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Combinatorial identities from complete i-partite graphs

LiMin Yang

Department of Applied Mathematics
Dalian University of Technology, Dalian 116024, P. R. China
e-mail: yanglm65@yahoo.com.cn

Abstract. In this paper, our ways are combinatorial counting methods. In the use of the number $N(G,k)$ of $S^{(n)}$-factors with exactly k components, the author gaines combinatorial identities related to equation from enumeration of complete i-partite graphs and present a combinatorial formula on enumeration of complete i-partite graphs Finally, a beautiful identity of order 4 on the Stirling number of the first kind is given .

1. Lemmas

Lemma 1[1] Let $N(\bar{G},k)$ be the number of $S^{(n)}$-factors with exactly k components in complementary graph $\bar{G}$. Then

$$\sum_{k=1}^{n}(n)_k N(\bar{G},k)\,t^k = e^{\int \frac{\sum_{k=1}^{n} k(n)_k N(\bar{G},k)\,t^k}{t\sum_{k=1}^{n}(n)_k N(\bar{G},k)\,t^k} d_t}.$$

Lemma 2[2] Suppose $G=(X_1,X_2,\cdots,X_i)$ is a complete i-partite graph. Then there exists the combinatorial identity on $N(G,k)$ as follows:

$$\sum_{k=1}^{n} k!N(G,k)\binom{n}{k} = \frac{(n!)^i}{[(i-1)n]!}\binom{(i-1)n}{n-|X_1|,n-|X_2|,\cdots,n-|X_i|},$$

Where $|X_1|+|X_2|+\cdots+|X_i|=n$.

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Lemma 3[2] Let $G = (X_1, X_2)$ be a complete 2-partite graph, and $|X_1| = |X_2| = m$. Then

$$\sum_{k=1}^{2m} k! N(G,k) \binom{2m}{k} = (2m)! \binom{2m}{m, m}.$$

2. Theorems

Theorem 1 If $N(G,k)$ is the number of $S^{(n)}$-factors with exactly k components in any complete i-partite graph G, then

$$e^{\int \frac{\sum_{k=1}^{n} k(n)_k N(G,k) t^k}{t \sum_{k=1}^{n} (n)_k N(G,k) t^k} dt}\Big|_{t=1} = \frac{(n!)^i}{[(i-1)n]!} \binom{(i-1)n}{n-n_1, n-n_2, \cdots, n-n_i},$$

where $n_1 + n_2 + \cdots + n_i = n$.

Proof: Suppose G is a complete i-partite graph, $G = (X_1, X_2, \cdots, X_i), |X_1| = n_1, |X_2| = n_2, \cdots, |X_i| = n_i$, and $n_1 + n_2 + \cdots + n_i = n$. Then by Lemma 2

$$\sum_{k=1}^{n} k! N(G,k) \binom{n}{k} = \frac{(n!)^i}{[(i-1)n]!} \binom{(i-1)n}{n-n_1, n-n_2, \cdots, n-n_i}, \tag{2.1}$$

where $|X_j| = n_j, 1 \le j \le i, n_1 + n_2 + \cdots + n_i = n$.

Suppose G is any graph, $\bar{G}$ is the complementary graph of G. Then by Lemma 1 we obtain

$$\sum_{k=1}^{n} (n)_k N(\bar{G},k) t^k = e^{\int \frac{\sum_{k=1}^{n} k(n)_k N(\bar{G},k) t^k}{t \sum_{k=1}^{n} (n)_k N(\bar{G},k) t^k} dt} \tag{2.2}$$

(see[1]).

On the other hand, let $H = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_i}$, $K_p \cap K_q = \phi$, any $p, q, p \neq q, 1 \le p, q \le i$, then $\bar{H}$ is the complete i-partite graph $(X_1, X_2, \cdots, X_i)$, where $|X_j| = n_j, 1 \le j \le i$. From (2.2) then

$$\sum_{k=1}^{n} (n)_k N(G,k) t^k = e^{\int \frac{\sum_{k=1}^{n} k(n)_k N(G,k) t^k}{t \sum_{k=1}^{n} (n)_k N(G,k) t^k} dt} \tag{2.3}$$

where $G = \bar{H} = K_{n_1, n_2, \cdots, n_i}$, and by (2.1) the left

$$\sum_{k=1}^{n} k! N(G,k) \binom{n}{k} = \sum_{k=1}^{n} N(G,k)(n)_k = \sum_{k=1}^{n} (n)_k N(G,k),$$

let $t=1$ in the left of (2.3), obtain

$$\sum_{k=1}^{n}(n)_k N(G,k)t^k|_{t=1}=\sum_{k=1}^{n}(n)_k N(G,k),$$

where $G=K_{n_1,n_2,\cdots,n_i}$. By using (2.1) and (2.3), we derive the combinatorial identity related to equation

$$e^{\int\frac{\sum_{k=1}^{n}k(n)_kN(G,k)t^k}{t\sum_{k=1}^{n}(n)_kN(G,k)t^k}dt}\Big|_{t=1}=\frac{(n!)^i}{[(i-1)n]!}\binom{(i-1)n}{n-n_1,n-n_2,\cdots,n-n_i},$$

where $N(G,k)$ is the number of $S^{(n)}$-factors with exactly k components in the complete i-partite graph, and $n_1+n_2+\cdots+n_i=n$.

