)_k\binom{k}{n-k}}\ .$$

when n is odd,

$$\mu(G)=\frac{\sum\limits_{k=[\frac{n}{2}]+1}^{n} k(n)_k\binom{k}{n-k}}{\sum\limits_{k=[\frac{n}{2}]+1}^{n} (n)_k\binom{k}{n-k}}\ .$$

Finally,

$$\mu(G)=\frac{\sum\limits_{k=m}^{n} k(n)_k\binom{k}{n-k}}{\sum\limits_{k=m}^{n} (n)_k\binom{k}{n-k}}\ ,$$

where $m=[\frac{n}{2}]$, $n\in N$. □

Corollary 5.3. *If G is n-3-regular graph with n vertices, and* $\bar{G}=C_{i_1}\bigcup C_{i_2}\bigcup\cdots\bigcup C_{i_q}$, $i_1+i_2+\cdots+i_q=n$,$3\le i_j\le n, 1\le j\le q, q\ge 1$,*then*

$$\mu(G)=\frac{\sum\limits_{k=1}^{n} k(n)_k\sum\limits_{l_1+l_2+\cdots+l_q=k}\prod_{j=1}^{q}N(C_{n_j},l_j)}{\sum\limits_{k=1}^{n} (n)_k\sum\limits_{l_1+l_2+\cdots+l_q=k}\prod_{j=1}^{q}N(C_{n_j},l_j)}\ ,$$

where

$$when\ n_j=3, N(C_3,l)=\begin{cases}1\,, & l=1\,,\\ 3\,, & l=2\,,\\ 1\,, & l=3\,,\end{cases}$$

$$when\ n_j\ge 4, N(C_{n_j},l_j)=\begin{cases}0\,, & 1\le l_j<\frac{n_j}{2}\,,\\ \frac{n_j}{l_j}\binom{l_j}{n_j-l_j}, & \frac{n_j}{2}\le l_j\le n_j\,.\end{cases}$$

Proof. For $\bar{G} = C_{n_1} \bigcup C_{n_2} \bigcup \cdots \bigcup C_{n_q}$, $n_1 + n_2 + \cdots + n_q = n$,$C_{n_i} \cap C_{n_j} = \phi$ for any i and j,$i \neq j$,$3 \leq i, j \leq q, 1 \leq j \leq q, q \geq 1$, then

$$\begin{aligned} N(\overline{G}, k) &= N(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_q}, k) \\ &= \sum_{l_1+l_2+\cdots+l_q=k} N(C_{n_1}, l_1) N(C_{n_2}, l_2) \cdots N(C_{n_q}, l_q) \\ &= \sum_{l_1+l_2+\cdots+l_q=k} \prod_{j=1}^{q} N(C_{n_j}, l_j) \end{aligned}$$

By Theorem 4.2, we have $\mu(G) = \frac{\sum\limits_{k=1}^{n} k(n)_k \sum\limits_{l_1+l_2+\cdots+l_q=k} \prod_{j=1}^{q} N(C_{n_j}, l_j)}{\sum\limits_{k=1}^{n} (n)_k \sum\limits_{l_1+l_2+\cdots+l_q=k} \prod_{j=1}^{q} N(C_{n_j}, l_j)}$,

$$when\ n_j = 3, N(C_3, l) = \begin{cases} 1\,, & l = 1\,, \\ 3\,, & l = 2\,, \\ 1\,, & l = 3\,, \end{cases}$$

$$when\ n_j \geq 4, N(C_{n_j}, l_j) = \begin{cases} 0\,, & 1 \leq l_j < \frac{n_j}{2}\,, \\ \frac{n_j}{l_j} \binom{l_j}{n_j - l_j}, & \frac{n_j}{2} \leq l_j \leq n_j\,. \end{cases}$$

□

Theorem 5.4. *If G is a tree with n vertices, then the chromatic polynomial of G is $f(T, t) = t(t-1)^{n-1} = \sum\limits_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} t^{k+1}$.*

Proof. The proof is completed from Theorem 4.2 and Theorem 4.5, omitted. □

Theorem 5.5. *If G is a complete d-partite graph $G = K_{n_1, n_2, \cdots, n_d}$ and $n_1 + n_2 + \cdots + n_d = n$, then $\mu(G) = \frac{\sum\limits_{k=1}^{n} k(n)_k \sum\limits_{l_1+\cdots+l_d=k} \prod\limits_{j=1}^{d} S(n_j, l_j)}{\sum\limits_{k=1}^{n} (n)_k \sum\limits_{l_1+\cdots+l_d=k} \prod\limits_{j=1}^{d} S(n_j, l_j)}$.*

Proof. The proof is completed By Theorem 4.2 and Theorem 4.6, omitted. □

Corollary 5.6. *If G is a complete tri-partite $G = K_{n,n,n}$ then the mean colour number is the following $\mu(G) = \frac{\sum\limits_{k=1}^{3n} k(3n)_k \sum\limits_{l_1+l_2+l_3=k} \prod\limits_{j=1}^{3} S(n, l_j)}{\sum\limits_{k=1}^{3n} (3n)_k \sum\limits_{l_1+l_2+l_3=k} \prod\limits_{j=1}^{3} S(n, l_j)}$, where$S(n, k)$ is the Stirling number of the second kind.*

Proof. Let $n_1 = n_2 = n_3 = n, d = 3$, the proof is derived. Omitted. □

Theorem 5.7. *Suppose G is a windgraph K_n^d, K_1 is a meet vertex of K_n with the number d, then* $\mu(G) = \frac{\sum_{k=2}^{d(n-1)+1} k(d(n-1)+1)_k detM_{k-1}}{\sum_{k=2}^{d(n-1)+1} (d(n-1)+1)_k detM_{k-1}}$ *, where M_{k-1} is corresponding with Theorem 4.7.*

Proof. By the proving course of Theorem 7, for $2 \leq k \leq d(n-1)+1, N(\bar{G}, k) = detM_{k-1}$, by Theorem 2 and $k = 1, N(\bar{G}, 1) = 0$, then we have $\mu(G) = \frac{\sum_{k=2}^{d(n-1)+1} k(d(n-1)+1)_k detM_{k-1}}{\sum_{k=2}^{d(n-1)+1} (d(n-1)+1)_k detM_{k-1}}$,where M_{k-1} is corresponding with Theorem 7. □

6. Conclusions and future work

In this paper, the authors have solved the representing formulas of $\alpha(G, k)$ and $\mu(G)$. Specially, we give the counting explicit formula of $\alpha(G, k)$ of any tree T, and present the explicit formula of n-3 regular graph $\mu(G)$. Here we have solved NP-hard problem on enumeration of k independent sets of graphs by using of the representing formula of $\alpha(G, k)$ and counting theory of $S^{(n)}$-factors. In future work, we will research the number of all $\alpha(G, k)$ and some new problems. Some results related to the mean colour numbers will be seen in [11] and [12].

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