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ASYMPTOTICS ON COMBINATORIAL ALTERNATIVE SUMS

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Abstract. In this article, we extend Rice's Lemma and give asymptotic value to the combinatorial alternative sums by the lemma and residue theorem.

Rice's method is designed to estimate the sums

$$Df_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k,$$

where the sequence f_k can be extended as an analytic function $f(k)$. Many problems in the analysis of algorithms lead to a sequence Df_n. In the sixties, Knuth encountered the sum $\sum_{k=2}^{n} \binom{n}{k} \frac{(-1)^k}{2^{k-1}-1}$ in the study of radix exchange sorting. One can find others examples in the analysis of digital structures or conflict resolution in broadcast communications.

There are two classical approaches to estimate such alternating sums: One can arrange the sum to obtain harmonic sums, which can be tackled by Mellin transforms. This is the standpoint of De Bruijn. Rice proposed a direct approach, which relies on the Rice's Lemma.

1. Preliminaries

Lemma 1 [1]. *Let $f(z)$ be an analytic function defined in a neighborhood Ω of the positive real axis $[0, \infty)$. Let $\mathbb{C}$ be a contour enclosing the integers $n_0, \ldots, n$ but no*

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singularity of $f(z)$. Then

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k f(k) = \frac{(-1)^n}{2\pi i} \int_{\mathbb{C}} f(z) \frac{n!dz}{z(z-1)\dots(z-n)}.$$

Lemma 2. *Let $f(z)$ be analytic in a domain D except for a finite number isolated singularities $z_1, z_2, \dots, z_n$. Let $\mathbb{C}$ be a positively oriented simple closed curve in the domain D which encircles all singularities, then*

$$\oint_{\mathbb{C}} f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}[f(z), z_k]\,.$$

Definition 1. *The Gamma function is a generalization of the factorial to complex numbers, one of its definitions is*

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt\,.$$

It satisfies the following relations:

$$\Gamma(n) = (n-1)!\,, \quad \forall n \in N_0\,, \tag{1.1}$$

$$\prod_{i=0}^{n} (s-i) = \frac{\Gamma(s+1)}{\Gamma(s-n)}\,, \tag{1.2}$$

$$\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} = n^\alpha (1 + O(\frac{1}{n})\,, \tag{1.3}$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} - \sum_{k=1}^{\infty} (\frac{1}{x+k} - \frac{1}{k})\,. \tag{1.4}$$

Definition 2 [1]. *The so called incomplete Hurwitz ζ function is:*

$$\zeta_n(r, \beta) = \sum_{i=0}^{n-1} \frac{1}{(i+\beta)^r}\,,$$

$\zeta_n(r, 1)$ defines the generalized harmonic $\zeta_n(r)$ and their limit ($n \to \infty$) is the famous Riemann ζ function.

From (4) it follows, that

$$\zeta_{n+1}(1, \beta) = \ln(n) - \frac{\Gamma'(\beta)}{\Gamma(\beta)} + O(\frac{1}{n})\,.$$

Definition 3 [3]. *The modified Bell polynomials $L_m = L_m(x_1, x_2, \dots, x_m)$ are defined as*

$$\exp(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}) = 1 + \sum_{k=1}^{\infty} L_m t^m. \tag{1.5}$$

It is rather technical than difficult to proof that in general

$$L_m(x_1, x_2, \dots, x_m) = \sum_{m_1+2m_2+\cdots=m} \frac{1}{m_1! m_2! \dots} \left(\frac{x_1}{1}\right)^{m_1} \left(\frac{x_2}{2}\right)^{m_2} \dots ,$$

and from this, we have

$$\exp(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}) = 1 + x_1 t + \left(\frac{x_2}{2} + \frac{x_1^2}{2}\right) t^2 + \left(\frac{x_3}{3} + \frac{x_1 x_2}{2} + \frac{x_1^3}{6}\right) t^3 + \left(\frac{x_4}{4} + \frac{x_1 x_3}{3} + \frac{x_2^2}{8} + \frac{x_2 x_1^2}{4} + \frac{x_1^4}{24}\right) t^4 + \cdots .$$

Definition 4 [1]. *A function $f(z)$ in an unbounded domain Ω is said to have polynomial growth , if for some r the formula $|f(z)| = O(|z|^r)$ holds as $z \to \infty$ in Ω. We also call r the degree of $f(z)$.*

Property 1 [1]. *If $f(z)$ is of polynomial growth (is of finite degree) in the half-plane $\Re(z) \geq c$ for some $c < n_0$, we have the alternative representation*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k f(k) = -\frac{(-1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(z) \frac{n! dz}{z(z-1)\dots(z-n)} , \qquad (1.6)$$

valid for n large enough, namely as soon as $n > r+1$.

2. Main results

We extend Rice's Lemma as follows.