Corollary 1 Let G be a complete 2-partite graph, $G=(X_1,X_2)$, and $|X_1|=m,|X_2|=m$. Then

$$e^{\int\frac{\sum_{k=1}^{2m}k(2m)_kN(G,k)t^k}{t\sum_{k=1}^{2m}(2m)_kN(G,k)t^k}dt}|_{t=1}=\frac{[(2m)!]^2}{(m!)^2}.$$

Proof: If G is a complete 2-partite graph, $G=(X_1,X_2)$, and $|X_1|=m,|X_2|=m$, then by Lemma 3

$$\sum_{k=1}^{2m}k!N(G,k)\binom{2m}{k}=(2m)!\binom{2m}{m,m}=(2m)!\frac{(2m)!}{m!m!}=\frac{[(2m)!]^2}{(m!)^2}.$$

the left

$$\sum_{k=1}^{2m}k!N(G,k)\binom{2m}{k}=\sum_{k=1}^{m}(2m)_kN(G,k),$$

by Theorem 1 as the following

$$\sum_{k=1}^{2m}k!N(G,k)\binom{2m}{k}=e^{\int\frac{\sum_{k=1}^{2m}k(2m)_kN(G,k)t^k}{t\sum_{k=1}^{2m}(2m)_kN(G,k)t^k}dt}\Big|_{t=1}=(2m)!\binom{2m}{m,m}=\frac{[(2m)!]^2}{(m!)^2}$$

The proof is completed.

Corollary 2 Suppose G is a complete 3-partite graph, $G=(X_1,X_2,X_3)$, and $|X_1|=n_1,|X_2|=n_2,|X_3|=n_3$. Then

$$e^{\int\frac{\sum_{k=1}^{n}k(n)_kN(G,k)t^k}{t\sum_{k=1}^{n}(n)_kN(G,k)t^k}dt}|_{t=1}=(n_1)!(n_2)!(n_3)!\binom{n}{n_1}\binom{n}{n_2}\binom{n}{n_3},$$

where $n_1 + n_2 + \cdots + n_i = n$.

Proof: Because G is a complete 3-partite graph, $G = (X_1, X_2, X_3)$, and $|X_1| = n_1, |X_2| = n_2, |X_3| = n_3$, by Lemma 2 we derive

$$\sum_{k=1}^{n} k! N(G,k) \binom{n}{k} = \frac{(n!)^3}{(2n)!} \binom{2n}{n-n_1, n-n_2, n-n_3},$$

Where $n_1 + n_2 + n_3 = n$.

$$\frac{(n!)^3}{(2n)!} \binom{2n}{n-n_1, n-n_2, n-n_3} = \frac{(n!)^3}{(2n)!} \cdot \frac{(2n)!}{(n-n_1)!(n-n_2)!(n-n_3)!},$$

where $n_1 + n_2 + n_3 = n$, $(n-n_1) + (n-n_2) + (n-n_3) = 2n$,

$$\begin{aligned} \frac{(n!)^3}{(2n)!} \cdot \frac{(2n)!}{(n-n_1)!(n-n_2)!(n-n_3)!} &= \frac{(n!)^3 (n_1)!(n_2)!(n_3)!}{(n-n_1)!(n_1)!(n-n_2)!(n_2)!(n-n_3)!(n_3)!} \\ &= (n_1)!(n_2)!(n_3)! \binom{n}{n_1} \binom{n}{n_2} \binom{n}{n_3}, \end{aligned}$$

and from Theorem 1 we gain

$$e^{\int \frac{\sum_{k=1}^{n} k(n)_k N(G,k) t^k}{t \sum_{k=1}^{n} (n)_k N(G,k) t^k} dt} \Big|_{t=1} = (n_1)!(n_2)!(n_3)! \binom{n}{n_1} \binom{n}{n_2} \binom{n}{n_3},$$

where G is a complete 3-partite graph, $n_1 + n_2 + n_3 = n$.

Corollary 3 Let G be a complete 3-partite graph, $G = (X_1, X_2, X_3) = K_{m,m,m}$, and $|X| = m$. Then

$$e^{\int \frac{\sum_{k=1}^{3m} k(3m)_k N(G,k) t^k}{t \sum_{k=1}^{3m} (3m)_k N(G,k) t^k} dt} \Big|_{t=1} = (m!)^3 \binom{3m}{2m, m}^3.$$

Proof: Because G is the complete 3-partite graph, $G = (X_1, X_2, X_3) = K_{m,m,m}$, and $|X| = m$, by the Corollary 2 then

$$\begin{aligned} e^{\int \frac{\sum_{k=1}^{3m} k(3m)_k N(G,k) t^k}{t \sum_{k=1}^{3m} (3m)_k N(G,k) t^k} dt} \Big|_{t=1} &= \frac{[(3m)!]^3}{(6m)!} \binom{6m}{2m, 2m, 2m} \\ &= \frac{[(3m)!]^3}{(6m)!} \frac{(6m)!}{(2m)!(2m)!(2m)!} = \frac{[(3m)!]^3}{(2m)!(2m)!(2m)!} \\ &= \frac{[(3m)!]^3 m! m! m!}{(2m)! m! (2m!) m! (2m)! m!} = (m!)^3 \binom{3m}{2m, m}^3. \end{aligned}$$

The proof is completed.

Corollary 4 If G is any complete i-partite graph, $G = (X_1, X_2, \cdots, X_i) = K_{m,m,\cdots,m}$, and $|X| = m$, then

$$e^{\int \frac{\sum_{k=1}^{im} k(im)_k N(G,k)t^k}{t\sum_{k=1}^{im}(im)_k N(G,k)t^k}dt}\Big|_{t=1} = \frac{[(im)!]^i}{[(i-1)im]!}\binom{(i-1)im}{(i-1)m, (i-1)m, \cdots, (i-1)m},$$

where $m \in N, i \in N$.

Proof: Because G is any complete i-partite graph, $G = (X_1, X_2, \cdots, X_i) = K_{m,m,\cdots,m}$, and $|X| = m$, $n = |X| + |X| + \cdots + |X| = im$, $n_j = m$, $1 \le j \le i$, by Theorem 1 we derive as follows:

$$e^{\int \frac{\sum_{k=1}^{im} k(im)_k N(G,k)t^k}{t\sum_{k=1}^{im}(im)_k N(G,k)t^k}dt}\Big|_{t=1} = \frac{[(im)!]^i}{[(i-1)im]!}\binom{(i-1)im}{(i-1)m, (i-1)m, \cdots, (i-1)m},$$

The proof is completed .

3. Identities from complete i-partite graphs

Theorem 2 There exists combinatorial identity from complete i-partite graphs

$$\sum_{k=1}^{n} k! \det M_k \binom{n}{k} = \frac{(n!)^i}{[(i-1)n]!}\binom{(i-1)n}{n-|X_1|, n-|X_2|, \cdots, n-|X_i|},$$

where $|X_1| + |X_2| + \cdots + |X_i| = n$,

$$M_k = \begin{pmatrix} s(1,1) & s(2,1) & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j = 1} \prod\limits_{j=1}^{i} s(|X_j|, l_j) & s(n-1,1) & s(n,1) \\ 0 & s(2,2) & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j = 2} \prod\limits_{j=1}^{i} s(|X_j|, l_j) & s(n-1,2) & s(n,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j = n-1} \prod\limits_{j=1}^{i} s(|X_j|, l_j) & s(n-1,n-1) & s(n,n-1) \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j = n} \prod\limits_{j=1}^{i} s(|X_j|, l_j) & 0 & s(n,n) \end{pmatrix}_{n\times n},$$

$s(n,k)$ is the Stirling of the first kind.

Proof: Because of

$$\sum_{k=1}^{n} k!N(G,k)\binom{n}{k} = \frac{(n!)^i}{[(i-1)n]!}\binom{(i-1)n}{n-|X_1|, n-|X_2|, \cdots, n-|X_i|}$$

and $N(G,k) = \det M_k$, (see[ ],counting formulas of complete i-partite graphs) then

$$\sum_{k=1}^{n} k!\det M_k\binom{n}{k} = \frac{(n!)^i}{[(i-1)n]!}\binom{(i-1)n}{n-|X_1|, n-|X_2|, \cdots, n-|X_i|},$$

where $|X_1| + |X_2| + \cdots + |X_i| = n$,

$$M_k = \begin{pmatrix} s(1,1) & s(2,1) & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j = 1} \prod\limits_{j=1}^{i} s(|X_j|, l_j) & s(n-1,1) & s(n,1) \\ 0 & s(2,2) & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j = 2} \prod\limits_{j=1}^{i} s(|X_j|, l_j) & s(n-1,2) & s(n,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j = n-1} \prod\limits_{j=1}^{i} s(|X_j|, l_j) & s(n-1,n-1) & s(n,n-1) \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j = n} \prod\limits_{j=1}^{i} s(|X_j|, l_j) & 0 & s(n,n) \end{pmatrix}_{n\times n},$$

$s(n,k)$ is the Stirling of the first kind.