Theorem 1. *Let $f(z)$ be an analytic function defined in a neighborhood Ω of the positive real axis $[0, \infty)$. Let $\mathbb{C}$ be a contour enclosing the integers $n_0, \dots, n, n+1, n+2, \dots, np+q (np > q)$ but does not include any of the integers $0, 1, \dots, n_0 - 1$ and not include singularity of $f(z)$. then*

$$\sum_{k=n_0}^{np+q} \binom{np+q}{k} (-1)^k f(k) = \frac{(-1)^{np+q}}{2\pi i} \int_{\mathbb{C}} f(z) \frac{(np+q)! dz}{z(z-1)\dots(z-np-q)} . \qquad (2.1)$$

Proof. We apply the residue theorem. Taking into account contribution of the simple poles at the integers $n_0, \dots, n, n+1, n+2, \dots, np+q$. The integral equals the sum of the residues of the integrand multiply $2\pi i$. Then we have:

$$\begin{aligned}
&\mathrm{Res}_{z=k} f(z)\frac{(np+q)!}{z(z-1)\dots(z-np-q)}\\
=&\mathrm{Res}_{z=k}\frac{1}{z-k}\left(f(z)\frac{(np+q)!}{z(z-1)\dots(z-k+1)(z-k-1)\dots(z-np-q)}\right)\\
=&f(k)\frac{(-1)^{np+q-k}(np+q)!}{k!(np+q-k)!}\,.
\end{aligned}$$

Simple summation for $k = n_0, \dots, n, n+1, n+2, \dots, np+q$, which completes the proof. It is the conclusion of Lemma 1 when $p = 1, q = 0$.

Next, we consider the basic case of a rational function.

Theorem 2. *Let $f(z)$ be a rational function which is analytic in a neighborhood of $[n_0, +\infty)$. If n is big enough we have*

$$\begin{aligned}
&\sum_{k=n_0}^{np+q}\binom{np+q}{k}(-1)^k f(k)\\
&\quad= -(-1)^{np+q}\sum_z \mathrm{Res} f(z)\frac{(np+q)!}{z(z-1)\dots(z-np-q)}\,,
\end{aligned} \tag{2.2}$$

where the sum is extended to all poles z of $\frac{f(z)}{z(z-1)\dots(z-np-q)}$ not on $[n_0, +\infty)$.

Proof. First, we use Theorem 1 and take as path of integration a large circle of radius R centered at the origin that avoids the poles. Then let $R \to +\infty$, and $n > r+1$ the integral on the right side of (2.1) tends to zero by a similar argument used for (2.2). By the residue Theorem, the integral also equals $\sum_{k=n_0}^{np+q}\binom{np+q}{k}(-1)^k f(k)$ plus the sum of the residues of (2.2) at the other poles of the integrand.

As a next step, we try to express the residues. As every rational function can be expressed as a linear combination of terms of the form $B(z-b)^{-r}$, where $r \in N_0$, we only have to consider function of this type.

Theorem 3. *Let α be a complex number not in N, and*

$$T(\alpha) = (-1)^{np+q}(np+q)!Res_\alpha \frac{1}{(z-\alpha)^r}\frac{1}{z(z-1)\dots(z-np-q)}\,.$$

Then $T(\alpha)$ has the following asymptotic

$$T(\alpha) = -\Gamma(-\alpha)(np)^\alpha \frac{(\ln np)^{r-1}}{(r-1)!}(1+O(\frac{1}{\ln np}))\,.$$

Proof.

$$\begin{aligned}
T(\alpha) &= -(np+q)![(z-\alpha)^{r-1}]\frac{1}{(-z)(1-z)\dots(np+q-z)} \\
&= -(np+q)![t^{r-1}]\frac{1}{(-t-\alpha)(1-t-\alpha)\dots(np+q-t-\alpha)} \\
&= -(np+q)![t^{r-1}]\exp(-\sum_{j=o}^{np+q}\ln(j-\alpha-t)) \\
&= \frac{-(np+q)!}{(-\alpha)(1-\alpha)\dots(np+q-\alpha)}[t^{r-1}]\exp(\sum_{m=1}^{\infty}\zeta_{np+q+1}(m,-\alpha)\frac{t^m}{m}) \\
&= -\frac{\Gamma(np+q+1)\Gamma(-\alpha)}{\Gamma(np+q+1-\alpha)}L_{r-1}(\zeta_{np+q+1}(1,-\alpha),\zeta_{np+q+1}(2,-\alpha),\dots) \\
&= -\Gamma(-\alpha)(np+q)^{\alpha}L_{r-1}(\ln(np+q)-\frac{\Gamma'(-\alpha)}{\Gamma(-\alpha)} \\
&+O(\frac{1}{np+q}),\zeta_{np+q+1}(2,-\alpha),\dots)(1+O(\frac{1}{np+q})) \\
&= -\Gamma(-\alpha)(np+q)^{\alpha}\frac{(\ln(np+q))^{r-1}}{(r-1)!}(1+O(\frac{1}{\ln(pn+q)})) \\
&= -\Gamma(-\alpha)(np)^{\alpha}(1+\frac{q}{np})^{\alpha}\frac{(\ln np(1+\frac{q}{np}))^{r-1}}{(r-1)!}(1+O(\frac{1}{\ln(np+q)})) \\
&= -\Gamma(-\alpha)(np)^{\alpha}\frac{(\ln np)^{r-1}}{(r-1)!}(1+O(\frac{1}{\ln(np)}))\,.
\end{aligned}$$

Applications on Theorem 2 and Theorem 3 are given as follows two Examples.

Example 1. Denote the sums $s(m)=\sum_{k=1}^{np}\binom{np}{k}\frac{(-1)^k}{k^m}$, for m any positive integer. Then

$$-s(m)=P_m(\ln(np))+O(\frac{\ln(np)^m}{np})\,,$$

where P_m is a polynomial of degree m.

by setting $q = 0$ and $f(k) = 1/k^m$ in Theorem 2:

$$\begin{aligned} s(m) &= -\mathrm{Res}_{z=0}\left(\frac{1}{z^{m+1}}\frac{(-1)^{np}(np)(np-1)\ldots 2\cdot 1}{(z-np)(z-np+1)\ldots(z-2)(z-1)}\right) \\ &= -\mathrm{Res}_{z=0}\left(\frac{1}{z^{m+1}}((1-\frac{z}{1})(1-\frac{z}{2})\ldots(1-\frac{z}{np}))^{-1}\right) \\ &= -[z^m]((1-\frac{z}{1})(1-\frac{z}{2})\ldots(1-\frac{z}{np}))^{-1} \\ &= -[z^m]\exp(-\sum_{j=1}^{np}\ln(1-\frac{z}{j})) \\ &= -[z^m]\exp(\sum_{j=1}^{np}\sum_{k=1}^{\infty}\frac{z^k}{kj^k}) = -[z^m]\exp(\sum_{k=1}^{\infty}\zeta_{np}(k)\frac{z^k}{k})\,. \\ -s(m) &= \sum_{m_1+2m_2+\ldots=m}\frac{1}{m_1!m_2!\ldots}(\frac{\zeta_{np}(1)}{1})^{m_1}(\frac{\zeta_{np}(2)}{2})^{m_2}\ldots \\ &= (1+O(\frac{1}{np}))\sum_{m_1+2m_2+\ldots=m}\frac{1}{m_1!m_2!\ldots}(\ln(np)+\gamma+O(\frac{1}{np}))^{m_1}(\frac{\zeta(2)}{2})^{m_2}\ldots\,. \end{aligned}$$

Since the $\zeta(k)$ are constants we have the asymtotics for a polynomial P_m of degree m.

$$-s(m) = P_m(\ln(np)) + O(\frac{\ln(np)^m}{np})\,.$$

For $m = 1, 2$, we get

$$-s(1) = \ln(np) + \gamma + O(\frac{1}{np})\,,$$
$$-s(2) = \frac{1}{2}(\ln(np))^2 + \gamma\ln(np) + \frac{\gamma}{2} + \frac{\pi^2}{12} + O(\frac{\ln np}{np})\,.$$

Example 2. Let the asymptotic analysis of the sequence of numbers

$$T(n) = \sum_{k=o}^{np+q}\binom{np+q}{k}\frac{(-1)^k}{k^2+1}\,.$$

Then

$$T(n) = \rho\cos(\theta_0 + \log(np)) + o(1).$$

We use Theorem 2 and Theorem 3:

$$\begin{aligned}
&\sum_{k=o}^{np+q} \binom{np+q}{k} \frac{(-1)^k}{k^2+1} \\
=&-(\Gamma(-\alpha_1)(np)^{\alpha_1}\frac{(\ln np)^{r-1}}{(r-1)!}+\Gamma(-\alpha_2)(np)^{\alpha_2}\frac{(\ln np)^{r-1}}{(r-1)!})(1+O(\frac{1}{\ln np})) \\
=&-(\Gamma(-i)(np)^{i}+\Gamma(i)(np)^{-i})(1+O(\frac{1}{\ln np})) \\
=&-(\Gamma(-i)(\cos\log(np)+i\sin\log(np)) \\
&+\Gamma(i)(\cos\log(np)-i\sin\log(np)))(1+O(\frac{1}{\ln np})) \\
=&-((\Gamma(-i)+\Gamma(i))\cos\log(np)+i(\Gamma(-i)-\Gamma(i))\sin\log(np))(1+O(\frac{1}{\ln np})) \\
=&-(\sqrt{\Gamma^2(-i)+\Gamma^2(i)})(\cos\theta_0\cos\log(np)-\sin\theta_0\sin\log(np))(1+O(\frac{1}{\ln np})) \\
=&\rho\cos(\theta_0+\log(np))+o(1)\,.
\end{aligned}$$

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