Corollary 5 There exists the combinatorial identity

$$\sum_{k=1}^{4m} k!\det M_k\binom{4m}{k} = \frac{[(4m)!]^4}{[(3m)!]^4},$$

where

$$M_k = \begin{pmatrix} s(1,1) & s(2,1) & \cdots & \sum\limits_{\sum_{1\le j\le 4} l_j=1} \prod\limits_{j=1}^{4} s(m,l_j) & s(n-1,1) & s(n,1) \\ 0 & s(2,2) & \cdots & \sum\limits_{\sum_{1\le j\le 4} l_j=2} \prod\limits_{j=1}^{4} s(m,l_j) & s(n-1,2) & s(n,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le 4} l_j=n-1} \prod\limits_{j=1}^{4} s(m,l_j) & s(n-1,n-1) & s(n,n-1) \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le 4} l_j=n} \prod\limits_{j=4}^{i} s(m,l_j) & 0 & s(n,n) \end{pmatrix}_{n\times n},$$

$s(n,k)$ is the Stirling of the first kind.

Proof: Let G be a complete 4-partite graph,and $G = (X_1, X_2, X_3, X_4) = K_{m,m,m,m}$, $|X| = m$, $n = 4m$, $n_j = m$, $1 \le j \le 4$. Then by Theorem 2 there exacts combinatorial identity

$$\sum_{k=1}^{4m} k!N(G,k)\binom{4m}{k} = \frac{[(4m)!]^4}{[(3m)!]^4},$$

where

$$M_k = \begin{pmatrix} s(1,1) & s(2,1) & \cdots & \sum\limits_{\sum_{1\le j\le 4} l_j=1} \prod\limits_{j=1}^{4} s(m,l_j) & s(n-1,1) & s(n,1) \\ 0 & s(2,2) & \cdots & \sum\limits_{\sum_{1\le j\le 4} l_j=2} \prod\limits_{j=1}^{4} s(m,l_j) & s(n-1,2) & s(n,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le 4} l_j=n-1} \prod\limits_{j=1}^{4} s(m,l_j) & s(n-1,n-1) & s(n,n-1) \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le 4} l_j=n} \prod\limits_{j=4}^{i} s(m,l_j) & 0 & s(n,n) \end{pmatrix}_{n\times n},$$

$s(n,k)$ is the Stirling of the first kind.

Corollary 6 There exists the combinatorial identity

$$\sum_{k=1}^{im} k! \det M_k \binom{im}{k} = \frac{[(im)!]^i}{[((i-1)m)!]^i},$$

where

$$M_k = \begin{pmatrix} s(1,1) & s(2,1) & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j=1} \prod\limits_{j=1}^{i} s(m,l_j) & s(n-1,1) & s(n,1) \\ 0 & s(2,2) & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j=2} \prod\limits_{j=1}^{i} s(m,l_j) & s(n-1,2) & s(n,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j=im-1} \prod\limits_{j=1}^{i} s(m,l_j) & s(n-1,n-1) & s(n,n-1) \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j=im} \prod\limits_{j=i}^{i} s(m,l_j) & 0 & s(n,n) \end{pmatrix}_{n\times n},$$

$s(n,k)$ is the Stirling of the first kind, $n = im$.

Proof: Let G be a complete i-partite graph, and $G = (X_1, X_2, \cdots, X_i) = K_{m,m,\cdots,m}$, and $|X| = m$, $n = |X| + |X| + \cdots + |X| = im$, $n_j = m$, $1 \le j \le i$. By Theorem 2 then

$$\sum_{k=1}^{im} k! \det M_k \binom{im}{k} = \frac{[(im)!]^i}{[((i-1)m)!]^i},$$

where

$$M_k = \begin{pmatrix} s(1,1) & s(2,1) & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j=1} \prod\limits_{j=1}^{i} s(m,l_j) & s(n-1,1) & s(n,1) \\ 0 & s(2,2) & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j=2} \prod\limits_{j=1}^{i} s(m,l_j) & s(n-1,2) & s(n,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j=im-1} \prod\limits_{j=1}^{i} s(m,l_j) & s(n-1,n-1) & s(n,n-1) \\ 0 & 0 & \cdots & \sum\limits_{\sum_{1\le j\le i} l_j=im} \prod\limits_{j=i}^{i} s(m,l_j) & 0 & s(n,n) \end{pmatrix}_{n\times n},$$

$s(n,k)$ is the Stirling of the first kind, and $n = im$.

4. Conclusions

In this paper, it is the main result that a beautiful identity of order 4 on the Stirling number of the first kind is given. It is very difficult and value in combinatorics from graphical enumeration.

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